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Rieffel deformation via crossed products ☆

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Abstract

We start from Rieffel data (A, Ψ, ρ), where A is a C*-algebra, ρ is an action of an abelian group Γ on A and Ψ is a 2-cocycle on the dual group. Using Landstad theory of crossed product we get a deformed C*-algebra A^{Ψ}. In the case of $\Gamma = \mathbb{R}^n$ we obtain a very simple proof of invariance of $\mathcal{K}$-groups under the deformation. In the general case we also get a very simple proof that nuclearity is preserved under the deformation. We show how our approach leads to quantum groups and investigate their duality. The general theory is illustrated by an example of the deformation of $SL(2, \mathbb{C})$. A description of it, in terms of noncommutative coordinates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, is given.
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Contents

1. Introduction

In [12] Rieffel described the method of deforming of C*-algebras known today as the *Rieffel deformation*. Having an action of $\mathbb{R}^d$ on a C*-algebra A and a skew symmetric operator $J : \mathbb{R}^d \mapsto \mathbb{R}^d$ Rieffel defined a new product that gave rise to the deformed C*-algebra A^J. In [13] the Rieffel deformation was applied to the C*-algebra of continuous functions vanishing at infinity on a Lie group G. An action of $\mathbb{R}^n$ was constructed using the left and right shifts along a fixed abelian Lie subgroup Γ. Having the deformed C*-algebra Rieffel introduced a comultiplication, a coinverse and a counit, showing that it is a locally compact quantum group.

M. Enock and L. Vainerman in [3] gave a method of deforming of the dual object associated with the locally compact group G that is $(C_r^*(G), \hat{\Delta})$ where $C_r^*(G)$ is the reduced group C*-algebra and $\hat{\Delta}$ is the canonical comultiplication on it. Using an abelian subgroup $\Gamma \subset G$ and a 2-cocycle Ψ on the Pontryagin dual group $\hat{\Gamma}$ they twisted the canonical comultiplication $\hat{\Delta}$ on the reduced group C*-algebra $C_r^*(G)$ obtaining a new quantum group. They also presented a formula for a multiplicative unitary and described a Haar measure for this new quantum group.

The existence of these two methods of deforming of objects related to a group G prompts the question about the relations between them. In this paper it is shown that they are dual versions of the same mathematical procedure. Let us note that the deformation framework of Enock and Vainerman is in a sense more general than the one of Rieffel: instead of a skew symmetric matrix on $\mathbb{R}^n$ they use a 2-cocycle Ψ on the abelian subgroup Γ. This suggests that it should be possible to perform the Rieffel deformation of a C*-algebra A acted on by an abelian group Γ with a 2-cocycle Ψ on $\hat{\Gamma}$. A formulation of Rieffel deformation in that context is one of the results of this paper.

Let us briefly describe the contents of the whole paper. In the next section we revise the Landstad theory of crossed products. We prove a couple of useful results that we could not find in the literature. In Section 3 we use the Landstad's theory to give a new approach to the Rieffel deformation of C*-algebras. In Section 4 we apply the Rieffel deformation to locally compact groups. We show that Enock–Vainerman's and Rieffel's approach give mutually dual, locally compact quantum groups. Moreover, a formula for a Haar measure on a quantized algebra of functions is given. In the last section we use our scheme to deform $SL(2, \mathbb{C})$. The subgroup Γ consists of diagonal matrices. We show that the deformed C*-algebra A is generated in the sense of Woronowicz by four affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ and give a detailed description of the commutation relations they satisfy. Moreover, we show that the comultiplication $\Delta^\Psi \in \mathrm{Mor}(A; A \otimes A)$ acts on the generators in the standard way:

$$\Delta^\Psi(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha} \dotplus \hat{\beta} \otimes \hat{\gamma}, \qquad \Delta^\Psi(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} \dotplus \hat{\beta} \otimes \hat{\delta},$$
$$\Delta^\Psi(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\alpha} \dotplus \hat{\delta} \otimes \hat{\gamma}, \qquad \Delta^\Psi(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta} \dotplus \hat{\gamma} \otimes \hat{\beta}. \tag{1}$$

Throughout the paper we will freely use the language of C*-algebras and the theory of locally compact quantum groups. For the notion of multipliers, affiliated elements, algebras generated by a family of affiliated elements and morphism of C*-algebras we refer the reader to [18] and [15]. For the theory of locally compact quantum groups we refer to [6] and [9]. For the theory of quantum groups given by a multiplicative unitary we refer to [19].

Some remarks about the notation. For a subset X of a Banach space B, X^{cls} denotes the closed linear span of X. Let A be a C*-algebra and A^* be its Banach dual. A^* is an A-bimodule where for $\omega \in A^*$ and $b, b' \in A$ we define $b \cdot \omega \cdot b'$ by the formula:

$$b \cdot \omega \cdot b'(a) = \omega(b'ab)$$

for any $a \in A$.

2. Landstad theory of crossed products

Let us start this section with a definition of Γ-product. For a detailed treatment of this notion see [10].

Definition 2.1. Let Γ be a locally compact abelian group, $\hat{\Gamma}$ its Pontryagin dual, B a C*-algebra, λ a homomorphism of Γ into the unitary group of $\mathrm{M}(B)$ continuous in the strict topology of $\mathrm{M}(B)$ and let $\hat{\rho}$ be a continuous action of $\hat{\Gamma}$ on B. The triple $(B, \lambda, \hat{\rho})$ is called a Γ-product if:

$$\hat{\rho}_{\hat{\gamma}}(\lambda_\gamma) = \langle \hat{\gamma}, \gamma \rangle \lambda_\gamma$$

for any $\hat{\gamma} \in \hat{\Gamma}$ and $\gamma \in \Gamma$.

The unitary representation $\lambda : \Gamma \to \mathrm{M}(B)$ gives rise to a morphism of C*-algebras $\lambda \in \mathrm{Mor}(\mathrm{C}^*(\Gamma); B)$. Identifying $\mathrm{C}^*(\Gamma)$ with $\mathrm{C}_\infty(\hat{\Gamma})$ via the Fourier transform, we get a morphism $\lambda \in \mathrm{Mor}(\mathrm{C}_\infty(\hat{\Gamma}); B)$. Let $\tau_{\hat{\gamma}} \in \mathrm{Aut}(\mathrm{C}_\infty(\hat{\Gamma}))$ denote the shift automorphism:

$$\tau_{\hat{\gamma}}(f)(\hat{\gamma}') = f(\hat{\gamma}' + \hat{\gamma}) \quad \text{for all } f \in \mathrm{C}_\infty(\hat{\Gamma}).$$

It is easy to see that λ intertwines the action $\hat{\rho}$ with τ:

$$\lambda(\tau_{\hat{\gamma}}(f)) = \hat{\rho}_{\hat{\gamma}}(\lambda(f)) \tag{2}$$

for any $f \in \mathrm{C}_\infty(\hat{\Gamma})$. The following lemma seems to be known but we could not find any reference.

Lemma 2.2. *Let $(B, \lambda, \hat{\rho})$ be a Γ-product. Then the morphism $\lambda \in \mathrm{Mor}(\mathrm{C}_\infty(\hat{\Gamma}); B)$ is injective.*

Proof. The kernel of the morphism λ is an ideal in $\mathrm{C}_\infty(\hat{\Gamma})$ hence it is contained in a maximal ideal. Therefore there exists $\hat{\gamma}_0$ such that $f(\hat{\gamma}_0) = 0$ for all $f \in \ker \lambda$. Eq. (2) implies that $\ker \lambda$ is τ invariant. Hence $f(\hat{\gamma}_0 + \hat{\gamma}) = 0$ for all $\hat{\gamma}$. This shows that $f = 0$ and $\ker \lambda = \{0\}$. $\square$

In what follows we usually treat a C*-algebra $\mathrm{C}_\infty(\hat{\Gamma})$ as a subalgebra of $\mathrm{M}(B)$ and we will not use the embedding λ explicitly.

Definition 2.3. Let $(B, \lambda, \hat{\rho})$ be a Γ-product and $x \in \mathrm{M}(B)$. We say that x satisfies the Landstad conditions if:

$$
\begin{cases}
\text{(i)} & \hat{\rho}_{\hat{\gamma}}(x) = x \text{ for all } \hat{\gamma} \in \hat{\Gamma}; \\
\text{(ii)} & \text{the map } \Gamma \ni \gamma \mapsto \lambda_\gamma x \lambda_\gamma^* \in \mathrm{M}(B) \text{ is norm continuous;} \\
\text{(iii)} & f x g \in B \text{ for all } f, g \in \mathrm{C}_\infty(\hat{\Gamma}).
\end{cases}
\tag{3}
$$

In computations it is useful to smear unitary elements $\lambda_\gamma \in \mathrm{M}(B)$ with a function $h \in L^1(\Gamma)$:

$$
\lambda_h = \int_\Gamma h(\gamma) \lambda_\gamma \, d\gamma \in \mathrm{M}(B).
$$

Note that $\lambda_h \in \mathrm{M}(B)$ coincides with the Fourier transform of h: $\mathfrak{F}(h) \in \mathrm{C}_\infty(\hat{\Gamma})$. In the original form of Definition 2.3 given by Landstad the third condition had the form:

$$
\lambda_f x, x \lambda_f \in B \quad \text{for all } f, g \in L^1(\Gamma).
\tag{4}
$$

Our conditions are simpler to check, which turns out to be useful in the example considered at the end of the paper. As shown below both definitions of invariants are in fact equivalent. The argument is very similar to the one given in [11] which shows that the third Landstad condition can be also replaced by

$$
\lambda_f x \in B \quad \text{for all } f \in L^1(\Gamma).
$$

Assume that $x \in \mathrm{M}(B)$ satisfies the Landstad conditions (3). Choose $\varepsilon \geqslant 0$ and a function $f \in L^1(\Gamma)$. By the second Landstad condition we can find a finite volume neighborhood $\mathcal{O}$ of the neutral element $e \in \Gamma$ such that:

$$
\|\lambda_\gamma x - x \lambda_\gamma\| \leqslant \varepsilon \quad \text{for all } \gamma \in \mathcal{O}.
\tag{5}
$$

Let $\chi_\mathcal{O}$ denote the normalized characteristic function of $\mathcal{O} \subset \Gamma$:

$$
\chi_\mathcal{O}(\gamma) = \begin{cases} \frac{1}{vol\,\mathcal{O}} & \text{if } \gamma \in \mathcal{O}, \\ 0 & \text{if } \gamma \in \Gamma \setminus \mathcal{O}. \end{cases}
$$

Then by (5) we have:

$$
\|\lambda_{\chi_\mathcal{O}} x - x \lambda_{\chi_\mathcal{O}}\| \leqslant \varepsilon.
\tag{6}
$$

If necessary, we can choose a smaller neighborhood and assume also that:

$$
\|\lambda_f \lambda_{\chi_\mathcal{O}} - \lambda_f\| \leqslant \varepsilon.
\tag{7}
$$

The calculation below is self-explanatory

$$
\lambda_f x = \lambda_f x - \lambda_f \lambda_{\chi_\mathcal{O}} x + \lambda_f \lambda_{\chi_\mathcal{O}} x = (\lambda_f x - \lambda_f \lambda_{\chi_\mathcal{O}} x) + (\lambda_f \lambda_{\chi_\mathcal{O}} x - \lambda_f x \lambda_{\chi_\mathcal{O}}) + \lambda_f x \lambda_{\chi_\mathcal{O}}
$$

and together with (6) and (7) shows that:

$$\|\lambda_f x - \lambda_f x \lambda_{\chi_\mathcal{O}}\| \leqslant \varepsilon \big(\|x\| + \|\lambda_f\|\big).$$

Hence we can approximate $\lambda_f x$ by elements of the form $\lambda_f x \lambda_{\chi_\mathcal{O}} \in B$. This shows that $\lambda_f x \in B$. A similar argument proves the second inclusion: $x\lambda_f \in B$.

The set of elements satisfying Landstad's conditions is a C*-algebra. We shall call it the Landstad algebra and denote it by A. It follows from Definition 2.3, that if $a \in A$ then $\lambda_\gamma a \lambda_\gamma^* \in A$ and the map $\Gamma \in \gamma \mapsto \lambda_\gamma a \lambda_\gamma^* \in A$ is norm continuous. An action of Γ on A defined in this way will be denoted by ρ.

It can be shown that the embedding of A into M(B) is a morphism of C*-algebras (cf. [8, Section 2]). Hence the multipliers algebra M(A) can also be embedded into M(B). Let $x \in$ M(B). Then $x \in$ M(A) if and only if it satisfies the following two conditions:

$$(8) \quad \begin{cases} \text{(i)} & \hat{\rho}_{\hat{\gamma}}(x) = x \text{ for all } \hat{\gamma} \in \hat{\Gamma}; \\ \text{(ii)} & \text{for all } a \in A, \text{ the map} \\ & \Gamma \ni \gamma \mapsto \lambda_\gamma x \lambda_\gamma^* a \in \text{M}(B) \\ & \text{is norm continuous.} \end{cases}$$

Note that the first and the second condition of (3) imply conditions (8).

Examples of Γ-products can be obtained via the crossed-product construction. Let A be a C*-algebra with an action ρ of Γ on A. There exists the standard action $\hat{\rho}$ of the group $\hat{\Gamma}$ on $A \rtimes_\rho \Gamma$ and a unitary representation $\lambda_\gamma \in$ M($A \rtimes_\rho \Gamma$) such that the triple ($A \rtimes_\rho \Gamma, \lambda_\gamma, \hat{\rho}$) is a Γ-product. It turns out that all Γ-products ($B, \lambda, \hat{\rho}$) are crossed-products of the Landstad algebra A by the action ρ implemented by λ. The following theorem is due to Landstad [10, Theorem 7.8.8]:

Theorem 2.4. *A triple* ($B, \lambda, \hat{\rho}$) *is a* Γ*-product if and only if there is a* C*-*dynamical system* (A, Γ, ρ) *such that* $B = A \rtimes_\rho \Gamma$. *This system is unique up to isomorphism and* A *consist of the elements in* M(B) *that satisfy Landstad conditions while* $\rho_\gamma(a) = \lambda_\gamma a \lambda_\gamma^*$.

Remark 2.5. The main problem in the proof of the above theorem is to show that the Landstad algebra is not small. It is solved by integrating the action $\hat{\rho}$ over the dual group. More precisely, we say that an element $x \in$ M(B)$_+$ is $\hat{\rho}$-integrable if there exists $y \in$ M(B)$_+$ such that

$$\omega(y) = \int d\hat{\gamma} \, \omega\big(\hat{\rho}_{\hat{\gamma}}(x)\big)$$

for any $\omega \in$ M(B)$_+^*$. We denote y by $\mathfrak{E}(x)$. If $x \in$ M(B) is not positive then we say that it is $\hat{\rho}$-integrable if it can be written as a linear combination of positive $\hat{\rho}$-integrable elements. The set of $\hat{\rho}$-integrable elements will be denoted by $D(\mathfrak{E})$. The averaging procedure induces a map:

$$\mathfrak{E} : D(\mathfrak{E}) \mapsto \text{M}(B).$$

It can be shown that for a large class of $x \in D(\mathfrak{E})$, $\mathfrak{E}(x)$ is an element of the Landstad algebra A. This is the case for $f_1 b f_2$ where $b \in B$ and $f_1, f_2 \in C_\infty(\hat{\Gamma})$ are square integrable. Moreover the

map:

$$B \ni b \mapsto \mathfrak{E}(f_1 b f_2) \in \mathrm{M}(B) \tag{9}$$

is continuous with the following estimate for norms

$$\left\| \mathfrak{E}(f_1 b f_2) \right\| \leqslant \|f_1\|_2 \|b\| \|f_2\|_2 \tag{10}$$

where $\| \cdot \|_2$ is the L^2-norm. Furthermore, we have

$$\left\{ \mathfrak{E}(f_1 b f_2) : b \in B, \ f_1, f_2 \in \mathrm{C}_\infty(\hat{\Gamma}) \cap L^2(\hat{\Gamma}) \right\}^{\mathrm{cls}} = A. \tag{11}$$

The last equality was not proven in [10]. We shall need it at some point so let us give a proof here. Let $a \in A$ and f_1, f_2, f_3, f_4 be continuous, compactly supported functions on Γ. Consider an element $x = \lambda_{f_1} \lambda_{f_2} a \lambda_{f_3} \lambda_{f_4}$. Clearly $x = \mathfrak{F}(f_1)\mathfrak{F}(f_2)a\mathfrak{F}(f_3)\mathfrak{F}(f_4)$ and $\mathfrak{F}(f_1), \ldots, \mathfrak{F}(f_4) \in L^2(\hat{\Gamma})$, hence by (10) $x \in D(\mathfrak{E})$. We compute:

$$
\begin{aligned}
\mathfrak{E}(x) &= \int d\hat{\gamma} \, \hat{\rho}_{\hat{\gamma}} (\lambda_{f_1} \lambda_{f_2} a \lambda_{f_3} \lambda_{f_4}) \\
&= \int d\hat{\gamma} \, \hat{\rho}_{\hat{\gamma}} \left(\int d\gamma_1 \, d\gamma_2 \, d\gamma_3 \, d\gamma_4 \, f_1(\gamma_1) f_2(\gamma_2) \lambda_{\gamma_1 + \gamma_2} a \lambda_{\gamma_3 + \gamma_4} f_3(\gamma_3) f_4(\gamma_4) \right) \\
&= \int d\hat{\gamma} \int d\gamma_1 \, d\gamma_2 \, d\gamma_3 \, d\gamma_4 \big(\langle \hat{\gamma}, \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \rangle f_1(\gamma_1) f_2(\gamma_2) \\
&\quad \times \lambda_{\gamma_1 + \gamma_2} a \lambda_{\gamma_3 + \gamma_4} f_3(\gamma_3) f_4(\gamma_4) \big).
\end{aligned}
$$

Using properties of the Fourier transform we obtain:

$$\mathfrak{E}(x) = \int d\gamma_1 \, d\gamma_2 \, d\gamma_3 \, \rho_{\gamma_1 + \gamma_2}(a) f_1(\gamma_1) f_2(\gamma_2) f_3(\gamma_3) f_4(-\gamma_1 - \gamma_2 - \gamma_3). \tag{12}$$

If f_1, f_2, f_3 approximate the Dirac delta function and $f_4(0) = 1$ then using (12) we see that elements $\mathfrak{E}(\lambda_{f_1} \lambda_{f_2} a \lambda_{f_3} \lambda_{f_4})$ approximate a in norm. This proves (11).

