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# Currents on superconducting strings at finite chemical potential and temperature

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## Abstract

We consider the model giving rise to Witten's superconducting cosmic strings at finite fermion chemical potential and temperature. We demonstrate how various symmetries of the Hamiltonian can be used to exactly compute the fermion electric current in the string background. We show that the current along the string is not sensitive to the profiles of the string fields, and at fixed chemical potential and temperature depends only on the string winding number, the total gauge flux through the vortex and, possibly, the fermion mass at infinity.

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## 1. Introduction

Ever since Witten's pioneering paper [1], it has been known that cosmic strings can possess fermion zero modes concentrated in the string core. One remarkable feature of this system, is that an application of a constant electric field in the string direction induces an electric current along the string carried by the zero modes. This current will grow linearly with time, while the electric field is turned on and will persist even after the field is turned off. The string, thus, becomes superconducting. It must be noted, however, that the behaviour of the system is known precisely only for induced currents smaller than a certain critical current—once the current exceeds this critical value, the energies of the zero modes become larger than the fermion mass at infinity  $m$ , and it becomes possible for the charge carriers to move off the string, quenching the superconductivity. The question of build up of charge and current on the superconducting string in an external electric field has been analyzed extensively in [1–6].

In this Letter, we investigate a very different mechanism for inducing a current on the string. Namely, we compute the current on a superconducting string in the presence of a non-zero fermion chemical potential  $\mu$  and

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temperature  $T$ . It is a rather trivial exercise to calculate the current  $J$  along the string for  $\mu \lesssim m$ ,  $T \ll m$ , when then the low energy dynamics of the fermion–string system are governed by an effective  $(1 + 1)$ -dimensional theory of zero modes moving along the string. In this case it is straightforward to show that the current for each fermion species is  $J = \frac{e\mu n}{2\pi}$ , where  $e$  is the fermion charge and  $n$  is the winding number of the string. However, we make a much stronger statement: it is possible to calculate exactly the total electric current in the string direction for *any* value of the fermion chemical potential  $\mu$  and temperature  $T$ . We shall show that the result is topological in nature, and is independent of the particular profiles of the background string fields. The result will depend crucially on whether the string is local (as considered by Witten) or global (as, for instance, in the case of axion strings). In particular, if the string is local, the naive prediction for  $J$  of the effective  $(1 + 1)$ -dimensional theory, remains valid for any value of  $\mu$ ,  $T$ .

The appearance of quantum numbers (particularly of fermion number) on topological defects is a very well-developed subject with known computational methods [7], such as trace identities and adiabatic expansion. At zero chemical potential, the fermion charge induced on defects is usually a topological quantity and, frequently, can be evaluated exactly. However, at arbitrary finite chemical potential, the fermion charge induced is generally not topological [8], and difficult to compute exactly. In view of this, our result is particularly interesting, since we show that quantum numbers such as total current can remain topological and exactly calculable at arbitrary fermion chemical potential. Mathematically, our analysis can be easily generalized to a large class of Hamiltonians involving fermions in  $d + 1$  dimensions in the background of a  $d$ -dimensional defect, which is uniform in the  $(d + 1)$ st direction.

As far as we know, the problem of computation of electric current on a superconducting string in the background of an arbitrary fermion chemical potential and temperature has not been considered before, although some of the techniques we use have been previously discussed in conjunction to index theorem [9] for string zero modes, and charge induced on the string by an electric field [4,5]. Although the present problem could be of some interest in application to cosmology, this Letter was largely motivated by related problems in dense quark matter. It is well known that quark matter at large baryon chemical potential, which might be realized in the cores of neutron stars, breaks certain symmetries of QCD [10], and may supports several kinds of strings [11]. Recently, the method of fictitious axial anomalies has been used [12] to derive the following effective action for the interaction of an electromagnetic field  $A_\mu$  with an axial string in dense quark matter:

$$S = \sum_a \frac{e_a \mu_a n}{2\pi} \frac{Q_a}{q} \int A_i dt^i. \quad (1)$$

Here the line integral is along the string, the index  $a$  runs over all species of quarks, and the fraction  $Q_a/q$  denotes the flavor content of the condensate that supports the axial string. This implies that the axial string in dense QCD carries an electric current of  $J = \frac{e_a \mu_a n}{2\pi} \frac{Q_a}{q}$  for each quark species. We wish to understand this phenomenological result microscopically. The method for computation of current on strings at finite chemical potential developed in [12] is sensitive only to the pattern of symmetry breaking and, thus, would yield in application to the present model the same result,  $J = \frac{e\mu n}{2\pi}$ . As we see below, in some cases this result remains correct for all  $\mu$ , while in other cases it receives corrections of order 1.

## 2. Currents on strings

### 2.1. The model

Consider the following model of a Dirac fermion  $\psi$  coupled to a string:

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \left( \partial_\mu - ie A_\mu - \frac{iq}{2} R_\mu \gamma^5 \right) \psi - h \bar{\psi} \left( \phi^* \frac{1 + \gamma^5}{2} + \phi \frac{1 - \gamma^5}{2} \right) \psi. \quad (2)$$

Here  $A_\mu$  and  $R_\mu$  are gauge fields and  $\phi$  is a complex scalar field. The model has the following classical gauge symmetries:

$$U(1): \quad \psi \rightarrow e^{ie\alpha(x)}\psi, \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha, \quad \phi \rightarrow \phi, \quad (3)$$

$$\tilde{U}(1): \quad \psi \rightarrow e^{iq\theta(x)\gamma^5/2}\psi, \quad R_\mu \rightarrow R_\mu + \partial_\mu\theta, \quad \phi \rightarrow e^{iq\theta(x)}\phi. \quad (4)$$

This model is exactly equivalent to Witten's model of superconducting cosmic strings with a particular choice of gauge charges.<sup>1</sup> By convention, we associate the vector field  $A_\mu$  with electromagnetism. We note that the above model suffers from gauge anomalies, which can be removed, for example, by adding another fermion  $\hat{\psi}$  to the model with the opposite  $R$  charge  $\hat{q} = -q$  and the electric charge  $\hat{e}$ , such that  $\hat{e}^2 = e^2$ . The Lagrangian for the  $\hat{\psi}$  fermion is then:

$$\hat{\mathcal{L}} = \hat{\psi} i \gamma^\mu \left( \partial_\mu - i \hat{e} A_\mu + \frac{i q}{2} R_\mu \gamma^5 \right) \hat{\psi} - h \hat{\psi} \left( \phi \frac{1 + \gamma^5}{2} + \phi^* \frac{1 - \gamma^5}{2} \right) \hat{\psi}. \quad (5)$$

Notice that  $\hat{\psi}$  now couples to  $\phi^*$  rather than  $\phi$ .

We could also consider the situation when the  $\tilde{U}(1)$  symmetry is global, such that the gauge field  $R_\mu$  is absent, the Lagrangian (2) is by itself anomaly free, and the addition of the  $\hat{\psi}$  field is unnecessary. In our calculations, we will recover this case by taking  $q = 0$ .

We assume that the  $\tilde{U}(1)$  symmetry is spontaneously broken, the  $\phi$  field acquires a non-zero expectation value and strings of the  $\phi$  field are possible. We wish to consider the fermion  $\psi$  in the background of an infinitely long static string uniform in the  $z$  direction. The string is characterized by a non-zero winding number  $n$  of the scalar field:

$$n = \int \frac{dl^a}{2\pi i} \frac{\phi^* \partial_a \phi}{|\phi|^2}, \quad (6)$$

where the integral is over a contour in the  $xy$  plane at infinity, and the absolute value of the scalar field  $|\phi|$  tends to some constant  $\phi_0$  as  $r \rightarrow \infty$  in the  $xy$  plane.

If the  $\tilde{U}(1)$  symmetry is local, then in most models (such as the Abelian Higgs model), the condition that the string energy is finite, implies that  $D_\mu \phi = (\partial_\mu - iq R_\mu) \phi \rightarrow 0$  fast enough as  $r \rightarrow \infty$  in the  $xy$  plane. This in turn implies the quantization of the string flux:

$$\Phi = \frac{q}{4\pi} \int d^2x \epsilon^{ab} R_{ab} = n. \quad (7)$$

From here on  $a, b = 1, 2$  and  $R_{\mu\nu}$  is the usual field strength tensor. It must be noted that the condition (7) is not present in the case of global strings, so we will throughout our calculations keep the flux  $\Phi$  arbitrary and at the end set  $\Phi = n$  for local strings and  $\Phi = 0$  for global strings.