The following lemma is simple but very useful:

Lemma 2.6. *Let $(B, \lambda, \hat{\rho})$ be a Γ-product, A its Landstad algebra, ρ an action of Γ on A implemented by λ and $\mathcal{V} \subset A$ a subset of the Landstad algebra which is invariant under the action ρ and such that $(\mathrm{C}_\infty(\hat{\Gamma})\mathcal{V}\mathrm{C}_\infty(\hat{\Gamma}))^{\mathrm{cls}} = B$. Then $\mathcal{V}^{\mathrm{cls}} = A$.*

Proof. This proof is similar to the proof of formula (11). Let f_1, f_2, f_3, f_4 be continuous, compactly supported functions on Γ. Using (10) and (11) we get:

$$A = \left\{ \mathfrak{E}\big(\lambda_{f_1} (\lambda_{f_2} v \lambda_{f_3}) \lambda_{f_4} \big) : v \in \mathcal{V} \right\}^{\mathrm{cls}}. \tag{13}$$

A simple calculation shows that:

$$\mathfrak{E}\big(\lambda_{f_1}(\lambda_{f_2}v\lambda_{f_3})\lambda_{f_4}\big) = \int d\gamma_1\, d\gamma_2\, d\gamma_3\, \rho_{\gamma_1+\gamma_2}(v)f_1(\gamma_1)f_2(\gamma_2)f_3(\gamma_3)f_4(-\gamma_1-\gamma_2-\gamma_3).$$

Note that the integrand

$$\Gamma \times \Gamma \ni (\gamma_1,\gamma_2,\gamma_3) \mapsto \rho_{\gamma_1+\gamma_2}(v)f_1(\gamma_1)f_2(\gamma_2)f_3(\gamma_3)f_4(-\gamma_1-\gamma_2-\gamma_3) \in \mathcal{V}$$

is a norm continuous, compactly supported function, hence

$$\mathfrak{E}(\lambda_{f_1}v\lambda_{f_2}\lambda_{f_3}) \in \mathcal{V}^{\mathrm{cls}}.$$

From this and (13) we get $\mathcal{V}^{\mathrm{cls}} = A$. $\quad\square$

The next proposition shows that morphisms of Γ-products induce morphisms of their Landstad algebras. The below result can be, to some extent, deduced from the results of paper [4].

Proposition 2.7. *Let $(B,\lambda,\hat{\rho})$ and $(B',\lambda',\hat{\rho}')$ be Γ-products and let A, A' be Landstad algebras for B and B' respectively. Assume that $\pi \in \mathrm{Mor}(B,B')$ satisfies*:

- $\pi(\lambda_\gamma) = \lambda'_\gamma$;
- $\pi(\hat{\rho}_{\hat{\gamma}}(b)) = \hat{\rho}'_{\hat{\gamma}}(\pi(b))$.

Then $\pi(A) \subset \mathrm{M}(A')$ and $\pi|_A \in \mathrm{Mor}(A,A')$. Moreover, if $\pi(B) \subset B'$ then $\pi(A) \subset A'$. If $\pi(B) = B$ then $\pi(A) = A'$.

Proof. We start by showing that $\pi(A) \subset \mathrm{M}(A')$. Let $a \in A$. Then

$$\hat{\rho}'_{\hat{\gamma}}\big(\pi(a)\big) = \pi\big(\hat{\rho}_{\hat{\gamma}}(a)\big) = \pi(a).$$

Hence $\pi(a)$ is $\hat{\rho}'$ invariant. Moreover the map:

$$\Gamma \ni \gamma \mapsto \lambda'_\gamma\pi(a)\lambda'^*_\gamma = \pi\big(\lambda_\gamma a\lambda^*_\gamma\big) \in \mathrm{M}(A')$$

is norm continuous. This shows that $\pi(a)$ satisfies the first and the second Landstad condition of (3) which guaranties that $\pi(a) \in \mathrm{M}(A')$.

To prove that the homomorphism π restricted to A is in fact a morphism from A to A' we have to check that the set $\pi(A)A'$ is linearly dense in A'. We know that $\pi(B)B'$ is linearly dense in B'. Using this fact in the last equality below, we get

$$\big(\mathrm{C}^*(\Gamma)\pi(A)A'\mathrm{C}^*(\Gamma)\big)^{\mathrm{cls}} = \big(\pi\big(\mathrm{C}^*(\Gamma)A\big)A'\mathrm{C}^*(\Gamma)\big)^{\mathrm{cls}}$$
$$= \big(\pi(B)B'\big)^{\mathrm{cls}} = B'.$$

Moreover

$$\lambda'_\gamma\pi(a)a'\lambda'^*_\gamma = \pi\big(\lambda_\gamma a\lambda^*_\gamma\big)\lambda'_\gamma a'\lambda'^*_\gamma,$$

hence the set $\pi(A)A'$ is ρ'-invariant (remember that ρ' is the action of Γ implemented by λ'). This shows that $\pi(A)A'$ satisfies the assumptions of Lemma 2.6 and gives the density of $\pi(A)A'$ in A'.

Assume now that $\pi(B) \subset B'$. Let $a \in A$ satisfy Landstad conditions (3). Then as was shown at the beginning of the proof, $\pi(a)$ satisfies the first and the second Landstad condition. Moreover $f\pi(a)g = \pi(fag) \in B'$ for all $f, g \in C_\infty(\hat{\Gamma})$, hence $\pi(a)$ also satisfies the third Landstad condition. Therefore $\pi(a) \in A'$.

If $\pi(B) = B'$ then the equality

$$\mathfrak{E}\big(f_1\pi(b)f_2\big) = \pi\big(\mathfrak{E}(f_1bf_2)\big) \tag{14}$$

and property (11) shows that $\pi(A) = A'$.

To prove (14) take $\omega \in M(B')^*$. Then

$$\omega\big(\pi\big(\mathfrak{E}(f_1bf_2)\big)\big) = \omega \circ \pi\big(\mathfrak{E}(f_1bf_2)\big) = \int d\hat{\gamma}\, \omega \circ \pi\big(\hat{\rho}_{\hat{\gamma}}(f_1bf_2)\big)$$

$$= \int d\hat{\gamma}\, \omega\big(\hat{\rho}'_{\hat{\gamma}}\big(f_1\pi(b)f_2\big)\big) = \omega\bigg(\int d\hat{\gamma}\, \hat{\rho}'_{\hat{\gamma}}\big(f_1\pi(b)f_2\big)\bigg)$$

$$= \omega\big(\mathfrak{E}\big(f_1\pi(b)f_2\big)\big).$$

Hence $\pi(\mathfrak{E}(f_1bf_2)) = \mathfrak{E}(f_1\pi(b)f_2)$. $\square$

Let Γ' be an abelian locally compact group and $\phi : \Gamma \mapsto \Gamma'$ a continuous homomorphism. For $\hat{\gamma}' \in \hat{\Gamma}'$ we set $\phi^T(\hat{\gamma}') = \hat{\gamma}' \circ \phi \in \hat{\Gamma}$. The map

$$\phi^T : \hat{\Gamma}' \mapsto \hat{\Gamma}, \quad \phi^T(\hat{\gamma}') = \hat{\gamma}' \circ \phi$$

is a continuous group homomorphism called the dual homomorphism. We have a version of Proposition 2.7 with two different groups.

Proposition 2.8. *Let $(B, \lambda, \hat{\rho})$ be a Γ-product, $(B', \lambda', \hat{\rho}')$ a Γ'-product, $\phi : \Gamma \mapsto \Gamma'$ a surjective continuous homomorphism and $\phi^T : \hat{\Gamma}' \mapsto \hat{\Gamma}$ the dual homomorphism. Assume that $\pi \in \mathrm{Mor}(B, B')$ satisfies*:

- $\pi(\lambda_\gamma) = \lambda'_{\phi(\gamma)}$;
- $\hat{\rho}'_{\hat{\gamma}'}(\pi(b)) = \pi(\hat{\rho}_{\phi^T(\hat{\gamma}')}(b))$.

Then $\pi(A) \subset M(A')$ and $\pi|_A \in \mathrm{Mor}(A, A')$. Moreover, if $\pi(B) \subset B'$ then $\pi(A) \subset A'$. If $\pi(B) = B$ then $\pi(A) = A'$.

Let $\pi \in \mathrm{Mor}(B, B')$ be a morphism of C*-algebras satisfying the assumptions of Proposition 2.7 such that $\pi(B) = B'$. We have an exact sequence of C*-algebras:

$$0 \to \ker\pi \to B \overset{\pi}{\to} B' \to 0. \tag{15}$$

The C*-algebra $\ker \pi$ has a canonical Γ-product structure. Indeed, consider a morphism $\alpha \in \mathrm{Mor}(B; \ker \pi)$ associated with the ideal $\ker \pi \subset B$:

$$\alpha(b)j = bj$$

where $b \in \mathrm{B}$ and $j \in \ker \pi$. Note that

$$\alpha(b) = b \quad \text{for any } b \in \ker \pi \subset B. \tag{16}$$

For all $\gamma \in \Gamma$ we set $\tilde{\lambda}_\gamma = \alpha(\lambda_\gamma) \in \mathrm{M}(\ker \pi)$. The map $\Gamma \ni \gamma \mapsto \tilde{\lambda}_\gamma \in \mathrm{M}(\ker \pi)$ is a strictly continuous representation of Γ on $\ker \pi$. Moreover $\ker \pi$ is invariant under the action $\hat{\rho}$. The restriction of $\hat{\rho}$ to $\ker \pi$ will also be denoted by $\hat{\rho}$. It is easy to check that $\hat{\rho}_{\hat{\gamma}}(\tilde{\lambda}_\gamma) = \langle \hat{\gamma}, \gamma \rangle \tilde{\lambda}_\gamma$ which shows that the triple $(\ker \pi, \tilde{\lambda}, \hat{\rho})$ is a Γ-product. Let $\mathcal{I}$, A, A' be Landstad algebras for the Γ-products $(\ker \pi, \tilde{\lambda}, \hat{\rho})$, $(B, \lambda, \hat{\rho})$, $(B', \lambda', \hat{\rho}')$ respectively. Our objective is to show that the exact sequence (15) induces an exact sequence of Landstad algebras:

$$0 \to \mathcal{I} \to A \to A' \to 0. \tag{17}$$

Let $\bar{\pi} \in \mathrm{Mor}(A; A')$ denote a morphism of Landstad algebras induced by π. We assumed that π is surjective, hence by Proposition 2.7 $\bar{\pi}(A) = A'$ and we have an exact sequence of C*-algebras:

$$0 \to \ker \bar{\pi} \to A \overset{\bar{\pi}}{\to} A' \to 0.$$

It is easy to check that the morphism $\alpha \in \mathrm{Mor}(B; \ker \pi)$ satisfies the assumptions of Proposition 2.7, hence $\alpha(A) \subset \mathrm{M}(\mathcal{I})$. If we show that α restricted to $\ker \bar{\pi}$ identifies it with the Landstad algebra $\mathcal{I}$, then the existence of the exact sequence (17) will be proven. There are two conditions to be checked:

(i) $\alpha(\ker \bar{\pi}) = \mathcal{I}$;
(ii) if $x \in \ker \bar{\pi}$ and $\alpha(x) = 0$ then $x = 0$.

Ad(i) Let $a \in \ker \bar{\pi}$ and $f \in \mathrm{C}_\infty(\hat{\Gamma})$. Then $af \in B \cap \ker \pi$, hence

$$\alpha(a)f = \alpha(af) = af \in \ker \pi, \tag{18}$$

where we used (16). This shows that $\alpha(a)$ satisfies the third Landstad condition for Γ-product $(\ker \pi, \tilde{\lambda}, \hat{\rho})$. As in Proposition 2.7 we check that $\alpha(a)$ also satisfies the first and the second Landstad condition, hence $\alpha(\ker \bar{\pi}) \subset \mathcal{I}$. Furthermore, $\mathfrak{E}(f_1 b f_2) \in \ker \bar{\pi}$ for all $b \in \ker \pi$, and we have

$$\alpha\big(\mathfrak{E}(f_1 b f_2) = \mathfrak{E}\big(f_1 \alpha(b) f_2\big)\big) = \mathfrak{E}(f_1 b f_2).$$

Using (11) we see that $\alpha(\ker \bar{\pi}) = \mathcal{I}$.

Ad(ii) Assume that $a \in \ker \bar{\pi}$ and $\alpha(a) = 0$. Note that $af \in \ker \pi$ for any $f \in \mathrm{C}_\infty(\hat{\Gamma})$. Using (16) we get $af = \alpha(af) = \alpha(a)\alpha(f) = 0$. Hence $af = 0$ for any $f \in \mathrm{C}_\infty(\hat{\Gamma})$, which implies that $a = 0$. We can summarize the above considerations in the following:

Proposition 2.9. *Let* $(B, \lambda, \hat{\rho})$, $(B', \lambda', \hat{\rho}')$ *be* Γ-*products with Landstad algebras* A, A' *respectively,* $\pi \in \mathrm{Mor}(B, B')$ *a surjective morphism intertwining* $\hat{\rho}$ *and* $\hat{\rho}'$ *such that* $\pi(\lambda_\gamma) = \lambda'_\gamma$. *Let* $(\ker \pi, \tilde{\lambda}, \hat{\rho})$ *be the* Γ-*product described after Proposition* 2.8 *and let* $\mathcal{I} \subset \mathrm{M}(\ker \pi)$ *be its Landstad algebra. Then* $\mathcal{I}$ *can be embedded into* A *and we have a* Γ-*equivariant exact sequence*:

$$0 \to \mathcal{I} \to A \xrightarrow{\bar{\pi}} A' \to 0$$

where $\bar{\pi} = \pi|_A$.

3. Rieffel deformation of C*-algebras

3.1. Deformation procedure

Let $(B, \lambda, \hat{\rho})$ be a Γ-product. A 2-cocycle on the group $\hat{\Gamma}$ is a continuous function $\Psi : \hat{\Gamma} \times \hat{\Gamma} \to \mathbb{T}^1$ satisfying:

(i) $\Psi(e, \hat{\gamma}) = \Psi(\hat{\gamma}, e) = 1$ for all $\hat{\gamma} \in \hat{\Gamma}$;
(ii) $\Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2, \hat{\gamma}_3) = \Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3)\Psi(\hat{\gamma}_1, \hat{\gamma}_2)$ for all $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \in \hat{\Gamma}$.

(For the theory of 2-cocycles we refer to [5].)