Our objective is to calculate the expectation value of the electromagnetic fermion current in the string direction,

$$J^3 = e \int d^2x \langle \bar{\psi} \gamma^3 \psi \rangle \quad (8)$$

at finite fermion chemical potential  $\mu$  and temperature  $T$ . Note that if the  $\tilde{U}(1)$  symmetry is local, there is an additional contribution to the electromagnetic current from the  $\hat{\psi}$  fermions. This, however, can be obtained from the result for the  $\psi$  fermions by setting  $e \rightarrow \hat{e}$ ,  $q \rightarrow \hat{q} = -q$ ,  $\phi \rightarrow \phi^*$ , which translates into  $\Phi \rightarrow -\Phi$ ,  $n \rightarrow -n$ .

<sup>1</sup> We could have easily considered the completely general version of Witten's model, however, to simplify the algebra slightly we concentrate on the above choice of gauge charges, which makes the  $A_\mu$  field couple to the vector current and the  $R_\mu$  field couple to the axial current.

## 2.2. The spectrum

Let us start by analyzing the spectrum of our fermions in the string background. The one-particle Hamiltonian is:

$$H = -i\alpha^i \left( \partial_i - ieA_i - \frac{iq}{2} R_i \gamma^5 \right) + h\gamma^0 \left( \phi^* \frac{1+\gamma^5}{2} + \phi \frac{1-\gamma^5}{2} \right), \quad (9)$$

where  $\alpha^i = \gamma^0 \gamma^i$  and  $i = 1, 2, 3$ . For a static background string uniform in the third direction,  $A_i = 0$ ,  $R_3 = 0$ , and hence,

$$H = -i\partial_3 \alpha^3 + H^\perp, \quad (10)$$

$$H^\perp = -i\alpha^a \left( \partial_a - \frac{iq}{2} R_a \gamma^5 \right) + h\gamma^0 \left( \phi^* \frac{1+\gamma^5}{2} + \phi \frac{1-\gamma^5}{2} \right). \quad (11)$$

Since all the fields are assumed uniform in the third direction, we can choose  $-i\partial_3 \psi = k\psi$  and work at fixed  $k$ .<sup>2</sup> In each  $k$  sector,

$$H_k = k\alpha^3 + H^\perp \quad (12)$$

and the operator  $H_k$  now acts solely in the transverse  $xy$  plane. At this point, we make our choice of the  $\gamma$  matrices to be:

$$\alpha^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha^a = \begin{pmatrix} 0 & i\sigma^a \\ -i\sigma^a & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}. \quad (13)$$

The operator  $H^\perp$  then takes the form:

$$H^\perp = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix}, \quad (14)$$

where

$$\mathcal{D} = \partial_a \sigma^a + \frac{q}{2} R_a \epsilon^{ab} \sigma^b + h \left( \frac{1+\sigma^3}{2} \phi + \frac{1-\sigma^3}{2} \phi^* \right). \quad (15)$$

Let us discuss the properties of the operator  $H^\perp$ . Since  $|\phi| \rightarrow \phi_0$  as  $r \rightarrow \infty$ , the continuum spectrum of  $H^\perp$  starts at eigenvalues  $|\lambda| = m = |h|\phi_0$ .  $H^\perp$  may also have bound states. We let  $m_b$  be the smallest positive eigenvalue of  $H^\perp$ . By dimensional reasons,  $m_b \sim m$ . Now, observe,