For $\hat{\gamma}, \hat{\gamma}_1$ we set $\Psi_{\hat{\gamma}}(\hat{\gamma}_1) = \Psi(\hat{\gamma}_1, \hat{\gamma})$. It defines a family of functions $\Psi_{\hat{\gamma}} : \hat{\Gamma} \mapsto \mathbb{T}^1$. Using the embedding $\lambda \in \mathrm{Mor}(\mathrm{C}_\infty(\hat{\Gamma}); B)$ we get a strictly continuous family of unitary elements

$$U_{\hat{\gamma}} = \lambda(\Psi_{\hat{\gamma}}) \in \mathrm{M}(B). \tag{19}$$

The 2-cocycle condition for Ψ gives:

$$U_{\hat{\gamma}_1 + \hat{\gamma}_2} = \overline{\Psi(\hat{\gamma}_1, \hat{\gamma}_2)} U_{\hat{\gamma}_1} \hat{\rho}_{\hat{\gamma}_1}(U_{\hat{\gamma}_2}). \tag{20}$$

Theorem 3.1. *Let* $(B, \lambda, \hat{\rho})$ *be a* Γ-*product and let* Ψ *be a 2-cocycle on* $\hat{\Gamma}$. *For any* $\hat{\gamma} \in \hat{\Gamma}$ *the map*

$$\hat{\rho}_{\hat{\gamma}}^{\Psi} : B \ni b \mapsto \hat{\rho}_{\hat{\gamma}}^{\Psi}(b) = U_{\hat{\gamma}}^* \hat{\rho}_{\hat{\gamma}}(b) U_{\hat{\gamma}} \in B$$

is an automorphism of C*-*algebra* B. *Moreover,*

$$\hat{\rho}^{\Psi} : \hat{\Gamma} \ni \hat{\gamma} \mapsto \hat{\rho}_{\hat{\gamma}}^{\Psi} \in \mathrm{Aut}(B)$$

is a strongly continuous action of $\hat{\Gamma}$ *on* B *and the triple* $(B, \lambda_\gamma, \hat{\rho}^{\Psi})$ *is a* Γ-*product.*

Proof. Using Eq. (20) we get

$$\begin{aligned}
\hat{\rho}_{\hat{\gamma}_1 + \hat{\gamma}_2}^{\Psi}(b) &= U_{\hat{\gamma}_1 + \hat{\gamma}_2}^* \hat{\rho}_{\hat{\gamma}_1 + \hat{\gamma}_2}(b) U_{\hat{\gamma}_1 + \hat{\gamma}_2} \\
&= \overline{\Psi(\hat{\gamma}_1, \hat{\gamma}_2)} U_{\hat{\gamma}_1}^* \hat{\rho}_{\hat{\gamma}_1}(U_{\hat{\gamma}_2})^* \hat{\rho}_{\hat{\gamma}_1}(\hat{\rho}_{\hat{\gamma}_2}(b)) \hat{\rho}_{\hat{\gamma}_1}(U_{\hat{\gamma}_2}) U_{\hat{\gamma}_1} \Psi(\hat{\gamma}_1, \hat{\gamma}_2) \\
&= U_{\hat{\gamma}_1}^* \hat{\rho}_{\hat{\gamma}_1}(U_{\hat{\gamma}_2}^* \hat{\rho}_{\hat{\gamma}_2}(b) U_{\hat{\gamma}_2}) U_{\hat{\gamma}_1} = \hat{\rho}_{\hat{\gamma}_1}^{\Psi}(\hat{\rho}_{\hat{\gamma}_2}^{\Psi}(b)).
\end{aligned}$$

This shows that $\hat{\rho}^\Psi$ is an action of $\hat{\Gamma}$ on B. Applying $\hat{\rho}^\Psi$ to λ_γ we get:

$$\hat{\rho}^\Psi_{\hat{\gamma}}(\lambda_\gamma) = U^*_{\hat{\gamma}}\hat{\rho}_{\hat{\gamma}}(\lambda_\gamma)U_{\hat{\gamma}} = \langle \hat{\gamma}, \gamma \rangle U^*_{\hat{\gamma}}\lambda_\gamma U_{\hat{\gamma}} = \langle \hat{\gamma}, \gamma \rangle \lambda_\gamma.$$

The last equality follows from commutativity of Γ. Hence the triple $(B, \lambda_\gamma, \hat{\rho}^\Psi)$ is a Γ-product. $\quad\square$

The above theorem leads to the following procedure of deformation of C*-algebras. The data needed to perform the deformation is a triple (A, ρ, Ψ) consisting of a C*-algebra A, an action ρ of a locally compact abelian group Γ and a 2-cocycle Ψ on $\hat{\Gamma}$. Such a triple is called deformation data. The resulting C*-algebra will be denoted A^Ψ. The procedure is carried out in three steps:

(1) Construct the crossed product $B = A \rtimes_\rho \Gamma$. Let $(B, \lambda, \hat{\rho})$ be the standard Γ-product structure of the crossed product.
(2) Introduce a Γ-product $(B, \lambda, \hat{\rho}^\Psi)$ as described in Theorem 3.1.
(3) Let A^Ψ be the Landstad algebra of the Γ-product $(B, \lambda, \hat{\rho}^\Psi)$.

Note that A^Ψ still carries an action ρ^Ψ of Γ given by

$$\rho^\Psi_\gamma(x) = \lambda_\gamma x \lambda^*_\gamma.$$

In this case it is not the formula defining the action itself, but its domain of definition that changes under deformation. The triple $(A^\Psi, \Gamma, \rho^\Psi)$ will be called a *twisted dynamical system*. The procedure of deformation described above is called the Rieffel deformation. Using Theorem 2.4 we immediately get

Proposition 3.2. *Let (A, ρ, Ψ) be deformation data and $(A^\Psi, \Gamma, \rho^\Psi)$ be the twisted dynamical system considered above. Then*

$$A \rtimes_\rho \Gamma = A^\Psi \rtimes_{\rho^\Psi} \Gamma.$$

In what follows we investigate the dependence of the Rieffel deformation on the choice of a 2-cocycle. Let $f : \hat{\Gamma} \mapsto \mathbb{T}^1$ be a continuous function such that $f(e) = 1$. For all $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$ we set

$$\partial f(\hat{\gamma}_1, \hat{\gamma}_2) = \frac{f(\hat{\gamma}_1 + \hat{\gamma}_2)}{f(\hat{\gamma}_1)f(\hat{\gamma}_2)}.$$

One can check that the map

$$\partial f : \hat{\Gamma} \times \hat{\Gamma} \ni (\hat{\gamma}_1, \hat{\gamma}_2) \mapsto \frac{f(\hat{\gamma}_1 + \hat{\gamma}_2)}{f(\hat{\gamma}_1)f(\hat{\gamma}_2)} \in \mathbb{T}$$

is a 2-cocycle. 2-cocycles of this form are considered to be trivial. We say that a pair of 2-cocycles Ψ_1, Ψ_2 is in the same cohomology class if they differ by a trivial 2 cocycle: $\Psi_2 = \Psi_1 \partial f$.

Theorem 3.3. *Let (A, ρ, Ψ) be deformation data, giving rise to a Landstad algebra A^Ψ. Then the isomorphism class of the Landstad algebra A^Ψ depends only on the cohomology class of Ψ.*

Theorem 3.3 easily follows from the next two lemmas.

Lemma 3.4. *Let (A, ρ, Ψ) be deformation data with a trivial 2-cocycle $\Psi = \partial f$. Then A and A^Ψ are isomorphic. More precisely, treating f as an element of C^*-algebra $M(A \rtimes_\rho \Gamma)$ we have $A^\Psi = \{faf^*: a \in A\}$.*

Proof. Fixing the second variable in Ψ we get a family $\Psi_{\hat\gamma}$ of the form

$$\Psi_{\hat\gamma} = \overline{f(\hat\gamma)} f^* \tau_{\hat\gamma}(f), \tag{21}$$

where $\tau_{\hat\gamma}(f)(\hat\gamma') = f(\hat\gamma + \hat\gamma')$. Let $U_{\hat\gamma} \in M(A \rtimes_\rho \Gamma)$ be the unitary element given by $\Psi_{\hat\gamma}$ (cf. (19)). The function f can be embedded into $M(A \rtimes_\rho \Gamma)$ and using (21) we get:

$$U_{\hat\gamma} = \overline{f(\hat\gamma)} f^* \hat\rho_{\hat\gamma}(f). \tag{22}$$

Assume that $a \in A$. Then

$$\begin{aligned}
\hat\rho_{\hat\gamma}^\Psi \left(faf^*\right) &= U_{\hat\gamma}^* \hat\rho_{\hat\gamma}\left(faf^*\right) U_{\hat\gamma} \\
&= U_{\hat\gamma}^* \hat\rho_{\hat\gamma}(f) a \hat\rho_{\hat\gamma}(f)^* U_{\hat\gamma}.
\end{aligned}$$

Using Eq. (22) we see that

$$\hat\rho_{\hat\gamma}^\Psi \left(faf^*\right) = f(\hat\gamma) f \hat\rho_{\hat\gamma}(f)^* \hat\rho_{\hat\gamma}(f) a \hat\rho_{\hat\gamma}(f)^* \overline{f(\hat\gamma)} f^* \hat\rho_{\hat\gamma}(f) = faf^*$$

which means that the element faf^* satisfies the first Landstad condition for the Γ-product $(A \rtimes_\rho \Gamma, \lambda, \hat\rho^\Psi)$. It is easy to check that it also satisfies the second and third Landstad condition, hence $fAf^* \subset A^\Psi$. An analogous reasoning proves the opposite inclusion $fAf^* \supset A^\Psi$. $\quad\square$

Let Ψ_1, Ψ_2 be a pair of 2-cocycles on $\hat\Gamma$. Their product $\Psi_1\Psi_2$ is also a 2-cocycle. Let (A, Γ, ρ) be a dynamical system. The deformation data (A, ρ, Ψ_1) gives rise to the twisted dynamical system $(A^{\Psi_1}, \Gamma, \rho^{\Psi_1})$. Furthermore, the triple $(A^{\Psi_1}, \rho^{\Psi_1}, \Psi_2)$ is deformation data which gives rise to the C^*-algebra $(A^{\Psi_1})^{\Psi_2}$. At the same time, using the deformation data $(A, \rho, \Psi_1\Psi_2)$ we can introduce the C^*-algebra $A^{\Psi_1\Psi_2}$.

Lemma 3.5. *Let (A, Γ, ρ) be a Γ-product and let Ψ_1, Ψ_2 be 2-cocycles on the group $\hat\Gamma$. Let $(A^{\Psi_1})^{\Psi_2}$ be a C^*-algebra constructed from the deformation data $(A^{\Psi_1}, \rho^{\Psi_1}, \Psi_2)$ and let $A^{\Psi_1\Psi_2}$ be a C^*-algebra constructed from the deformation data $(A, \rho, \Psi_1\Psi_2)$. Then*

$$A^{\Psi_1\Psi_2} \simeq \left(A^{\Psi_1}\right)^{\Psi_2}.$$

Proof. The algebras $A^{\Psi_1 \Psi_2}$ and $(A^{\Psi_1})^{\Psi_2}$ can be embedded into $M(A \rtimes_\rho \Gamma)$: they are Landstad algebras of the Γ-products $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho}^{\Psi_1 \Psi_2})$ and $(A \rtimes_\rho \Gamma, \lambda, (\hat{\rho}^{\Psi_1})^{\Psi_2})$ respectively. Note that $U_{\hat{\gamma}}^{\Psi_1 \Psi_2} = U_{\hat{\gamma}}^{\Psi_1} U_{\hat{\gamma}}^{\Psi_2}$, hence

$$
\begin{aligned}
\hat{\rho}_{\hat{\gamma}}^{\Psi_1 \Psi_2}(b) &= U_{\hat{\gamma}}^{\Psi_1 \Psi_2 *} \hat{\rho}_{\hat{\gamma}}(b) U_{\hat{\gamma}}^{\Psi_1 \Psi_2} \\
&= U_{\hat{\gamma}}^{\Psi_2 *} U_{\hat{\gamma}}^{\Psi_1 *} \hat{\rho}_{\hat{\gamma}}(b) U_{\hat{\gamma}}^{\Psi_1} U_{\hat{\gamma}}^{\Psi_2} = \left(\hat{\rho}^{\Psi_1}\right)_{\hat{\gamma}}^{\Psi_2}(b).
\end{aligned}
$$

This shows that $\hat{\rho}^{\Psi_1 \Psi_2} = (\hat{\rho}^{\Psi_1})^{\Psi_2}$ and implies that the $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho}^{\Psi_1 \Psi_2})$ and $(A \rtimes_\rho \Gamma, \lambda, (\hat{\rho}^{\Psi_1})^{\Psi_2})$ are in fact the same Γ-products. Therefore their Landstad algebras coincide. $\quad\square$

3.2. Functorial properties of the Rieffel deformation

Let $(B, \lambda, \hat{\rho})$ be a Γ-product, Ψ a 2-cocycle on the dual group $\hat{\Gamma}$ and H a Hilbert space. Using Theorem 3.1 we introduce the twisted Γ-product $(B, \lambda, \hat{\rho}^\Psi)$. Let A, $A^\Psi \subset M(B)$ be Landstad algebras of $(B, \lambda, \hat{\rho})$ and $(B, \lambda, \hat{\rho}^\Psi)$ respectively and $\pi \in \text{Rep}(B; H)$ a representation of the C*-algebra B. The representation of B extends to multipliers $M(B)$ and can be restricted to A and A^Ψ.

Theorem 3.6. *Let $(B, \lambda, \hat{\rho})$, $(B, \lambda, \hat{\rho}^\Psi)$ be Γ-products considered above, A, A^Ψ their Landstad algebras and π a representation of C*-algebra B on a Hilbert space H. Then π is faithful on A if and only if it is faithful on A^Ψ.*

Proof. Assume that π is faithful on A and let $a \in A^\Psi$ be such that $\pi(a) = 0$. Invariance of a with respect to the action $\hat{\rho}^\Psi$ implies that $\hat{\rho}_{\hat{\gamma}}(a) = U_{\hat{\gamma}} a U_{\hat{\gamma}}^*$. Hence

$$
\hat{\rho}_{\hat{\gamma}}(f) U_{\hat{\gamma}} a U_{\hat{\gamma}}^* \hat{\rho}_{\hat{\gamma}}\left(g f^* \lambda_{-\gamma}\right) = \hat{\rho}_{\hat{\gamma}}\left(f a g f^* \lambda_{-\gamma}\right) \tag{23}
$$

for all $f, g \in C_\infty(\hat{\Gamma})$. The element $a \in A^\Psi$ belongs to $\ker \pi$ therefore

$$
\pi\left(\hat{\rho}_{\hat{\gamma}}(f) U_{\hat{\gamma}} a U_{\hat{\gamma}}^* \hat{\rho}_{\hat{\gamma}}\left(g f^* \lambda_{-\gamma}\right)\right) = 0.
$$

Combining it with Eq. (23) we obtain $\pi(\hat{\rho}_{\hat{\gamma}}(f a g f^* \lambda_{-\gamma})) = 0$.

Assume now that $f, g \in L^2(\hat{\Gamma}) \cap C_\infty(\hat{\Gamma})$. Let $\mathfrak{E}$ denote the averaging map with respect to undeformed action $\hat{\rho}$. Then $f a g f^* \lambda_{-\gamma} \in D(\mathfrak{E})$ and $\mathfrak{E}(f a g f^* \lambda_{-\gamma}) = 0$. Indeed, let $\omega \in B(H)_*$. Then

$$
\begin{aligned}
\omega\left(\pi\left(\mathfrak{E}(f a g f^* \lambda_{-\gamma})\right)\right) &= \omega \circ \pi\left(\mathfrak{E}(f a g f^* \lambda_{-\gamma})\right) \\
&= \int d\hat{\gamma}\, \omega\left(\pi\left(\hat{\rho}(f a g f^* \lambda_{-\gamma})\right)\right) = 0.
\end{aligned}
$$

Hence $\omega(\pi(\mathfrak{E}(f a g f^* \lambda_{-\gamma}))) = 0$ for any $\omega \in B(H)_*$ and $\pi(\mathfrak{E}(f a g f^* \lambda_{-\gamma})) = 0$. But $\mathfrak{E}(f a g f^* \lambda_{-\gamma}) \in A$ and π is faithful on A hence

$$
\mathfrak{E}\left(f a g f^* \lambda_{-\gamma}\right) = 0 \quad \text{for all } f, g \in L^2(\hat{\Gamma}) \cap C_\infty(\hat{\Gamma}). \tag{24}
$$

We will show that the above equation may be satisfied only if $a = 0$. Let $f_\epsilon \in L^1(\Gamma)$ be an approximation of the Dirac delta function as used in Theorem 7.8.7 of [10]. This theorem says that for any y of the form $y = fagf^*$ we have the following norm convergence:

$$\lim_{\varepsilon \to 0} \int \mathfrak{E}(y\lambda_{-\gamma})\lambda_\gamma \lambda_{f_\epsilon} \, d\gamma = y.$$

Using (24) we get $\mathfrak{E}(y\lambda_{-\gamma}) = \mathfrak{E}(fagf^*\lambda_{-\gamma}) = 0$ hence $fagf^* = 0$ for all $f, g \in L^2(\hat{\Gamma}) \cap C_\infty(\hat{\Gamma})$. This immediately implies that $a = 0$ and shows that π is faithful on A^Ψ. A similar argument shows that faithfulness of π on A^Ψ implies its faithfulness on A. $\square$

Definition 3.7. Let (A, ρ, Ψ), (A', ρ', Ψ') be deformation data with groups Γ and Γ' respectively. Let $\phi : \Gamma \mapsto \Gamma'$ be a surjective continuous homomorphism, $\phi^T : \hat{\Gamma}' \mapsto \hat{\Gamma}$ the dual homomorphism and $\pi \in \mathrm{Mor}(A, A')$. We say that (π, ϕ) is a morphism of deformation data (A, ρ, Ψ) and (A', ρ', Ψ') if:

- $\Psi \circ (\phi^T \times \phi^T) = \Psi'$;
- $\rho'_{\phi(\gamma)}\pi(a) = \pi(\rho_\gamma(a))$.

Using universal properties of crossed products, we see that a morphism (π, ϕ) of the deformation data induces the morphism $\pi^\phi \in \mathrm{Mor}(A \rtimes \Gamma; A' \rtimes \Gamma')$ of crossed products. One can check that π^ϕ satisfies the assumptions of Proposition 2.8 with the Γ-product $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho})$ and the Γ'-product $(A' \rtimes_{\rho'} \Gamma', \lambda', \hat{\rho}')$. This property is not spoiled by the deformation procedure. Applying Proposition 2.8 and Theorem 3.6 to the morphism $\pi^\phi \in \mathrm{Mor}(A \rtimes_\rho \Gamma; A' \rtimes_{\rho'} \Gamma')$, Γ-product $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho}^\Psi)$ and Γ'-product $(A' \rtimes_{\rho'} \Gamma', \lambda', \hat{\rho}'^{\Psi'})$ we get

Proposition 3.8. *Let (π, ϕ) be a morphism of deformation data (A, ρ, Ψ) and (A', ρ', Ψ') and let $\pi^\phi \in \mathrm{Mor}(A \rtimes_\rho \Gamma; A' \rtimes_{\rho'} \Gamma')$ be the induced morphism of the crossed products considered above. Then $\pi^\phi(A^\Psi) \subset \mathrm{M}(A'^{\Psi'})$ and $\pi^\phi|_{A^\Psi} \in \mathrm{Mor}(A^\Psi; A'^{\Psi'})$. Morphism $\pi \in \mathrm{Mor}(A; A')$ is injective if and only if so is $\pi^\phi|_{A^\Psi} \in \mathrm{Mor}(A^\Psi; A'^{\Psi'})$ and $\pi(A) = A'$ if and only if $\pi^\phi(A^\Psi) = A'^{\Psi'}$.*

Let $(\mathcal{I}, \Gamma, \rho_\mathcal{I})$, (A, Γ, ρ), (A', Γ, ρ') be dynamical systems and let

$$0 \to \mathcal{I} \to A \xrightarrow{\pi} A' \to 0 \tag{25}$$

be an exact sequence of C*-algebras which is Γ-equivariant. Morphism π induces a surjective morphism $\pi \in \mathrm{Mor}(A \rtimes_\rho \Gamma; A' \rtimes_{\rho'} \Gamma)$. It sends A to A' by means of π and it is identity on C*(Γ). Its kernel can be identified with $\mathcal{I} \rtimes_{\rho_\mathcal{I}} \Gamma$ so we have an exact sequence of crossed product C*-algebras:

$$0 \to \mathcal{I} \rtimes_{\rho_\mathcal{I}} \Gamma \to A \rtimes_\rho \Gamma \xrightarrow{\pi} A' \rtimes_{\rho'} \Gamma \to 0. \tag{26}$$

Note that $\pi \in \mathrm{Mor}(A \rtimes_\rho \Gamma; A' \rtimes_{\rho'} \Gamma)$ satisfies the assumptions of Proposition 2.9 with the Γ-products $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho})$ and $(A' \rtimes_{\rho'} \Gamma, \lambda', \hat{\rho}')$. This property is not spoiled by the deformation procedure. Hence applying Proposition 2.9 to the Γ-products $(A \rtimes_\rho \Gamma, \lambda, \hat{\rho}^\Psi)$ and $(A' \rtimes_{\rho'} \Gamma, \lambda', \hat{\rho}'^\Psi)$ we obtain

Theorem 3.9. *Let $(\mathcal{I}, \Gamma, \rho_{\mathcal{I}})$, (A, Γ, ρ), (A', Γ, ρ') be dynamical systems. Let*

$$0 \to \mathcal{I} \to A \xrightarrow{\pi} A' \to 0$$

be an exact sequence of C*-algebras which is Γ-equivariant, Ψ a 2-cocycle on the dual group $\hat{\Gamma}$ and $\mathcal{I}^{\Psi}$, A^{Ψ}, A'^{Ψ} the Landstad algebras constructed from the deformation data $(\mathcal{I}, \rho_{\mathcal{I}}, \Psi)$, (A, ρ, Ψ), (A', ρ', Ψ). Then we have the Γ-equivariant exact sequence:

$$0 \to \mathcal{I}^{\Psi} \to A^{\Psi} \xrightarrow{\pi^{\Psi}} A'^{\Psi} \to 0$$

where the morphism $\pi^{\Psi} \in \mathrm{Mor}(A^{\Psi}; A'^{\Psi})$ is the restriction of the morphism $\pi \in \mathrm{Mor}(A \rtimes_{\rho} \Gamma, A' \rtimes_{\rho'} \Gamma)$ to the Landstad algebra $A^{\Psi} \subset \mathrm{M}(A \rtimes_{\rho} \Gamma)$.

3.3. Preservation of nuclearity

Theorem 3.10. *Let (A, ρ, Ψ) be the deformation data which gives rise to the Landstad algebra A^{Ψ}. C*-algebra A is nuclear if and only if A^{Ψ} is.*

The proof follows from the equality $A \rtimes_{\rho} \Gamma = A^{\Psi} \rtimes_{\rho^{\Psi}} \Gamma$ (Proposition 3.2) and the following:

Theorem 3.11. *Let A be a C*-algebra with an action ρ of an abelian group Γ. Then A is nuclear if and only if $A \rtimes_{\rho} \Gamma$ is nuclear.*

The above theorem can be deduced from Theorem 3.3 and Theorem 3.16 of [14].

3.4. $\mathcal{K}$-theory in the case of $\Gamma = \mathbb{R}^n$

In this section we will prove the invariance of $\mathcal{K}$-groups under the Rieffel deformation in the case of $\Gamma = \mathbb{R}^n$. The tool we use is the analogue of the Thom isomorphism due to Connes [2]:

Theorem 3.12. *Let A be a C*-algebra, and ρ an action of $\mathbb{R}^n$ on A. Then*

$$\mathcal{K}_i(A) \simeq \mathcal{K}_{i+n}\big(A \rtimes_{\rho} \mathbb{R}^n\big).$$

Theorem 3.13. *Let $(A, \mathbb{R}^n, \rho)$ be a dynamical system and let (A, ρ, Ψ) be the deformation data giving rise to the Landstad algebra A^{Ψ}. Then*

$$\mathcal{K}_i(A) \simeq \mathcal{K}_i\big(A^{\Psi}\big).$$

Proof. Proposition 3.2 asserts that

$$A \rtimes_{\rho} \mathbb{R}^n \simeq A^{\Psi} \rtimes_{\rho^{\Psi}} \mathbb{R}^n.$$

Hence using Theorem 3.12 we get

$$\mathcal{K}_i(A) \simeq \mathcal{K}_{i+n}\big(A \rtimes_{\rho} \mathbb{R}^n\big) \simeq \mathcal{K}_{i+n}\big(A^{\Psi} \rtimes_{\rho^{\Psi}} \mathbb{R}^n\big) \simeq \mathcal{K}_i\big(A^{\Psi}\big). \qquad \square$$

4. Rieffel deformation of locally compact groups

4.1. From an abelian subgroup with a dual 2-cocycle to a quantum group

In this section we shall apply our deformation procedure to the algebra of functions on a locally compact group G. First we shall fix a notation and introduce auxiliary objects. Let $G \ni g \mapsto R_g \in \mathrm{B}(L^2(G))$ be the right regular representation of G on Hilbert space $L^2(G)$ of the right invariant Haar measure. Let $\mathrm{C}_\infty(G) \subset \mathrm{B}(L^2(G))$ be the C*-algebra of continuous functions on G vanishing at infinity, $\mathrm{C}_r^*(G) \subset \mathrm{B}(L^2(G))$ the reduced group C*-algebra generated by R_g and $V \in \mathrm{B}(L^2(G \times G))$ the Kac–Takesaki operator: $Vf(g, g') = f(gg', g')$ for any $f \in L^2(G \times G)$. By $\Delta_G \in \mathrm{Mor}(\mathrm{C}_\infty(G); \mathrm{C}_\infty(G) \otimes \mathrm{C}_\infty(G))$ we will denote the comultiplication on $\mathrm{C}_\infty(G)$. It is known that the Kac–Takesaki operator V is an element of $\mathrm{M}(\mathrm{C}_r^*(G) \otimes \mathrm{C}_\infty(G))$ which implements comultiplication:

$$\Delta_G(f) = V(f \otimes 1)V^*$$

for any $f \in \mathrm{C}_\infty(G)$. Let $\Gamma \subset G$ be an abelian subgroup of G, $\hat{\Gamma}$ its dual group and $\Delta_{\hat{\Gamma}} \in \mathrm{Mor}(\mathrm{C}_\infty(\hat{\Gamma}); \mathrm{C}_\infty(\hat{\Gamma}) \otimes \mathrm{C}_\infty(\hat{\Gamma}))$ the comultiplication on $\mathrm{C}_\infty(\hat{\Gamma})$. Let $\pi^R \in \mathrm{Mor}(\mathrm{C}^*(\Gamma); \mathrm{C}_r^*(G))$ be a morphism induced by the following representation of the group Γ:

$$\Gamma \ni \gamma \mapsto R_\gamma \in \mathrm{M}\big(\mathrm{C}_r^*(G)\big).$$

Identifying $\mathrm{C}^*(\Gamma)$ with $\mathrm{C}_\infty(\hat{\Gamma})$ we get $\pi^R \in \mathrm{Mor}(\mathrm{C}_\infty(\hat{\Gamma}); \mathrm{C}_r^*(G))$.

Let us fix a 2-cocycle Ψ on the group $\hat{\Gamma}$. Our objective is to show that an action of Γ^2 on the C*-algebra $\mathrm{C}_\infty(G)$ given by the left and right shifts and a 2-cocycle on $\hat{\Gamma}^2$ determined by Ψ, give rise to a quantum group. We shall describe this construction step by step.

Let ρ^R be the action of Γ on $\mathrm{C}_\infty(G)$ given by right shifts: $\rho_\gamma^R(f)(g) = f(g\gamma)$ for any $f \in \mathrm{C}_\infty(G)$. Let B^R be the crossed product C*-algebra $\mathrm{C}_\infty(G) \rtimes_{\rho^R} \Gamma$ and $(B^R, \lambda, \hat{\rho})$ the standard Γ-product structure on it. The standard embeddings of $\mathrm{C}_\infty(G)$ and $\mathrm{C}_\infty(\hat{\Gamma})$ into $\mathrm{M}(B^R)$ enable us to treat $(\pi^R \otimes \mathrm{id})\Psi$ and $V^*(1 \otimes f)V$ (where $f \in \mathrm{C}_\infty(\hat{\Gamma})$) as elements of $\mathrm{M}(\mathrm{C}_r^*(G) \otimes B^R)$. One can show that $V^*(1 \otimes \lambda_\gamma)V = R_\gamma \otimes \lambda_\gamma$ for all $\gamma \in \Gamma$, which implies that

$$V^*(1 \otimes f)V = \big(\pi^R \otimes \mathrm{id}\big)\Delta_{\hat{\Gamma}}(f) \tag{27}$$

for any $f \in \mathrm{C}_\infty(\hat{\Gamma})$. Using Ψ we deform the standard Γ-product structure on B^R to $(B^R, \lambda, \hat{\rho}^\Psi)$.

Proposition 4.1. *Let $(B^R, \lambda, \hat{\rho}^\Psi)$ be the deformed Γ-product and $V(\pi^R \otimes \mathrm{id})\Psi \in \mathrm{M}(\mathrm{C}_r^*(G) \otimes B^R)$ the unitary element considered above. Then $V(\pi^R \otimes \mathrm{id})\Psi$ is invariant with respect to the action $\mathrm{id} \otimes \hat{\rho}^\Psi$.*

Proof. The 2-cocycle equation for Ψ implies that:

$$\big(\mathrm{id} \otimes \hat{\rho}_{\hat{\gamma}}^\Psi\big)\Psi = (\mathrm{id} \otimes \hat{\rho}_{\hat{\gamma}})\Psi$$

$$= (I \otimes U_{\hat{\gamma}})^* \Delta_{\hat{\Gamma}}(U_{\hat{\gamma}})\Psi.$$

The second leg of V is invariant with respect to the action $\hat{\rho}$ hence

$$\big(\mathrm{id} \otimes \hat{\rho}^{\Psi}_{\hat{\gamma}}\big)V = \big(I \otimes U^*_{\hat{\gamma}}\big)\big((\mathrm{id} \otimes \hat{\rho}_{\hat{\gamma}})V\big)(I \otimes U_{\hat{\gamma}})$$
$$= \big(I \otimes U^*_{\hat{\gamma}}\big)V(I \otimes U_{\hat{\gamma}}) = V\big(\pi^R \otimes \mathrm{id}\big)\Delta_{\hat{F}}(U_{\hat{\gamma}})^*(I \otimes U_{\hat{\gamma}}).$$

The last equality follows from (27). Finally

$$\big(\mathrm{id} \otimes \hat{\rho}^{\Psi}_{\hat{\gamma}}\big)\big[V\big(\pi^R \otimes \mathrm{id}\big)\Psi\big]$$
$$= V\big(\pi^R \otimes \mathrm{id}\big)\Delta_{\hat{F}}(U_{\hat{\gamma}})^*(I \otimes U_{\hat{\gamma}})(I \otimes U_{\hat{\gamma}})^*\big(\pi^R \otimes \mathrm{id}\big)\Delta_{\hat{F}}(U_{\hat{\gamma}})\big(\pi^R \otimes \mathrm{id}\big)\Psi$$
$$= V\big(\pi^R \otimes \mathrm{id}\big)\Psi$$

where in the last equality we used the fact that $U_{\hat{\gamma}}$ is unitary. $\quad\square$

Let ρ^L be the action of Γ on $C_\infty(G)$ given by left shifts: $\rho^L_\gamma(f)(g) = f(\gamma^{-1}g)$ for any $f \in C_\infty(G)$. Let B^L be the crossed product C*-algebra $C_\infty(G) \rtimes_{\rho^L} \Gamma$ and let $(B^L, \lambda, \hat{\rho})$ be the standard Γ-product structure on it. For any $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$ we set

$$\Psi^\star(\hat{\gamma}_1, \hat{\gamma}_2) = \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_2)}.$$

This defines a function $\Psi^\star \in C_b(\hat{\Gamma}^2)$. The standard embeddings of $C_\infty(G)$ and $C_\infty(\hat{\Gamma})$ into $M(B^L)$ enable us to treat $(\pi^R \otimes \mathrm{id})\Psi^\star V$ and $V(1 \otimes f)V^*$ (where $f \in C_\infty(\hat{\Gamma})$) as elements of $M(C^*_r(G) \otimes B^R)$. One can show that $V(1 \otimes \lambda_\gamma)V^* = R_\gamma \otimes \lambda_\gamma$ for all $\gamma \in \Gamma$, which implies that

$$V(1 \otimes f)V^* = \big(\pi^R \otimes \mathrm{id}\big)\Delta_{\hat{F}}(f) \tag{28}$$

for any $f \in C_\infty(\hat{\Gamma})$. Let $\widetilde{\Psi}$ denote a 2-cocycle defined by the formula:

$$\widetilde{\Psi}(\hat{\gamma}_1, \hat{\gamma}_2) \equiv \overline{\Psi(-\hat{\gamma}_1, -\hat{\gamma}_2)}$$

for any $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$. Using $\widetilde{\Psi}$ we deform the standard Γ-product structure on B^L to $(B^L, \lambda, \hat{\rho}^{\widetilde{\Psi}})$.