$$\{\alpha^3, H^\perp\} = 0. \quad (16)$$

Thus,  $\alpha^3$  maps a properly normalized eigenstate  $|\lambda\rangle$  of  $H^\perp$  with eigenvalue  $\lambda$  into a properly normalized eigenstate of  $H^\perp$  with eigenvalue  $-\lambda$ . Moreover, since  $\alpha^3$  maps zero modes of  $H^\perp$  into zero modes of  $H^\perp$ , all the zero modes of  $H^\perp$  can be classified by their eigenvalue under  $\alpha^3$ . Writing,  $\lambda(x) = (u(x), v(x))$ , we note that the zero modes of  $H^\perp$  with  $\alpha^3 = 1$  satisfy  $v = 0$ ,  $\mathcal{D}^\dagger u = 0$ , while the zero modes of  $H^\perp$  with  $\alpha^3 = -1$  satisfy  $u = 0$ ,  $\mathcal{D}v = 0$ . So letting  $N_+$  be the number of  $\alpha^3 = 1$  zero modes, and  $N_-$  the number of  $\alpha^3 = -1$  zero modes, we have,

$$N = N_+ - N_- = \dim(\ker(\mathcal{D}^\dagger)) - \dim(\ker(\mathcal{D})) = \text{Index}(H^\perp). \quad (17)$$

<sup>2</sup> We take the third direction  $z$  to be compact of length  $L$  so that the eigenvalues  $k$  are discrete. As usual, we will take  $L \rightarrow \infty$  at the end of the calculation.

Hence,  $N$  is the index of an elliptic operator, which is usually a strongly topological quantity.  $N$  has been first computed explicitly for a particular background string configuration in [13] to be:

$$N = n. \tag{18}$$

This result was later generalized [4,9] to arbitrary background string fields.

We now return to the operator  $H_k$ . Observe,  $[H_k, H^{\perp 2}] = 0$ . So, we can obtain the spectrum of  $H_k$  from the spectrum of  $H^{\perp}$  in the following way. Let,

$$H^{\perp} \lambda(x) = \lambda \lambda(x). \tag{19}$$

First, suppose,  $\lambda > 0$ . Then, the state,

$$\psi(x) = c_1 \lambda(x) + c_2 \alpha^3 \lambda(x) \tag{20}$$

is going to be an eigenstate of  $H_k$  with eigenvalue  $E$ , provided that,

$$\begin{pmatrix} \lambda & k \\ k & -\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{21}$$

The eigenvalues of the above equation are,

$$E = \pm \sqrt{\lambda^2 + k^2} \tag{22}$$

and the eigenvectors,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_{\pm} = \frac{1}{(2(\lambda^2 + k^2)^{\frac{1}{2}})^{\frac{1}{2}}} \begin{pmatrix} \pm \operatorname{sgn}(k)((\lambda^2 + k^2)^{\frac{1}{2}} \pm \lambda)^{\frac{1}{2}} \\ ((\lambda^2 + k^2)^{\frac{1}{2}} \mp \lambda)^{\frac{1}{2}} \end{pmatrix}. \tag{23}$$

Thus, each eigenstate of  $H^{\perp}$  with positive eigenvalue, generates one positive energy and one negative energy eigenstate of  $H_k$ . However, this correspondence has to be taken with a grain of salt, since most eigenstates of  $H^{\perp}$  are continuum states, and the “1 to 2” map discussed above between eigenstates of  $H^{\perp}$  and eigenstates of  $H_k$  need not preserve the density of states.

The zero modes of  $H^{\perp}$  are also simultaneously eigenstates of  $H_k$ . These have the dispersion,

$$E = k \alpha^3. \tag{24}$$

So the zero modes of  $H^{\perp}$  become chiral fermions moving up or down the string depending on the sign of their eigenvalue under  $\alpha^3$ .

### 2.3. Current—naive approach

We now proceed to the computation of electric current at finite  $\mu, T$ . This is given by:

$$J^3 = e \int d^2x \langle \bar{\psi} \gamma^3 \psi \rangle = e \int d^2x \operatorname{tr} \langle x | \alpha^3 n(H) | x \rangle, \tag{25}$$

where

$$n(E) = \frac{\operatorname{sgn}(E)}{e^{\beta(E-\mu)\operatorname{sgn}(E)} + 1} \tag{26}$$

is the usual Fermi–Dirac distribution. Summing over each momentum sector  $k$ , we obtain,<sup>3</sup>

$$J^3 = e \frac{1}{L} \sum_k \int d^2x \operatorname{tr} \langle x | \alpha^3 n(H_k) | x \rangle = e \frac{1}{L} \sum_k \operatorname{Tr}(\alpha^3 n(H_k)). \tag{27}$$

<sup>3</sup> Here  $\operatorname{tr}$  denotes matrix trace and  $\operatorname{Tr}$  denotes a general operator trace.