Proposition 4.2. *Let $(B^L, \lambda, \hat{\rho}^{\widetilde{\Psi}})$ be the deformed Γ-product and $(\pi^R \otimes \mathrm{id})\Psi^\star V \in M(C^*_r(G) \otimes B^L)$ the unitary element considered above. Then $(\pi^R \otimes \mathrm{id})\Psi^\star V \in M(C^*_r(G) \otimes B^L)$ is invariant with respect to the action $\mathrm{id} \otimes \hat{\rho}^{\widetilde{\Psi}}$.*

Proof. One can check that

$$\Psi^\star(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}) = \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma})} = \widetilde{\Psi}(\hat{\gamma}_2, \hat{\gamma})\overline{\widetilde{\Psi}(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma})}\Psi^\star(\hat{\gamma}_1, \hat{\gamma}_2).$$

Hence

$$\big(\mathrm{id} \otimes \hat{\rho}^{\widetilde{\Psi}}_{\hat{\gamma}}\big)\Psi^\star = (I \otimes \widetilde{U}_{\hat{\gamma}})\Delta_{\hat{F}}(\widetilde{U}_{\hat{\gamma}})^*\Psi^\star.$$

Moreover

$$\left(\mathrm{id} \otimes \hat{\rho}_{\hat{\gamma}}^{\widetilde{\Psi}}\right) V = (I \otimes \widetilde{U}_{\hat{\gamma}})^* V (I \otimes \widetilde{U}_{\hat{\gamma}}) = (I \otimes \widetilde{U}_{\hat{\gamma}})^* \left(\pi^R \otimes \mathrm{id}\right) \Delta_{\hat{\Gamma}} (\widetilde{U}_{\hat{\gamma}}) V.$$

Following the proof of Proposition 4.1 we get our assertion. $\quad \square$

Let ρ denote the action of Γ^2 on $\mathrm{C}_\infty(G)$ given by the left and right shifts, B the crossed product C*-algebra $\mathrm{C}_\infty(G) \rtimes_\rho \Gamma^2$ and $(B, \lambda, \hat{\rho})$ the standard Γ^2-product. The standard embedding of $\mathrm{C}_\infty(G)$ into $\mathrm{M}(B)$ applied to the second leg of $V \in \mathrm{M}(\mathrm{C}_r^*(G) \otimes \mathrm{C}_\infty(G))$ embeds V into $\mathrm{M}(\mathrm{C}_r^*(G) \otimes B)$. We have two embeddings λ^L and λ^R of $\mathrm{C}_\infty(\hat{\Gamma})$ into $\mathrm{M}(B)$ corresponding to the left and the right action of Γ. Moreover by Eqs. (27) and (28) we have:

$$V\left(1 \otimes \lambda^L(f)\right) V^* = \left(\pi^R \otimes \lambda^L\right) \Delta_{\hat{\Gamma}}(f),$$
$$V^*\left(1 \otimes \lambda^R(f)\right) V = \left(\pi^R \otimes \lambda^R\right) \Delta_{\hat{\Gamma}}(f) \tag{29}$$

for any $f \in \mathrm{C}_\infty(\hat{\Gamma})$. Note also that:

$$(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) V \left(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*\right) = (R_{-\gamma_1} \otimes I) V (R_{\gamma_2} \otimes I). \tag{30}$$

Let us introduce elements Ψ^L and Ψ^R:

$$\Psi^L = \left(\pi^R \otimes \lambda^L\right)(\Psi^\star), \quad \Psi^R = \left(\pi^R \otimes \lambda^R\right)(\Psi) \in \mathrm{M}\left(\mathrm{C}_r^*(G) \otimes B\right). \tag{31}$$

Multiplying Ψ^L, V and Ψ^R we get the unitary element:

$$V^\Psi = \Psi^L V \Psi^R \in \mathrm{M}\left(\mathrm{C}_r^*(G) \otimes B\right). \tag{32}$$

Using the 2-cocycle $\widetilde{\Psi} \otimes \Psi$ on $\hat{\Gamma}^2$ we deform the standard Γ^2-product structure on B to $(B, \lambda, \hat{\rho}^{\widetilde{\Psi} \otimes \Psi})$.

Proposition 4.3. *Let* $(B, \lambda, \hat{\rho}^{\widetilde{\Psi} \otimes \Psi})$ *be the deformed* Γ^2*-product structure and* $V^\Psi \in \mathrm{M}(\mathrm{C}_r^*(G) \otimes B)$ *the unitary element given by* (32). *Then* V^Ψ *is invariant with respect to the action* $\mathrm{id} \otimes \hat{\rho}^{\widetilde{\Psi} \otimes \Psi}$. *Moreover, for any* $\gamma_1, \gamma_2 \in \Gamma$ *we have*

$$(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) V^\Psi \left(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*\right) = (R_{-\gamma_1} \otimes I) V^\Psi (R_{\gamma_2} \otimes I). \tag{33}$$

Proof. Invariance of V^Ψ with respect to the action $\mathrm{id} \otimes \hat{\rho}^{\widetilde{\Psi} \otimes \Psi}$ follows easily from Propositions 4.1 and 4.2. The group Γ is abelian, hence

$$(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) V^\Psi \left(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*\right) = (\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) \Psi^L V \Psi^R \left(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*\right)$$
$$= \Psi^L (\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) V \left(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*\right) \Psi^R.$$

Using (30) we get

$$(\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}) V^\Psi (\mathrm{id} \otimes \lambda_{\gamma_1, \gamma_2}^*) = \Psi^L (R_{-\gamma_1} \otimes I) V (R_{\gamma_2} \otimes I) \Psi^R$$
$$= (R_{-\gamma_1} \otimes I) \Psi^L V \Psi^R (R_{\gamma_2} \otimes I)$$
$$= (R_{-\gamma_1} \otimes I) V^\Psi (R_{\gamma_2} \otimes I).$$

This proves (33). □

The first leg of V^Ψ belongs to $\mathrm{C}_r^*(G)$ so it acts on $L^2(G)$. It is well known that slices of Kac–Takesaki operator V by normal functionals $\omega \in \mathrm{B}(L^2(G))_*$ give a dense subspace of $\mathrm{C}_\infty(G)$ (see [1, Section 2]). We will show that slices of V^Ψ give a dense subspace of $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$.

Theorem 4.4. *Let $(B, \lambda, \hat{\rho}^{\widetilde{\Psi} \otimes \Psi})$ be the deformed Γ^2-product structure and $V^\Psi \in \mathrm{M}(\mathrm{C}_r^*(G) \otimes B)$ the unitary operator given by* (32). *Then*

$$\mathcal{V} = \{(\omega \otimes \mathrm{id}) V^\Psi : \omega \in \mathrm{B}(L^2(G))_*\}$$

is a norm dense subset of $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$.

Proof. We need to check that for any $\omega \in \mathrm{B}(L^2(G))_*$ the element $(\omega \otimes \mathrm{id}) V^\Psi \in \mathrm{M}(B)$ satisfies Landstad conditions for Γ^2-product $(B, \lambda, \hat{\rho}^{\widetilde{\Psi} \otimes \Psi})$. The first Landstad condition is equivalent to the invariance of the second leg of V^Ψ with respect to the action $\hat{\rho}^{\widetilde{\Psi} \otimes \Psi}$ (Proposition 4.3). Using (33) we get

$$\lambda_{\gamma_1, \gamma_2}[(\omega \otimes \mathrm{id}) V^\Psi] \lambda_{\gamma_1, \gamma_2}^* = (R_{\gamma_2} \cdot \omega \cdot R_{-\gamma_1} \otimes \mathrm{id}) V^\Psi \tag{34}$$

for any $\gamma_1, \gamma_2 \in \Gamma$. The norm continuity of the map

$$\Gamma^2 \ni (\gamma_1, \gamma_2) \mapsto R_{\gamma_2} \cdot \omega \cdot R_{-\gamma_1} \in \mathrm{B}(L^2(G))_*$$

implies that $(\omega \otimes \mathrm{id}) V^\Psi$ satisfies the second Landstad condition. To check the third Landstad condition we need to show that

$$f_1[(\omega \otimes \mathrm{id}) V^\Psi] f_2 \in B \tag{35}$$

for any $f_1, f_2 \in \mathrm{C}_\infty(\hat{\Gamma} \times \hat{\Gamma})$. Let us consider the set

$$\mathcal{W} = \{f_1[(\omega \otimes \mathrm{id}) V^\Psi] f_2 : f_1, f_2 \in \mathrm{C}_\infty(\hat{\Gamma} \times \hat{\Gamma}), \ \omega \in \mathrm{B}(L^2(G))_*\}^{\mathrm{cls}}. \tag{36}$$

We will prove that $\mathcal{W} = B$ which is a stronger property than (35). Taking for $\omega \in \mathrm{B}(L^2(G))_*$ elements of the form $\pi^R(h_3) \cdot \mu \cdot \pi^R(h_4)$, for $f_1 \in \mathrm{C}_\infty(\hat{\Gamma} \times \hat{\Gamma})$ elements $\lambda^R(h_1) \lambda^L(h_2)$ where $h_1, h_2 \in \mathrm{C}_\infty(\hat{\Gamma})$ and similarly for f_2 we do not change the closed linear span. Thus we have:

$$\mathcal{W} = \{\lambda^R(h_1) \lambda^L(h_2) [((\pi^R(h_3) \cdot \mu \cdot \pi^R(h_4)) \otimes \mathrm{id})(V^\Psi)] \lambda^R(h_5) \lambda^L(h_6) :$$
$$h_1, h_2, \dots, h_6 \in \mathrm{C}_\infty(\hat{\Gamma}), \mu \in \mathrm{B}(L^2(G))_*\}^{\mathrm{cls}}.$$

Note that

$$\lambda^R(h_1)\lambda^L(h_2)\big[\big(\big(\pi^R(h_3)\cdot\mu\cdot\pi^R(h_4)\big)\otimes\mathrm{id}\big)\big(V^\Psi\big)\big]\lambda^R(h_5)\lambda^L(h_6)$$
$$=\lambda^R(h_1)\big[(\mu\otimes\mathrm{id})\big(\pi^R\otimes\lambda^L\big)\big(\Psi^\star(h_4\otimes h_2)\big)V\big(\pi^R\otimes\lambda^R\big)\big(\Psi(h_3\otimes h_5)\big)\big]\lambda^L(h_6),$$

hence $\mathcal{W}$ coincides with the following set:

$$\big\{\lambda^R(h_1)\big[(\mu\otimes\mathrm{id})\big(\pi^R\otimes\lambda^L\big)\big(\Psi^\star(h_4\otimes h_2)\big)V\big(\pi^R\otimes\lambda^R\big)\big(\Psi(h_3\otimes h_5)\big)\big]\lambda^L(h_6):$$
$$h_1,h_2,\dots,h_6\in\mathrm{C}_\infty(\hat{\Gamma}),\mu\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}.$$

Using the fact that Ψ and $\Psi^\star$ are unitary we get

$$\mathcal{W}=\big\{\lambda^R(h_1)\big[(\mu\otimes\mathrm{id})\big(\pi^R\otimes\lambda^L\big)(h_4\otimes h_2)V\big(\pi^R\otimes\lambda^R\big)(h_3\otimes h_5)\big]\lambda^L(h_6):$$
$$h_1,h_2,\dots,h_6\in\mathrm{C}_\infty(\hat{\Gamma}),\mu\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}$$
$$=\big\{\lambda^R(h_1)\lambda^L(h_2)\big[\big(\big(\pi^R(h_3)\cdot\mu\cdot\pi^R(h_4)\big)\otimes\mathrm{id}\big)(V)\big]\lambda^R(h_5)\lambda^L(h_6):$$
$$h_1,h_2,\dots,h_6\in\mathrm{C}_\infty(\hat{\Gamma}),\mu\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}.$$

Now again

$$\big\{\lambda^R(h_1)\lambda^L(h_2)\big[\big(\big(\pi^R(h_3)\cdot\mu\cdot\pi^R(h_4)\big)\otimes\mathrm{id}\big)(V)\big]\lambda^R(h_5)\lambda^L(h_6):$$
$$h_1,h_2,\dots,h_6\in\mathrm{C}_\infty(\hat{\Gamma}),\mu\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}$$
$$=\big\{f_1\big[(\omega\otimes\mathrm{id})V\big]f_2:f_1,f_2\in\mathrm{C}_\infty(\hat{\Gamma}\times\hat{\Gamma}),\omega\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}$$

hence we get

$$\mathcal{W}=\big\{f_1\big[(\omega\otimes\mathrm{id})V\big]f_2:f_1,f_2\in\mathrm{C}_\infty(\hat{\Gamma}\times\hat{\Gamma}),\omega\in\mathrm{B}\big(L^2(G)\big)_*\big\}^{\mathrm{cls}}.$$

The set $\{(\omega\otimes\mathrm{id})V:\omega\in\mathrm{B}(L^2(G))_*\}$ is dense in $\mathrm{C}_\infty(G)$ which shows that $\mathcal{W}=B$ and proves formula (36).

We see that the elements of the set $\mathcal{V}$ satisfy the Landstad conditions. To prove that $\mathcal{V}$ is dense in $\mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$ we use Lemma 2.6. According to (34), $\mathcal{V}$ is a $\rho^{\widetilde{\Psi}\otimes\Psi}$-invariant subspace of $\mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$. Moreover we have that $(\mathrm{C}^*(\Gamma^2)\mathcal{V}\mathrm{C}^*(\Gamma^2))^{\mathrm{cls}}=\mathcal{W}=B$. Hence the assumptions of Lemma 2.6 are satisfied and we get the required density. $\quad\square$

Remark 4.5. The representation of $\mathrm{C}_\infty(G)$ on $L^2(G)$ is covariant. The action of Γ^2 is implemented by the left and right shifts: $L_{\gamma_1},R_{\gamma_2}\in\mathrm{B}(L^2(G))$, where by $L_g\in\mathrm{B}(L^2(G))$ we understand the unitarized left shift. More precisely, let $\delta:G\to\mathbb{R}^+$ be the modular function for the right Haar measure. Then $L_g\in\mathrm{B}(L^2(G))$ is a unitary given by:

$$(L_gf)(g')=\delta(g)^{\frac{1}{2}}f\big(g^{-1}g'\big)$$

for any $g, g' \in G$ and $f \in L^2(G)$. This covariant representation of $C_\infty(G)$ induces the representation of crossed product $B = C_\infty(G) \rtimes_\rho \Gamma^2$, which we denote by π^{can}. Clearly it is faithful on $C_\infty(G)$, hence by Theorem 3.6 it is faithful on $C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$.

Let us introduce the unitary operator:

$$W = \left(\mathrm{id} \otimes \pi^{\mathrm{can}}\right) V^\Psi \in \mathrm{B}\left(L^2(G) \otimes L^2(G)\right). \tag{37}$$

Theorem 4.6. *The unitary operator $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$ considered above satisfies the pentagonal equation*:

$$W_{12}^* W_{23} W_{12} = W_{13} W_{23}.$$

Remark 4.7. A similar construction of the operator W and the proof that it satisfies the pentagonal equation was given by Enock and Vainerman in [3] and independently by Landstad in [7]. We included the following proof for the completeness of the exposition.

Proof. Let us introduce two unitary operators $X, Y \in \mathrm{B}(L^2(G) \otimes L^2(G))$:

$$X = \left(\mathrm{id} \otimes \pi^{\mathrm{can}}\right)\left(\Psi^R\right), \qquad Y = \left(\mathrm{id} \otimes \pi^{\mathrm{can}}\right)\left(\Psi^L\right) \tag{38}$$

where $\Psi^R, \Psi^L \in \mathrm{M}(C_r^*(G) \otimes B)$ are elements defined by (31). Note that

$$X \in \mathrm{M}\left(C_r^*(G) \otimes C_r^*(G)\right), \qquad Y \in \mathrm{M}\left(C_r^*(G) \otimes C_l^*(G)\right), \tag{39}$$

hence $W = YVX \in \mathrm{M}(C_r^*(G) \otimes \mathcal{K})$ where $\mathcal{K}$ is the algebra of compact operators acting on $L^2(G)$. Inserting $\hat{\gamma}_3 \to (-\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3)$ into the 2-cocycle condition

$$\Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2, \hat{\gamma}_3) = \Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3)\Psi(\hat{\gamma}_1, \hat{\gamma}_2) \tag{40}$$

and taking the complex conjugate we get

$$\overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_3)}\,\overline{\Psi(\hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_3 - \hat{\gamma}_2)} = \overline{\Psi(\hat{\gamma}_1, \hat{\gamma}_2)}\,\overline{\Psi(\hat{\gamma}_1 + \hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3)}.$$

This implies that

$$\overline{\Psi}(\hat{\gamma}_1, \hat{\gamma}_2)\Psi^\star(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3) = \Psi^\star(\hat{\gamma}_1, \hat{\gamma}_3)\Psi^\star(\hat{\gamma}_2, \hat{\gamma}_1 + \hat{\gamma}_3) \tag{41}$$

where $\Psi^\star(\hat{\gamma}_1, \hat{\gamma}_2) = \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_2)}$. Using Eqs. (38), (40), (41) and the fact that V implements the coproduct we obtain:

$$X_{12}^* V_{12}^* Y_{23} V_{12} = Y_{13} V_{13} Y_{23} V_{13}^*,$$
$$V_{12}^* X_{23} V_{12} X_{12} = V_{23}^* X_{13} V_{23} X_{23}.$$

Now we can check the pentagonal equation:

$$
\begin{aligned}
W_{12}^* W_{23} W_{12} &= X_{12}^* V_{12}^* Y_{12}^* W_{23} Y_{12} V_{12} X_{12} \\
&= X_{12}^* V_{12}^* Y_{23} V_{23} X_{23} V_{12} X_{12} \\
&= \left(X_{12}^* V_{12}^* Y_{23} V_{12}\right)\left(V_{12}^* V_{23} V_{12}\right)\left(V_{12}^* X_{23} V_{12} X_{12}\right) \\
&= \left(Y_{13} V_{13} Y_{23} V_{13}^*\right)(V_{13} V_{23})\left(V_{23}^* X_{13} V_{23} X_{23}\right) \\
&= Y_{13} V_{13} Y_{23} X_{13} V_{23} X_{23} = (Y_{13} V_{13} X_{13})(Y_{23} V_{23} X_{23}) = W_{13} W_{23}.
\end{aligned}
$$

In the second equality we used the fact that the second leg of element Y commutes with the first leg of W (see (39)). $\square$

Our next aim is to show that W is manageable. For all $\hat\gamma \in \hat\Gamma$ we set $u(\hat\gamma) = \Psi(-\hat\gamma, \hat\gamma)$. It defines a function $u \in C_b(\hat\Gamma)$. Applying $\pi^R \in \mathrm{Mor}(C_\infty(\hat\Gamma); C_r^*(G))$ to $u \in M(C_\infty(\hat\Gamma))$ we get the unitary operator:

$$
J = \pi^R(u) \in M\left(C_r^*(G)\right) \subset B\left(L^2(G)\right). \tag{42}
$$

Theorem 4.8. *Let $W \in B(L^2(G) \otimes L^2(G))$ be the multiplicative unitary and $J \in B(L^2(G))$ be the unitary operator* (42). *Then W is manageable. Operators Q and $\widetilde{W}$ entering Definition* 1.2 *of* [19] *equal respectively* 1 *and* $(J \otimes 1)W^*(J^* \otimes 1)$.