Using the correspondence between spectra of  $H_k$  and  $H^\perp$ , we may *schematically* write the operator trace (27) as:

$$\begin{aligned} J^3 &= e \frac{1}{L} \sum_k \sum_{E(H_k)} \langle \psi_E | \alpha^3 | \psi_E \rangle n(E) \\ &= e \frac{1}{L} \sum_k \left( \sum_{\lambda(H^\perp) > 0, s = \pm} \langle \psi_{\lambda, k, s} | \alpha^3 | \psi_{\lambda, k, s} \rangle n(E_s(\lambda, k)) + \sum_{\lambda(H^\perp) = 0} \langle \lambda | \alpha^3 | \lambda \rangle n(E(\lambda, k)) \right). \end{aligned} \quad (28)$$

Here  $E(H_k)$  denote eigenstates of  $H_k$ ,  $\lambda(H^\perp)$  denote eigenstates of  $H^\perp$ , and  $\psi_{\lambda, k, \pm}$  denote eigenstates of  $H_k$  generated by an eigenstate  $|\lambda\rangle$  of  $H^\perp$ , with energies  $E_\pm(\lambda, k) = \pm \sqrt{\lambda^2 + k^2}$ . Again, we stress that the above representation would have been absolutely correct if all the states contributing to the operator trace were discrete, and normalizable (for instance, if  $T = 0$  and  $\mu < m$ ). In our case, this is not generally so, but we choose for now to ignore this problem, in order to illustrate the general idea behind the computation. We will later return to take the continuum states into consideration more carefully.

For the moment suppose,  $T = 0$ ,  $0 < \mu < m_b$ . Then  $n(E) = \theta(E)\theta(\mu - E)$ . Hence, only states generated by zero modes contribute to the sum in (28), as all the other states have energies  $|E| = \sqrt{\lambda^2 + k^2} \geq |\lambda| \geq m_b > \mu$ . The zero modes are eigenstates of  $\alpha^3$ , and thus, satisfy,  $E = \alpha^3 k$  and  $\langle \lambda | \alpha^3 | \lambda \rangle = \alpha^3$ . Thus,

$$J^3 = e \frac{1}{L} \sum_k (N_+ \theta(\mu - k) \theta(k) - N_- \theta(\mu + k) \theta(-k)) = eN \int \frac{dk}{2\pi} \theta(k) \theta(\mu - k) = \frac{e\mu N}{2\pi} = \frac{e\mu n}{2\pi}, \quad (29)$$

where we have used the fact that index  $N$  is equal to the winding number of the vortex  $n$ .

Now, let us relax our assumption and work at arbitrary  $T$ ,  $\mu$ . We first need to evaluate the matrix element  $\langle \psi_{\lambda, k, s} | \alpha^3 | \psi_{\lambda, k, s} \rangle$  (in what follows we suppress the indices  $\lambda, k, s$ ). Using Eq. (20) and  $(\alpha^3)^2 = 1$ , we see  $\langle \psi | \alpha^3 | \psi \rangle = (|c_1|^2 + |c_2|^2) \langle \lambda | \alpha^3 | \lambda \rangle + (c_1^* c_2 + c_2^* c_1)$ . Recalling  $\alpha^3 | \lambda \rangle = |-\lambda\rangle$ , we obtain,  $\langle \psi | \alpha^3 | \psi \rangle = c_1^* c_2 + c_2^* c_1 = \frac{k}{E}$ , where we have used Eq. (23). Hence,

$$J^3 = e \frac{1}{L} \sum_k \left( \sum_{\lambda(H^\perp) > 0, s = \pm} \frac{k}{E_s(\lambda, k)} n(E_s(\lambda, k)) + N_+ n(k) - N_- n(-k) \right). \quad (30)$$

Now, observe that  $E_s(\lambda, k) = E_s(\lambda, -k)$  for  $\lambda > 0$ . Hence, the first sum in the brackets in Eq. (30) is odd in  $k$ , and, thus, the contribution to  $J^3$  from non-zero modes of  $H^\perp$  cancels out exactly, leading to:

$$J^3 = eN \frac{1}{L} \sum_k n(k) = en \int \frac{dk}{2\pi} n(k) = en n_0(\mu, T). \quad (31)$$

Here  $n_0(\mu, T)$  is the number density of a free massless chiral fermion in 1 dimension, at finite chemical potential  $\mu$  and temperature  $T$ . It is a peculiar fact that  $n_0(\mu, T)$  is temperature independent and equals  $\frac{\mu}{2\pi}$ , so that,

$$J^3 = \frac{e\mu n}{2\pi}. \quad (32)$$

#### 2.4. Current—corrections from polarized continuum

Although, the result (32) is very attractive, it is actually generally *incorrect*. We know that this result is exact for  $T = 0$ ,  $\mu < m$ , when  $J^3$  receives contributions only from normalizable eigenstates of  $H_k$ . We will now show, that the presence of long range vortex fields polarizes the continuum eigenstates of  $H_k$  in a way, which might significantly modify the result (32) for  $\mu > m$ .

Let us return to the trace (27). We can rewrite this expression in terms of spectral current density as:

$$J^3 = \int dE n(E) j^3(E), \quad (33)$$

$$j^3(E) = e \frac{1}{L} \sum_k \text{Tr}(\alpha^3 \delta(H_k - E)). \quad (34)$$

We use the following representation of the delta function,  $\delta(x) = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} (\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon})$ , to rewrite,

$$j^3(E) = \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0^+} e \frac{1}{L} \sum_k \text{Tr} \left( \alpha^3 \left( \frac{1}{H_k + z^+} - \frac{1}{H_k + z^-} \right) \right), \quad (35)$$

where  $z^+ = -E + i\epsilon$ ,  $z^- = -E - i\epsilon$ . From here on, the limit  $\epsilon \rightarrow 0^+$  is implied. Simplifying (35),

$$\begin{aligned} j^3(E) &= \frac{1}{L} \sum_k \frac{ie}{2\pi} \text{Tr} \left( \alpha^3 \left( \frac{H_k - z^+}{H_k^2 - (z^+)^2} - \frac{H_k - z^-}{H_k^2 - (z^-)^2} \right) \right) \\ &= \frac{1}{L} \sum_k \frac{ie}{2\pi} \text{Tr} \left( \alpha^3 \left( \frac{k\alpha^3 + H^\perp - z^+}{H^{\perp 2} + k^2 - (z^+)^2} - \frac{k\alpha^3 + H^\perp - z^-}{H^{\perp 2} + k^2 - (z^-)^2} \right) \right), \end{aligned} \quad (36)$$

where we have used  $H_k^2 = H^{\perp 2} + k^2$ . The terms in (36), which are odd in  $k$  cancel out, and we obtain,

$$j^3(E) = \frac{1}{L} \sum_k \frac{ie}{2\pi} \text{Tr} \left( \alpha^3 \left( \frac{H^\perp - z^+}{H^{\perp 2} + k^2 - (z^+)^2} - \frac{H^\perp - z^-}{H^{\perp 2} + k^2 - (z^-)^2} \right) \right). \quad (37)$$

Now,  $\{\alpha^3, H^\perp\} = 0$ . Hence, for any function  $f$ ,  $\text{Tr}(\alpha^3 H^\perp f(H^{\perp 2})) = -\text{Tr}(H^\perp f(H^{\perp 2})\alpha^3) = -\text{Tr}(\alpha^3 H^\perp \times f(H^{\perp 2})) = 0$ , and,

$$j^3(E) = \frac{1}{L} \sum_k \frac{ie}{2\pi} \text{Tr} \left( \alpha^3 \left( \frac{-z^+}{H^{\perp 2} + k^2 - (z^+)^2} - \frac{-z^-}{H^{\perp 2} + k^2 - (z^-)^2} \right) \right). \quad (38)$$

We now introduce the function  $g$ ,

$$g(M^2) = \text{Tr} \left( \alpha^3 \frac{M^2}{H^{\perp 2} + M^2} \right) = \text{Tr} \left( \frac{M^2}{\mathcal{D}\mathcal{D}^\dagger + M^2} \right) - \text{Tr} \left( \frac{M^2}{\mathcal{D}^\dagger\mathcal{D} + M^2} \right). \quad (39)$$