Remark 4.9. The presented proof seems to be simpler than the Landstad's proof given in [7]. In what follows we shall use the bracket notation for the scalar product: let H be a Hilbert space, $x, y \in H$, and $T \in B(H)$. Then $(x|T|y)$ denotes the scalar product $(x|Ty)$.

Proof. Let $x, y, z, t \in L^2(G)$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$. The Kac–Takesaki operator is manageable, therefore

$$
\begin{aligned}
&\left(x \otimes t \,|\, (R_{\gamma_1} \otimes L_{\gamma_2}) V \,|\, (R_{\gamma_3} \otimes R_{\gamma_4}) \,|\, z \otimes y\right) \\
&= (R_{-\gamma_1} x \otimes L_{-\gamma_2} t \,|\, V \,|\, R_{\gamma_3} z \otimes R_{\gamma_4} y) \\
&= \left(\overline{R_{\gamma_3} z} \otimes L_{-\gamma_2} t \,|\, V^* \,|\, \overline{R_{-\gamma_1} x} \otimes R_{\gamma_4} y\right) \\
&= \left(\bar z \otimes t \,|\, (R_{-\gamma_3} \otimes L_{\gamma_2}) V^* (R_{-\gamma_1} \otimes R_{\gamma_4}) \,|\, \bar x \otimes y\right).
\end{aligned}
$$

Using well-known equalities

$$
\begin{aligned}
V^*(I \otimes R_g)V &= R_g \otimes R_g, \\
V(I \otimes L_g)V^* &= R_g \otimes L_g
\end{aligned}
$$

and commutativity of Γ we get the following formula:

$$
\begin{aligned}
&\left(\bar z \otimes t \,|\, (R_{-\gamma_3} \otimes L_{\gamma_2}) V^* (R_{-\gamma_1} \otimes R_{\gamma_4}) \,|\, \bar x \otimes y\right) \\
&= \left(\bar z \otimes t \,|\, (R_{-\gamma_3+\gamma_4} \otimes R_{\gamma_4}) V^* (R_{-\gamma_1+\gamma_2} \otimes L_{\gamma_2}) \,|\, \bar x \otimes y\right). \tag{43}
\end{aligned}
$$

Hence:

$$\big(x \otimes t|(R_{\gamma_1} \otimes L_{\gamma_2})V(R_{\gamma_3} \otimes R_{\gamma_4})|z \otimes y\big)$$
$$= \big(\bar{z} \otimes t|(R_{-\gamma_3+\gamma_4} \otimes R_{\gamma_4})V^*(R_{-\gamma_1+\gamma_2} \otimes L_{\gamma_2})|\bar{x} \otimes y\big). \tag{44}$$

Using continuity arguments, this equality will be extended. We will repeatedly use the identifications $C^*(\Gamma^2) = C_\infty(\hat{\Gamma}^2) = C_\infty(\hat{\Gamma}) \otimes C_\infty(\hat{\Gamma})$, etc. Let u_γ be a unitary generator of $C^*(\Gamma)$. Let us define the following morphisms:

$$\Phi_1^R \in \mathrm{Mor}\big(C^*(\Gamma) \otimes C^*(\Gamma); C_r^*(G) \otimes C_r^*(G)\big): \quad \Phi_1^R(u_{\gamma_1} \otimes u_{\gamma_2}) = R_{\gamma_1} \otimes R_{\gamma_2},$$
$$\Phi_1^L \in \mathrm{Mor}\big(C^*(\Gamma) \otimes C^*(\Gamma); C_r^*(G) \otimes C_l^*(G)\big): \quad \Phi_1^L(u_{\gamma_1} \otimes u_{\gamma_2}) = R_{\gamma_1} \otimes L_{\gamma_2}$$

and automorphism $\Theta \in \mathrm{Aut}(C_\infty(\hat{\Gamma}^2))$ given by the formula:

$$\Theta(f)(\hat{\gamma}_1, \hat{\gamma}_2) = f(-\hat{\gamma}_1, \hat{\gamma}_1 + \hat{\gamma}_2)$$

for any $f \in C_\infty(\hat{\Gamma}^2)$. One can check that

$$\Theta(u_{\gamma_1} \otimes u_{\gamma_2}) = u_{-\gamma_1+\gamma_2} \otimes u_{\gamma_2}.$$

Using the above morphisms we reformulate (44):

$$\big(x \otimes t|\Phi_1^L(u_{\gamma_1} \otimes u_{\gamma_2})V\Phi_1^R(u_{\gamma_3} \otimes u_{\gamma_4})|z \otimes y\big)$$
$$= \big(\bar{z} \otimes t|\Phi_1^R \circ \Theta(u_{\gamma_3} \otimes u_{\gamma_4})V^*\Phi_l^L \circ \Theta(u_{\gamma_1} \otimes u_{\gamma_2})|\bar{x} \otimes y\big).$$

By linearity and continuity we get

$$\big(x \otimes t|\Phi_1^L(f)V\Phi_1^R(g)|z \otimes y\big)$$
$$= \big(\bar{z} \otimes t|\Phi_1^R \circ \Theta(g)V^*\Phi_1^L \circ \Theta(f)|\bar{x} \otimes y\big)$$

for any $f, g \in M(C_\infty(\hat{\Gamma}) \otimes C_\infty(\hat{\Gamma}))$. In particular

$$\big(x \otimes t|\Phi_1^L(\Psi^\star)V\Phi_1^R(\Psi)|z \otimes y\big)$$
$$= \big(\bar{z} \otimes t|\Phi_1^R \circ \Theta(\Psi)V^*\Phi_1^L \circ \Theta(\Psi^\star)|\bar{x} \otimes y\big).$$

It is easy to see that $X = \Phi_1^R(\Psi)$, $Y = \Phi_1^L(\Psi^\star)$ and $\Theta(\Psi) = \overline{\Psi}(u \otimes I)$ where X and Y are given by (38). Therefore

$$\Phi_1^R \circ \Theta(\Psi) = \Phi_1^R\big(\overline{\Psi}(u \otimes I)\big) = (J \otimes I)X^*. \tag{45}$$

Similarly we prove that

$$\Phi_1^L \circ \Theta(\Psi^\star) = Y^*\big(J^* \otimes I\big) \tag{46}$$

and finally we get

$$\bigl(x \otimes t \,|\, \Phi_1^L(\Psi^\star) V \Phi_1^R(\Psi) \,|\, z \otimes y\bigr)$$
$$= \bigl(\bar{z} \otimes t \,|\, (J \otimes I) X^* V^* Y^* (J^* \otimes I) \,|\, \bar{x} \otimes y\bigr).$$

This shows that

$$\widetilde{W} = \bigl((J \otimes I) Y V X (J^* \otimes I)\bigr)^* = (J \otimes I) W^* (J^* \otimes I) \quad \text{and} \quad Q = 1. \qquad \Box$$

Proposition 4.10. *Let* $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$ *and* $J \in \mathrm{B}(L^2(G))$ *be the unitaries defined in* (37) *and* (42) *respectively. Let* x, y *be vectors in* $L^2(G)$ *and* $\omega_{x,y} \in \mathrm{B}(L^2(G))_*$ *a functional given by* $\omega_{x,y}(T) = (x|T|y)$ *for any* $T \in \mathrm{B}(L^2(G))$. *Then we have*

$$\bigl[(\omega_{x,y} \otimes \mathrm{id}) W\bigr]^* = (\omega_{J^*\bar{x}, J^*\bar{y}} \otimes \mathrm{id})(W). \tag{47}$$

Proof. Using manageability of W we get:

$$(\omega_{x,y} \otimes \mathrm{id}) W = (\omega_{\bar{y},\bar{x}} \otimes \mathrm{id})(\widetilde{W}) = (\omega_{\bar{y},\bar{x}} \otimes \mathrm{id})\bigl((J \otimes 1) W^* (J^* \otimes 1)\bigr)$$
$$= (\omega_{J^*\bar{y}, J^*\bar{x}} \otimes \mathrm{id})(W^*) = \bigl[(\omega_{J^*\bar{x}, J^*\bar{y}} \otimes \mathrm{id})(W)\bigr]^*. \qquad \Box$$

Let A be a C*-algebra obtained by slicing the first leg of a manageable multiplicative unitary $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$:

$$A = \overline{\bigl\{(\omega \otimes \mathrm{id}) W \colon \omega \in \mathrm{B}\bigl(L^2(G)\bigr)_*\bigr\}}^{\|\cdot\|}.$$

Theorem 1.5 of [19] shows that A carry the structure of a quantum group. The comultiplication on A is given by the formula:

$$A \ni a \mapsto W(a \otimes I) W^* \in \mathrm{M}(A \otimes A).$$

At the same time, using the morphism $\pi^{\mathrm{can}} \in \mathrm{Rep}(B; L^2(G))$ introduced in Remark 4.5 we can faithfully represent $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ on $L^2(G)$. By Theorem 4.4 $\pi^{\mathrm{can}}(\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}) = A$, hence we can transport the structure of a quantum group from A to $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$. Our next objective is to present a useful formula for comultiplication on $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ which does not use multiplicative unitary W. The construction is done in two steps.

- Let ρ be the action of Γ^2 on $\mathrm{C}_\infty(G)$ given by left and right shifts along the subgroup $\Gamma \subset G$. The comultiplication is covariant:

$$\Delta_G\bigl(\rho_{\gamma_1,\gamma_2}(f)\bigr) = (\rho_{\gamma_1,0} \otimes \rho_{0,\gamma_2})\bigl(\Delta_G(f)\bigr)$$

for any $f \in \mathrm{C}_\infty(G)$. Therefore, it induces a morphism of crossed products:

$$\Delta \in \mathrm{Mor}\bigl(\mathrm{C}_\infty(G) \rtimes \Gamma^2; \mathrm{C}_\infty(G) \rtimes \Gamma^2 \otimes \mathrm{C}_\infty(G) \rtimes \Gamma^2\bigr).$$

Δ restricted to $\mathrm{C}_\infty(G) \subset \mathrm{M}(\mathrm{C}_\infty(G) \rtimes \Gamma^2)$ coincides with Δ_G and Δ restricted to $\mathrm{C}_\infty(\hat{\Gamma}^2) \subset \mathrm{M}(\mathrm{C}_\infty(G) \rtimes \Gamma^2)$ is given by

$$\Delta(h) = \left(\lambda^L \otimes \lambda^R\right) h \in M\big(C_\infty(G) \rtimes \Gamma^2 \otimes C_\infty(G) \rtimes \Gamma^2\big)$$

where $\lambda^L, \lambda^R \in \mathrm{Mor}(C_\infty(\hat{\Gamma}), C_\infty(G) \rtimes \Gamma^2)$ are morphisms introduced after the proof of Proposition 4.2 and $h \in C_\infty(\hat{\Gamma}^2)$.

- Let Ψ be a 2-cocycle on $\hat{\Gamma}$. Recall that $\Psi^\star \in M(C_\infty(\hat{\Gamma}^2))$ is defined by

$$\Psi^\star(\hat{\gamma}_1, \hat{\gamma}_2) = \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_2)}.$$

Let us introduce the unitary element $\Upsilon \in M(C_\infty(G) \rtimes \Gamma^2 \otimes C_\infty(G) \rtimes \Gamma^2)$:

$$\Upsilon = \left(\lambda^R \otimes \lambda^L\right)\Psi^\star$$

and a morphism $\Delta^\Psi \in \mathrm{Mor}(C_\infty(G) \rtimes \Gamma^2; C_\infty(G) \rtimes \Gamma^2 \otimes C_\infty(G) \rtimes \Gamma^2)$ given by the formula

$$\Delta^\Psi(a) = \Upsilon \Delta(a) \Upsilon^* \tag{48}$$

for any $a \in C_\infty(G) \rtimes \Gamma^2$.

Theorem 4.11. *Let* $\Delta^\Psi \in \mathrm{Mor}(C_\infty(G) \rtimes \Gamma^2; C_\infty(G) \rtimes \Gamma^2 \otimes C_\infty(G) \rtimes \Gamma^2)$ *be the morphism defined by formula* (48). *For all* $a \in C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ *we have*

$$\Delta^\Psi(a) \in M\big(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi} \otimes C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}\big)$$

and

$$\Delta^\Psi|_{C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}} \in \mathrm{Mor}\big(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}; C_\infty(G)^{\widetilde{\Psi} \otimes \Psi} \otimes C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}\big).$$

Moreover $\Delta^\Psi|_{C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}}$ *coincides with the comultiplication implemented by* W:

$$C_\infty(G)^{\widetilde{\Psi} \otimes \Psi} \ni a \mapsto W(a \otimes 1)W^* \in M\big(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi} \otimes C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}\big).$$

Proof. By Theorem 1.5 of [19] it is enough to show that

$$\left(\mathrm{id} \otimes \Delta^\Psi\right) V^\Psi = V^\Psi_{12} V^\Psi_{13}.$$

From the definition of Δ it follows that

$$
\begin{aligned}
(\mathrm{id} \otimes \Delta) V^\Psi &= (\mathrm{id} \otimes \Delta)\big(\left(\pi^R \otimes \lambda^L\right)\left(\Psi^\star\right) V \left(\pi^R \otimes \lambda^R\right)(\Psi)\big) \\
&= \big(\left(\pi^R \otimes \lambda^L\right)\Psi^\star V\big)_{12} \big(V\left(\pi^R \otimes \lambda^R\right)\Psi\big)_{13}.
\end{aligned}
$$

Hence

$$\left(\mathrm{id} \otimes \Delta^\Psi\right) V^\Psi = (1 \otimes \Upsilon)\big(\left(\pi^R \otimes \lambda^L\right)\Psi^\star V\big)_{12}\big(V\big(\left(\pi^R \otimes \lambda^R\right)\Psi\big)_{13}\big)(1 \otimes \Upsilon^*).$$

By Eq. (29) we get

$$(1 \otimes \Upsilon) V_{12} = V_{12} \big((\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\Delta_{\hat{\Gamma}} \otimes \mathrm{id}) (\Psi^\star) \big)$$

and

$$V_{13} (1 \otimes \Upsilon) = \big((\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_{\hat{\Gamma}}) (\Psi^\star) \big) V_{13}$$

where σ is the flip operator. Therefore

$$\begin{aligned}
(\mathrm{id} \otimes \Delta^\Psi) V^\Psi &= \big((\pi^R \otimes \lambda^L) \Psi^\star V \big)_{12} \big((\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\Delta_{\hat{\Gamma}} \otimes \mathrm{id}) (\Psi^\star) \big) \\
&\quad \times \big((\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_{\hat{\Gamma}}) (\Psi^\star)^* \big) \big(V (\pi^R \otimes \lambda^R) \Psi \big)_{13}.
\end{aligned}$$

We compute

$$\begin{aligned}
&\overline{\Psi(\hat{\gamma}_1 + \hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3)} \Psi(\hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3) \\
&\quad = \Psi(\hat{\gamma}_1, \hat{\gamma}_2) \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_3)} \overline{\Psi(\hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3)} \Psi(\hat{\gamma}_2, -\hat{\gamma}_1 - \hat{\gamma}_2 - \hat{\gamma}_3) \\
&\quad = \Psi(\hat{\gamma}_1, \hat{\gamma}_2) \overline{\Psi(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_3)}.
\end{aligned}$$

The above equality implies that

$$\begin{aligned}
&(\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\Delta_{\hat{\Gamma}} \otimes \mathrm{id}) (\Psi^\star) (\pi^R \otimes \lambda^R \otimes \lambda^L) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_{\hat{\Gamma}}) (\Psi^\star)^* \\
&\quad = \big((\pi^R \otimes \lambda^R) \Psi \big)_{12} \big((\pi^R \otimes \lambda^L) \Psi^\star \big)_{13}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\mathrm{id} \otimes \Delta^\Psi) V^\Psi &= \big((\pi^R \otimes \lambda^L) \Psi^\star V (\pi^R \otimes \lambda^R) \Psi \big)_{12} \big((\pi^R \otimes \lambda^L) \Psi^\star V (\pi^R \otimes \lambda^R) \Psi \big)_{13} \\
&= V^\Psi_{12} V^\Psi_{13}.
\end{aligned}$$

This ends the proof. □

4.2. Dual quantum group

Let G be a locally compact group, Γ an abelian subgroup of G and Ψ a 2-cocycle on $\hat{\Gamma}$. Using the results of previous sections we can construct the quantum group $(\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^\Psi)$ and the multiplicative unitary $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$. In this section we will investigate the dual quantum group in the sense of duality given by W. Our objective is to show that this is the twist, in the sense of M. Enock and L. Vainerman (see [3]), of the canonical quantum group structure on the reduced group C*-algebra $\mathrm{C}_r^*(G)$.