This function is very well known [7,9] as it satisfies,

$$N = \text{Index}(H^\perp) = \lim_{M^2 \rightarrow 0} g(M^2). \quad (40)$$

More generally,  $g(M^2)$  is related to the spectral asymmetry  $\sigma_k(E)$ , of the Hamiltonian  $H_k$  [7], and hence to its  $\eta$ -invariant as,

$$\sigma_k(E) = \frac{i}{2\pi} k (G(k^2 - (z^+)^2) - G(k^2 - (z^-)^2)), \quad (41)$$

$$\eta_k = 2 \int_0^\infty dE \sigma_k(E), \quad (42)$$

$$G(z) = \frac{g(z)}{z}. \quad (43)$$

Here,  $g(z)$  is understood as the analytic continuation of  $g$  from  $\mathfrak{R}_+$  to  $\mathcal{C}$ . From Eq. (38), we can express  $j^3(E)$  in terms of  $G$  as,

$$j^3(E) = \frac{1}{L} \sum_k \frac{-ie}{2\pi} (z^+ G(k^2 - (z^+)^2) - z^- G(k^2 - (z^-)^2)) = \frac{1}{L} \sum_k e \frac{E}{k} \sigma_k(E). \quad (44)$$

Following the technique of trace identities described in detail by [7,9], one can explicitly calculate  $g(M^2)$  to be:

$$g(M^2) = n - (n - \Phi) \frac{M^2}{m^2 + M^2}. \quad (45)$$

Hence, the index  $N = \lim_{M^2 \rightarrow 0} g(M^2) = n$ , in agreement with previous calculations [4,9,13]. Continuing  $g$  to the complex plane, we obtain,

$$G(z) = \frac{n}{z} - (n - \Phi) \frac{1}{z + m^2}. \quad (46)$$

Hence, generically,  $G$  has a pole at  $z = 0$ , and a pole at  $z = -m^2$ , i.e., at the continuum threshold. Notice, however, that the pole at  $z = m^2$  disappears when  $n = \Phi$ .

We can now substitute the result (46) into (41) and take the limit  $\epsilon \rightarrow 0^+$  to calculate the spectral asymmetry,

$$\sigma_k(E) = k \operatorname{sgn}(E) (n \delta(E^2 - k^2) - (n - \Phi) \delta(E^2 - k^2 - m^2)) \quad (47)$$

which yields the  $\eta$ -invariant,

$$\eta_k = n \operatorname{sgn}(k) - (n - \Phi) \frac{k}{\sqrt{k^2 + m^2}}. \quad (48)$$

We note that Eq. (48) is in agreement with previous calculation of the  $\eta$ -invariant [4].

Returning to the evaluation of current, we substitute the result (47) into Eq. (44) to obtain,

$$j^3(E) = \frac{1}{L} \sum_k \frac{e}{2} \left( n (\delta(E - k) + \delta(E + k)) - (n - \Phi) (\delta(E - \sqrt{k^2 + m^2}) + \delta(E + \sqrt{k^2 + m^2})) \right) \quad (49)$$

and the total current in the string direction (33) becomes,

$$J^3 = e \int \frac{dk}{2\pi} \left( nn(k) - \frac{n - \Phi}{2} (n(\sqrt{k^2 + m^2}) + n(-\sqrt{k^2 + m^2})) \right). \quad (50)$$

This can be conveniently rewritten as,

$$J^3 = e \left( nn_0(\mu, T) - \frac{n - \Phi}{2} n_m(\mu, T) \right), \quad (51)$$

where  $n_0(\mu, T) = \frac{\mu}{2\pi}$  is the familiar number density of one-dimensional chiral massless fermions, and,

$$n_m(\mu, T) = \int \frac{dk}{2\pi} (n(\sqrt{k^2 + m^2}) + n(-\sqrt{k^2 + m^2})) \quad (52)$$

is the number density of one-dimensional 2-component (Dirac) fermions of mass  $m$ .

Several comments are in order here. First of all, we see from Eq. (51) that the naive result (31) is generally modified by a contribution from modes located at the continuum threshold. Observe, that for  $\mu = 0$ , the current  $J^3$  vanishes for all temperatures. At non-zero chemical potential, two cases are of particular interest. The first case is that of a local string, satisfying the finite energy condition,  $D\phi \rightarrow 0$  faster than  $1/r$ , which implies  $\Phi = n$ . In this case, the contribution from continuum modes vanishes, and we recover our initial result (32), which is due solely to the zero modes,

$$J^3 = \frac{e\mu n}{2\pi}. \quad (53)$$

This ‘‘coincidence’’ can be explained as follows. If  $D\phi \rightarrow 0$  fast enough, the fields in the problem are, essentially, short range, and hence we can easily put the system in a box, making the spectrum discrete, so that the argument in Section 2.3 is correct.



Let us briefly discuss what happens when we add the second fermion  $\hat{\psi}$  to the problem. Recall, we used this fermion to cancel gauge anomalies of our model. As noted in Section 2.1, the contribution of  $\hat{\psi}$  to  $J^3$  can be obtained by taking  $e \rightarrow \hat{e}$ ,  $\mu \rightarrow \hat{\mu}$ ,  $n \rightarrow -n$ ,  $\Phi \rightarrow -\Phi$ . In particular, the continuum modes at threshold again cancel out, and  $\hat{J}^3 = -\frac{\hat{e}\hat{\mu}n}{2\pi}$ . In particular, if the chemical potentials of  $\psi$  and  $\hat{\psi}$  fermions are the same, we can obtain a non-vanishing total electromagnetic current along the string, by letting<sup>4</sup>  $\hat{e} = -e$ , so that,

$$J_{\text{EM}}^3 = \frac{e\mu n}{\pi}. \quad (54)$$

The second practically interesting case is that of a global string. This case can be recovered by taking  $\Phi \rightarrow 0$ . Then,

$$J^3 = en \left( n_0(\mu, T) - \frac{1}{2} n_m(\mu, T) \right). \quad (55)$$

In this case, the field  $\phi$  is long range, and there is a significant modification of the result (31). Note that  $n_m(\mu, T)$  is no longer temperature independent, so for simplicity, we choose to work at  $T = 0$ ,  $\mu > 0$ . Then,  $n(E) = \theta(E)\theta(\mu - E)$  and,

$$J^3 = \frac{en}{2\pi} (\mu - \theta(\mu - m) \sqrt{\mu^2 - m^2}). \quad (56)$$

Thus, for  $\mu < m$ , the current is governed by our original result (32), while for  $\mu > m$ , we also get a counterflow current from the states at continuum threshold. Thus,  $J^3(\mu)$  has a cusp at  $\mu = m$ , and for  $\mu \gg m$  falls off to 0 as  $\frac{emn}{4\pi} \frac{m}{\mu}$ .

### 3. Conclusion

In this Letter, we have found an exact expression for the electric current on superconducting strings as a function of fermion chemical potential and temperature. We have analyzed the case of both local and global strings, and our analysis has not been limited to a low energy theory of zero modes in the string core. Our ability to obtain such an exact result has been due to a cancellation (or partial cancellation) of contributions of all, but the zero fermion modes to the current. For local strings, we have seen that for *all* values of  $T$  and  $\mu$ , the current is due to zero modes in the string core. On the other hand, for global strings, the current receives contributions both from the zero modes and from certain states at continuum threshold. The latter contribution tends to cancel out the contribution from the zero modes as the fermion chemical potential becomes much larger than the fermion mass  $m$ . The results for  $\mu \gg m$ , might be particularly interesting in application to currents on axial strings in dense quark matter [12], where the gap  $\Delta \ll \mu$ .

We would like to note that the study of persistent topological currents and spin currents in conjunction with problems, such as, for example, quantum hall effect [14] and spin hall effect [15], has over the past years become an active subject of research in condensed matter physics. It would be interesting to investigate the relation of the phenomenon discussed in this Letter to problems in condensed matter systems. For instance, persistent supercurrents, are known to appear on vortices in superfluid  $^3\text{He-A}$  and, somewhat similarly to currents considered in this Letter, are due to chiral anomalies [16].

<sup>4</sup> This choice certainly respects the anomaly cancellation condition  $\hat{e}^2 = e^2$ .

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