Theorem 4.12. *Let $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$ be a manageable multiplicative unitary (37) and $(\hat{A}, \hat{\Delta}_{\hat{A}})$ a quantum group obtained by slicing the second leg of W:*

$$\hat{A} = \overline{\{ (\mathrm{id} \otimes \omega)(W^*) : \omega \in \mathrm{B}(L^2(G)) \}}^{\|\cdot\|}.$$

Then

1. $\hat{A} = C^*_r(G)$.
2. *The comultiplication on $\hat{A}$ is given by*

$$\hat{A} \ni a \mapsto \hat{\Delta}_{\hat{A}}(a) = \Sigma X^* \Sigma \hat{\Delta}(a) \Sigma X \Sigma \in M(\hat{A} \otimes \hat{A})$$

   where $\hat{\Delta}$ is the canonical comultiplication on $C^*_r(G)$ and X is given by (38).
3. *The coinverse on $\hat{A}$ is given by*

$$\hat{\kappa}_{\hat{A}}(a) = J\hat{\kappa}(a)J^*$$

where $\hat{\kappa}$ is the canonical coinverse on $C^*_r(G)$ and J is given by (42).

The proof was communicated to the author by S.L. Woronowicz.

Proof. Using Eq. (33) we get

$$R_{\gamma_1}\big[(\mathrm{id} \otimes \omega)W\big]R_{\gamma_2} = (\mathrm{id} \otimes R_{-\gamma_2}L_{\gamma_1} \cdot \omega \cdot L_{-\gamma_1}R_{\gamma_2})W \qquad (49)$$

for any $\gamma_1, \gamma_2 \in \Gamma$. Therefore $R_\gamma \in B(L^2(G))$ is a multiplier of $\hat{A}$ and representation:

$$\Gamma \ni \gamma \mapsto R_\gamma \in M(\hat{A})$$

is strictly continuous. This representation induces a morphism which we denote by $\chi \in \mathrm{Mor}(C_\infty(\hat{\Gamma}), \hat{A})$. Applying it to Ψ and $\Psi^\star$ we obtain

$$X = (\chi \otimes \chi)(\Psi) \in M(\hat{A} \otimes \hat{A}),$$
$$Y = (\chi \otimes \pi^L)(\Psi^\star) \in M(\hat{A} \otimes \mathcal{K}).$$

Recall that $W \in M(\hat{A} \otimes A)$, hence

$$V = Y^*WX^* \in M(\hat{A} \otimes \mathcal{K}) \qquad (50)$$

which immediately implies that $V^*_{12}V_{23}$, $V_{12}V^*_{23} \in M(\hat{A} \otimes \mathcal{K} \otimes C_\infty(G))$. The pentagonal equation for V together with (50) gives

$$V_{13} = V^*_{12}V_{23}V_{12}V^*_{23} \in M\big(\hat{A} \otimes \mathcal{K} \otimes C_\infty(G)\big),$$

therefore

$$V \in M\big(\hat{A} \otimes C_\infty(G)\big). \qquad (51)$$

Similarly we prove that

$$W \in M\big(C^*_r(G) \otimes A\big). \qquad (52)$$

Formula (51) and point 6 of Theorem 1.6 of [19] imply that the natural representation of $C_r^*(G)$ on $L^2(G)$ is in fact an element of $\mathrm{Mor}(C_r^*(G), \hat{A})$. Similarly, (52) implies that the natural representation of $\hat{A}$ on $L^2(G)$ is an element of $\mathrm{Mor}(\hat{A}, C_r^*(G))$. The general properties of morphisms gives

$$\overline{C_r^*(G)\hat{A}} = \hat{A},$$

$$\overline{\hat{A}C_r^*(G)} = C_r^*(G).$$

But $C_r^*(G)$ and $\hat{A}$ are closed under the star operation, hence $\hat{A} = \hat{A}^* = \overline{\hat{A}C_r^*(G)} = C_r^*(G)$, which proves point 1 of our theorem. To prove point 2 we recall that the comultiplication on $\hat{A}$ is implemented by $\Sigma W^* \Sigma$, hence

$$\begin{aligned}
\hat{\Delta}_{\hat{A}}(a) &= \Sigma X^* V^* Y^* (I \otimes a) Y V X \Sigma \\
&= \Sigma X^* V^* (I \otimes a) V X \Sigma \\
&= (\Sigma X^* \Sigma) \hat{\Delta}(a) (\Sigma X \Sigma).
\end{aligned}$$

Point three follows from Proposition 4.10. $\quad\square$

4.3. Haar measure

Let G be a locally compact group, Γ an abelian subgroup of G and Ψ a 2-cocycle on $\hat{\Gamma}$. Throughout this section we shall assume that the modular function δ on G restricted to Γ is identically equal to 1. Let $(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^{\Psi})$ be the quantum group that we considered previously. In what follows we will identify $C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ with its image in $B(L^2(G))$.

Definition 4.13. Let $f \in C_\infty(G)$ and $R_g \in B(L^2(G))$ be the right regular representation of group G. We say that f is quantizable if there exists $\omega \in B(L^2(G))_*$ such that $f(g) = \omega(R_g)$ for any $g \in G$. Given a quantizable function f we introduce an operator $\mathcal{Q}(f) \in C_\infty(G)^{\widetilde{\Psi} \otimes \Psi} \subset B(L^2(G))$ given by:

$$\mathcal{Q}(f) = (\omega \otimes \mathrm{id}) W \in C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}.$$

Note that the equation $f(g) = \omega(R_g)$ does not determine $\omega \in B(L^2(G))_*$. Nevertheless, the operator $\mathcal{Q}(f)$ does not depend on the choice of the functional that gives rise to f. It is easy to see that the vector space of quantizable functions equipped with the pointwise multiplication forms an algebra which in the literature is called the *Fourier Algebra*. We use the term *quantizable function* to stress that with such an f we can associate the operator $\mathcal{Q}(f) = (\omega \otimes \mathrm{id}) W \in B(L^2(G))$.

Theorem 4.14. *Let* $(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^{\Psi})$ *be the quantum group with multiplicative unitary* $W \in B(L^2(G) \otimes L^2(G))$ *considered above. Let* $f, h \in C_\infty(G)$ *be quantizable functions given by functionals* $\omega \in B(L^2(G))_*$ *and* $\mu \in B(L^2(G))_*$ *respectively. They yield operators* $\mathcal{Q}(f), \mathcal{Q}(h) \in B(L^2(G))$. *Assume that* $h \in L^2(G)$. *Then* $\mathcal{Q}(f)h \in L^2(G)$ *is a quantizable function and*

$$\mathcal{Q}(\mathcal{Q}(f)h) = \mathcal{Q}(f)\mathcal{Q}(h). \tag{53}$$

Proof. Note that

$$\mathcal{Q}(f)\mathcal{Q}(h) = (\omega \otimes \mathrm{id})(W)(\mu \otimes \mathrm{id})W$$
$$= (\omega \otimes \mu \otimes \mathrm{id})(W_{13}W_{23})$$
$$= (\omega \otimes \mu \otimes \mathrm{id})(W_{12}^* W_{23} W_{12}).$$

The above calculation shows that $\mathcal{Q}(f)\mathcal{Q}(h)$ is given by the quantization of the function $k \in C_\infty(G)$:

$$k(g) = (\omega \otimes \mu)(W^*(I \otimes R_g)W).$$

Using the identity $W^*(I \otimes R_g)W = X^*(R_g \otimes R_g)X$ we get

$$k(g) = (\omega \otimes \mu)(X^*(R_g \otimes R_g)X).$$

Therefore, to prove formula (53) we need to show that

$$(\omega \otimes \mu)(X^*(R_g \otimes R_g)X) = [\mathcal{Q}(f)h](g). \tag{54}$$

In order to do that we compute

$$(\omega \otimes \mathrm{id})((R_{\gamma_1} \otimes L_{\gamma_2})V(R_{\gamma_3} \otimes R_{\gamma_4}))h(g) = L_{\gamma_2} f(\gamma_1 \cdot \gamma_3) R_{\gamma_4} h(g)$$
$$= \delta^{\frac{1}{2}}(\gamma_2) f((\gamma_1 - \gamma_2)g\gamma_3)h((-\gamma_2)g\gamma_4).$$

Using the assumption that $\delta(\gamma) = 1$ for any $\gamma \in \Gamma$ we get

$$(\omega \otimes \mathrm{id})((R_{\gamma_1} \otimes L_{\gamma_2})V(R_{\gamma_3} \otimes R_{\gamma_4}))h(g) = f((\gamma_1 - \gamma_2)g\gamma_3)h((-\gamma_2)g\gamma_4).$$

The equality $h(g) = \mu(R_g)$ implies that

$$((\omega \otimes \mathrm{id})((R_{\gamma_1} \otimes L_{\gamma_2})V(R_{\gamma_3} \otimes R_{\gamma_4}))h)(g)$$
$$= (\omega \otimes \mu)((R_{\gamma_1 - \gamma_2} \otimes R_{-\gamma_2})(R_g \otimes R_g)(R_{\gamma_3} \otimes R_{\gamma_4})). \tag{55}$$

Let $\vartheta \in \mathrm{Aut}(C_\infty(\hat{\Gamma} \times \hat{\Gamma}))$ be the automorphism given by

$$\vartheta(f)(\hat{\gamma}_1, \hat{\gamma}_2) = f(\hat{\gamma}_1, -\hat{\gamma}_1 - \hat{\gamma}_2) \quad \text{for all } f \in C_\infty(\hat{\Gamma} \times \hat{\Gamma}).$$

By continuity, (55) extends to

$$[(\omega \otimes \mathrm{id})((\pi^R \otimes \pi^L)(f_1)V(\pi^R \otimes \pi^R)(f_2))]h(g)$$
$$= (\omega \otimes \mu)((\pi^R \otimes \pi^R)\vartheta(f_1)(R_g \otimes R_g)(\pi^R \otimes \pi^R)(f_2))$$

for any $f_1, f_2 \in C_b(\hat{\Gamma} \otimes \hat{\Gamma})$. Taking $f_1 = \Psi^\star$ and $f_2 = \Psi$ we obtain $\vartheta(\Psi^\star) = \overline{\Psi}$ and

$$\big[(\omega \otimes \mathrm{id})(YVX)\big]h(g) = (\omega \otimes \mu)\big(X^*(R_g \otimes R_g)X\big)$$

where X and Y were introduced in (38). Recall that $W = YVX$, hence

$$\mathcal{Q}(f)h(g) = (\omega \otimes \mu)\big(X^*(R_g \otimes R_g)X\big) = k(g).$$

This proves formula (54) and ends the proof of our theorem. $\square$

Let $f \in \mathrm{C}_\infty(G)$ be a quantizable function i.e. $f(g) = \omega(R_g)$ for some $\omega \in \mathrm{B}(L^2(G))_*$. Suppose that $\mathcal{Q}(f) = 0$. This means that $(\omega \otimes \mathrm{id})W = 0$ which together with Theorem 4.12 shows that $\omega(R_g) = 0$ for all $g \in G$. Hence $f(g) = 0$ for any $g \in G$, which shows that the quantization map $\mathcal{Q}$ is injective and its inverse is well defined. We shall show that the closure of this inverse is the GNS map for a Haar measure of $(\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^\Psi)$. Let us introduce $\mathfrak{N}_0 \subset \mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$:

$$\mathfrak{N}_0 = \big\{\mathcal{Q}(f)\colon f\text{-quantizable and } f \in L^2(G)\big\}.$$

For all $\mathcal{Q}(f) \in \mathfrak{N}_0$ we set $\eta_0(\mathcal{Q}(f)) = f$. This defines a map $\eta_0 : \mathfrak{N}_0 \to L^2(G)$.

Proposition 4.15. *Let η_0 be the map defined above. Then this is a densely defined, closable map from $\mathrm{C}_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ to $L^2(G)$.*

Proof. Let Ψ^Σ be a 2-cocycle obtained from Ψ by a flip of variables: $\Psi^\Sigma(\hat{\gamma}_1, \hat{\gamma}_2) = \Psi(\hat{\gamma}_2, \hat{\gamma}_1)$. Let $\mathcal{Q}^\Sigma$ be the quantization map related to Ψ^Σ. Using the equality

$$(\omega \otimes \mu)\big(X^*(R_g \otimes R_g)X\big) = (\mu \otimes \omega)\big(\Sigma X^* \Sigma (R_g \otimes R_g) \Sigma X \Sigma\big)$$

and Theorem 4.14 we see that for quantizable, square integrable functions $h, h' \in L^2(G)$ we have

$$\mathcal{Q}(h)h' = \mathcal{Q}^\Sigma(h')h. \tag{56}$$

Let us assume that $\lim_{n \to \infty} \mathcal{Q}(f_n) = 0$ and $\lim_{n \to \infty} \eta_0(\mathcal{Q}(f_n)) = f$. Using Eq. (56) we see that:

$$\mathcal{Q}^\Sigma(h)f = \lim_{n \to \infty} \mathcal{Q}^\Sigma(h)f_n = \lim_{n \to \infty} \mathcal{Q}(f_n)h = 0 \tag{57}$$

for all quantizable functions $h \in L^2(G)$. To conclude that f is 0 we have to show that the set of operators

$$\big\{\mathcal{Q}^\Sigma(h)\colon h \text{ is quantizable and } h \in L^2(G)\big\} \subset \mathrm{B}\big(L^2(G)\big)$$

separates elements of $L^2(G)$. In order to do that we introduce a multiplicative unitary W^Σ related to the 2-cocycle Ψ^Σ. The C*-algebra obtained by the slices of the first leg of W^Σ will be denoted by A^{Ψ^Σ}. By point 1 of Theorem 1.5 of [19], A^{Ψ^Σ} separates elements of $L^2(G)$, hence it is enough to note that:

$$A^{\Psi\Sigma} = \left\{ (\omega \otimes \mathrm{id}) W^{\Sigma} \colon \omega \in \mathrm{B}\big(L^2(G)\big) \right\}^{\mathrm{cls}}$$

$$= \left\{ (\omega_{x,y} \otimes \mathrm{id}) W^{\Sigma} \colon x, y \text{ are of compact support} \right\}^{\mathrm{cls}}$$

$$\subset \left\{ \mathcal{Q}^{\Sigma}(h) \colon h \text{ is quantizable and } h \in L^2(G) \right\}^{\mathrm{cls}}.$$

The last inclusion follows from the fact that, when x and y are of compact support, then the function f defined by $f(g) = \omega_{x,y}(R_g)$ is also of compact support. □

The closure of the map η_0 will be denoted by η and its domain will be denoted by $\mathfrak{N}$.

Proposition 4.16. *Let* $\eta : \mathfrak{N} \mapsto L^2(G)$ *be the map introduced above. Then* $\mathfrak{N}$ *is a left ideal in* $\mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$ *and* $\eta(ab) = a\eta(b)$ *for all* $a \in \mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$ *and* $b \in \mathfrak{N}$.

Proof. Let $b \in \mathfrak{N}$ and $a \in \mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$. Let us fix a sequence of quantizable functions f_n such that $a = \lim_{n\to\infty} \mathcal{Q}(f_n)$. Map η is the closure of η_0, therefore there exists a sequence $h_m \in \mathrm{C}_\infty(G)$ of quantizable functions such that:

$$\lim_{m\to\infty} \mathcal{Q}(h_m) = b \quad \text{and} \quad \lim_{m\to\infty} \eta\big(\mathcal{Q}(h_m)\big) = \eta(b).$$

Using Theorem 4.14 we get $\mathcal{Q}(f_n)\eta(\mathcal{Q}(h_m)) = \eta(\mathcal{Q}(f_n)\mathcal{Q}(h_m))$ and

$$a\eta\big(\mathcal{Q}(h_m)\big) = \lim_{n\to\infty} \mathcal{Q}(f_n)\eta\big(\mathcal{Q}(h_m)\big) = \lim_{n\to\infty} \eta\big(\mathcal{Q}(f_n)\mathcal{Q}(h_m)\big).$$

The closedness of the map η implies that

$$a\mathcal{Q}(h_m) \in \mathfrak{N} \quad \text{and} \quad \eta\big(a\mathcal{Q}(h_m)\big) = a\eta\big(\mathcal{Q}(h_m)\big).$$

Taking limits with respect to m and using the closedness of η once again we conclude that $ab \in \mathfrak{N}$ and $\eta(ab) = a\eta(b)$. □

The above proposition shows that the map $\eta : \mathfrak{N} \mapsto L^2(G)$ is a GNS map. To show that η corresponds to the Haar measure of $(\mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}, \Delta^{\Psi})$ we shall need the following

Proposition 4.17. *Let* $\eta : \mathfrak{N} \mapsto L^2(G)$ *be the map introduced above. For* $a \in \mathfrak{N}$ *and* $\varphi \in (\mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi})^*$ *let us consider their convolution* $\varphi * a = (\mathrm{id} \otimes \varphi)\Delta(a) \in \mathrm{C}_\infty(G)^{\widetilde{\Psi}\otimes\Psi}$. *Then* $\varphi * a$ *is an element of* $\mathfrak{N}$ *and*

$$\eta(\varphi * a) = \big[(\mathrm{id} \otimes \varphi) W\big]\eta(a). \tag{58}$$

Proof. Recall that with any normal functional $\omega \in \mathrm{B}(L^2(G))_*$ we can associate a function $f_\omega \in \mathrm{C}_\infty(G)$ where $f_\omega(g) = \omega(R_g)$. Assume that $a = \mathcal{Q}(f_\omega)$ for some $f_\omega \in L^2(G)$. In particular $a \in \mathfrak{N}$ and $\eta(a) = f_\omega$. We compute

$$\varphi * a = (\mathrm{id} \otimes \varphi)\big(W^*(a \otimes 1)W\big)$$
$$= (\mathrm{id} \otimes \varphi)\big(W^*\big((\omega \otimes \mathrm{id})(W) \otimes 1\big)W\big)$$
$$= (\omega \otimes \mathrm{id} \otimes \varphi)\big(W_{23}^* W_{12} W_{23}\big)$$
$$= (\omega \otimes \mathrm{id} \otimes \varphi)(W_{12} W_{13})$$
$$= (b \cdot \omega \otimes \mathrm{id})W \tag{59}$$

where $b = (\mathrm{id} \otimes \varphi)W \in \mathrm{M}(\mathrm{C}_r^*(G))$. Therefore to prove that $\varphi * a$ is an element of $\mathfrak{N}$ it is enough to show that $f_{b \cdot \omega} \in L^2(G)$ for all $b \in \mathrm{M}(\mathrm{C}_r^*(G))$. First we check it for $b = R_g$. Note that

$$f_{b \cdot \omega}(g') = b \cdot \omega(R_g) = \omega\big(R_g' R_g\big) = \omega(R_{g'g})$$
$$= f_\omega(g'g) = (R_g f_\omega)(g') = (b f_\omega)(g')$$

for any $g, g' \in G$, therefore $f_{b \cdot \omega} = b f_\omega$. By linearity this equality is satisfied for any $b \in \mathrm{span}\{R_g \colon g \in G\}$. We extend it using a continuity argument. There exists a net of operators $b_i \in \mathrm{lin\text{-}span}\{R_g \colon g \in G\}$ strongly convergent to $b \in \mathrm{M}(\mathrm{C}_r^*(G))$. Functional ω is strongly continuous hence $\lim_i b_i \cdot \omega = b \cdot \omega$ in the norm sense. Therefore $\lim_i f_{b_i \cdot \omega} = f_{b \cdot \omega}$ where $\lim_i$ is taken in the uniform sense. At the same time $\lim_i b_i f_\omega = b f_\omega$ in the L^2-norm, hence

$$f_{b \cdot \omega}(g) = \lim_i f_{b_i \cdot \omega}(g) = \lim_i b_i f_\omega(g) = b f_\omega(g)$$

for almost all $g \in G$. This shows that $f_{b \cdot \omega} \in L^2(G)$ and

$$f_{b \cdot \omega} = b f_\omega \tag{60}$$

for any $b \in \mathrm{M}(\mathrm{C}_r^*(G))$. Using (59) and (60) we get the following sequence of equalities:

$$\eta(\varphi * a) = f_{b \cdot \omega} = b f_\omega$$
$$= b \eta(a) = \big[(\mathrm{id} \otimes \varphi)W\big] \eta(a)$$

which proves (58) for $a = \mathcal{Q}(f_\omega)$. But the set

$$\big\{a = (\omega \otimes \mathrm{id})W \colon f_\omega \in L^2(G)\big\}$$

is a core for η, hence Eq. (58) is satisfied for any $a \in \mathfrak{N}$. $\quad\square$

Remark 4.18. Let $\pi^R, \pi^L \in \mathrm{Rep}(\mathrm{C}^*(\Gamma), L^2(G))$ be representations that send generators $u_\gamma \in \mathrm{M}(\mathrm{C}^*(\Gamma))$ to R_γ and $L_\gamma \in \mathrm{B}(L^2(G))$ respectively. Let $f, \tilde{f} \in \mathrm{M}(\mathrm{C}^*(\Gamma))$ and $\omega \in \mathrm{B}(L^2(G))_*$ be such that $f_\omega \in L^2(G)$. Then using a method similar to the one used in the proof of Proposition 4.17 we can show that

$$f_{\pi^R(f) \cdot \omega \cdot \pi^R(\tilde{f})} = \pi^R(f) \pi^L\big(\kappa(\tilde{f})\big)(f_\omega) \tag{61}$$

where κ is the coinverse on $\mathrm{C}^*(\Gamma)$.

With GNS-map $\eta : \mathfrak{N} \to L^2(G)$ we can associate a weight $h^\Psi : C_\infty(G)_+^{\widetilde{\Psi} \otimes \Psi} \mapsto \mathbb{R}_+$: $h^\Psi(a^*a) = (\eta(a)|\eta(a))$.

Proposition 4.19. *Let h^Ψ be the weight on $C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}$ introduced above. Then it is a faithful trace. In particular it is strictly faithful.*

Proof. Let $u \in M(C^*(\Gamma))$ be the unitary element which appears in formula (42). From the above remark and Proposition 4.10 it follows that

$$\eta\big(\big((\omega_{x,y} \otimes \mathrm{id})W\big)^*\big) = \eta\big(\big(\pi^R(u) \cdot \omega_{\bar{x},\bar{y}} \cdot \pi^R(u) \otimes \mathrm{id}\big)W\big)$$
$$= \pi^R(u)\pi^L\big(\kappa(u)\big)\eta\big((\omega_{\bar{x},\bar{y}} \otimes \mathrm{id})W\big)$$
$$= \pi^R(u)\pi^L\big(\kappa(u)\big)\overline{\eta\big((\omega_{x,y} \otimes \mathrm{id})W\big)}.$$

The set

$$\big\{a = (\omega \otimes \mathrm{id})W \colon f_\omega \in L^2(G)\big\}$$

is a core for η, hence we have

$$\eta\big(a^*\big) = \pi^R(u)\pi^L\big(\kappa(u)\big)\overline{\eta(a)}$$

for any $a \in \mathfrak{N}$. Now we can prove the trace property:

$$h^\Psi\big(a^*a\big) = \big(\eta(a)|\eta(a)\big) = \big(\overline{\eta(a)}|\overline{\eta(a)}\big)$$
$$= \big(\pi^L\big(\kappa(u)\big)^*\pi^R(u)^*\eta\big(a^*\big)|\pi^L\big(\kappa(u)\big)^*\pi^R(u)^*\eta\big(a^*\big)\big)$$
$$= \big(\eta\big(a^*\big)|\eta\big(a^*\big)\big) = h^\Psi\big(aa^*\big).$$

Let us prove the faithfulness of h^Ψ. Assume that $h^\Psi(a^*a) = 0$. Then

$$h^\Psi\big(a^*c^*ca\big) = 0 = h^\Psi\big(caa^*c^*\big)$$

hence $\eta(a^*c^*) = a^*\eta(c^*) = 0$. The set of elements of the form $\eta(c^*)$ is dense in $L^2(G)$, hence $a = 0$. The notion of strict faithfulness was introduced in [9]. It can be shown that a faithful trace is automatically strictly faithful. This ends our proof. $\quad\square$

Using Propositions 4.17 and 4.19 one can check that the assumptions of Theorem 3.9 of [9] are satisfied. Hence we get

Theorem 4.20. *Let $(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^\Psi)$ be the quantum group with the multiplicative unitary W and the weight h^Ψ considered above. Then h^Ψ is a Haar measure for $(C_\infty(G)^{\widetilde{\Psi} \otimes \Psi}, \Delta^\Psi)$ and W is the canonical multiplicative unitary.*

5. An example of quantization of $SL(2, \mathbb{C})$

In this section we use the Rieffel deformation to quantize the special linear group:

$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C}, \ \alpha\delta - \beta\gamma = 1 \right\}.$$

(In what follows $SL(2, \mathbb{C})$ will be denoted by G.) The resulting quantum group is the C*-algebraic version of one of the $*$-Hopf algebras introduced by S.L. Woronowicz and S. Zakrzewski in paper [17]. As a $*$-algebra it is generated by four elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, satisfying the following commutation relations:

$$\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha},$$
$$\hat{\alpha}\hat{\delta} = \hat{\delta}\hat{\alpha},$$
$$\hat{\alpha}\hat{\gamma} = \hat{\gamma}\hat{\alpha},$$
$$\hat{\beta}\hat{\gamma} = \hat{\gamma}\hat{\beta},$$
$$\hat{\beta}\hat{\delta} = \hat{\delta}\hat{\beta},$$
$$\hat{\gamma}\hat{\delta} = \hat{\delta}\hat{\gamma},$$
$$\hat{\alpha}\hat{\delta} = 1 + \hat{\beta}\hat{\gamma},$$

$$\hat{\alpha}\hat{\alpha}^* = \hat{\alpha}^*\hat{\alpha},$$
$$\hat{\alpha}\hat{\beta}^* = t\hat{\beta}^*\hat{\alpha}, \qquad \hat{\beta}\hat{\beta}^* = \hat{\beta}^*\hat{\beta},$$
$$\hat{\alpha}\hat{\gamma}^* = t^{-1}\hat{\gamma}^*\hat{\alpha}, \qquad \hat{\beta}\hat{\gamma}^* = \hat{\gamma}^*\hat{\beta}, \qquad \hat{\gamma}\hat{\gamma}^* = \hat{\gamma}^*\hat{\gamma},$$
$$\hat{\alpha}\hat{\delta}^* = \hat{\delta}^*\hat{\alpha}, \qquad \hat{\beta}\hat{\delta}^* = t^{-1}\hat{\delta}^*\hat{\beta}, \qquad \hat{\gamma}\hat{\delta}^* = t\hat{\delta}^*\hat{\gamma}, \qquad \hat{\delta}\hat{\delta}^* = \hat{\delta}^*\hat{\delta}, \qquad (62)$$

where t is a nonzero real parameter. The comultiplication, coinverse and counit act on them in the standard way:

$$\begin{aligned}
\Delta(\hat{\alpha}) &= \hat{\alpha} \otimes \hat{\alpha} + \hat{\beta} \otimes \hat{\gamma}, & \kappa(\hat{\alpha}) &= \hat{\delta}, & \varepsilon(\hat{\alpha}) &= 1, \\
\Delta(\hat{\beta}) &= \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\delta}, & \kappa(\hat{\beta}) &= -\hat{\beta}, & \varepsilon(\hat{\beta}) &= 0, \\
\Delta(\hat{\gamma}) &= \hat{\gamma} \otimes \hat{\alpha} + \hat{\delta} \otimes \hat{\gamma}, & \kappa(\hat{\gamma}) &= -\hat{\gamma}, & \varepsilon(\hat{\gamma}) &= 0, \\
\Delta(\hat{\delta}) &= \hat{\gamma} \otimes \hat{\beta} + \hat{\delta} \otimes \hat{\delta}, & \kappa(\hat{\delta}) &= \hat{\alpha}, & \varepsilon(\hat{\delta}) &= 1.
\end{aligned} \qquad (63)$$

The deformation procedure in our example is based on the abelian subgroup $\Gamma \subset G$ of diagonal matrices:

$$\Gamma = \left\{ \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} : w \in \mathbb{C}_* \right\}.$$

To simplify some calculations we pull back the action of Γ^2 on $C_\infty(G)$ to the action of $\mathbb{C}^2$ on $C_\infty(G)$. The resulting action is denoted by ρ:

$$(\rho_{z_1,z_2} f)(g) = f\left(\begin{pmatrix} e^{-z_1} & 0 \\ 0 & e^{z_1} \end{pmatrix} g \begin{pmatrix} e^{z_2} & 0 \\ 0 & e^{-z_2} \end{pmatrix}\right). \tag{64}$$

Let us fix a 2-cocycle on the dual group. The additive group $(\mathbb{C}, +)$ is self-dual, with the duality given by:

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto \exp\big(i \operatorname{Im}(z_1 z_2)\big) \in \mathbb{T}.$$

Let $s \in \mathbb{R}$. For any $z_1, z_2 \in \mathbb{C}$ we set

$$\Psi(z_1, z_2) = \exp\big(is \operatorname{Im}(z_1 \bar{z}_2)\big).$$

It is clear, that $\Psi \in C_b(\mathbb{C}^2)$ satisfies the 2-cocycle condition. Using results of Section 4 we deform the standard $\mathbb{C}^2$-product structure on $C_\infty(G) \rtimes_\rho \mathbb{C}^2$ to $(C_\infty(G) \rtimes_\rho \mathbb{C}^2, \lambda, \hat{\rho}^{\widetilde{\Psi} \otimes \Psi})$. In our case $\widetilde{\Psi}$ is just the complex conjugate of Ψ and the deformed action of the dual group is given by

$$\hat{\rho}^{\widetilde{\Psi} \otimes \Psi}_{z_1,z_2}(b) = \lambda_{-s\bar{z}_1, s\bar{z}_2} \hat{\rho}_{z_1,z_2}(b) \lambda^*_{-s\bar{z}_1, s\bar{z}_2}$$

for any $b \in C_\infty(G) \rtimes_\rho \mathbb{C}^2$. The Landstad algebra A of the deformed $\mathbb{C}^2$-product carries the structure of a quantum group. Our aim is to show that this quantum group is the C*-algebraic version of the Hopf *-algebra described above. The relation between parameters $s, t \in \mathbb{R}$ is $t = \exp(-2s)$.

5.1. C*-algebra structure

In this section we will construct four affiliated elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \, \eta \, A$ and show that they generate C*-algebra A.

Let $T_r, T_l \in \mathrm{C}^*(\mathbb{C}^2)^\eta \subset (C_\infty(G) \rtimes_\rho \mathbb{C}^2)^\eta$ be infinitesimal generators of the left and right shifts. By definition T_l and T_r are normal elements satisfying:

$$\lambda_{z_1,z_2} = \exp\big(i \operatorname{Im}(z_1 T_l)\big) \exp\big(i \operatorname{Im}(z_2 T_r)\big) \tag{65}$$

for any $z_1, z_2 \in \mathbb{C}$. Let $\alpha, \beta, \gamma, \delta$ be coordinate functions on G:

$$\alpha, \beta, \gamma, \delta \in C(G) = \big(C_\infty(G)\big)^\eta \subset \big(C_\infty(G) \rtimes_\rho \mathbb{C}^2\big)^\eta.$$

Consider also a unitary element:

$$U = \exp\big(is \operatorname{Im}(T_r^* T_l)\big) \in \mathrm{M}\big(\mathrm{C}^*(\mathbb{C}^2)\big) \subset \mathrm{M}\big(C_\infty(G) \rtimes_\rho \mathbb{C}^2\big). \tag{66}$$

We use it to define four normal elements affiliated with $C_\infty(G) \rtimes_\rho \mathbb{C}^2$:

$$\hat{\alpha} = U\alpha U^*, \qquad \hat{\beta} = U^*\beta U,$$
$$\hat{\gamma} = U^*\gamma U, \qquad \hat{\delta} = U\delta U^*. \qquad (67)$$

In the next lemma we present different formulas for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ which will be needed later.

Lemma 5.1. *Let* $\alpha, \beta, \gamma, \delta \in C_\infty(G)^\eta \subset (C_\infty(G) \rtimes_\rho \mathbb{C}^2)^\eta$ *be coordinate functions on G. Let T_l, T_r be infinitesimal generators defined by* (65) *and let*

$$\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in \left(C_\infty(G) \rtimes_\rho \mathbb{C}^2\right)^\eta$$

be normal elements (67). *Then*

$$\begin{cases} 1. & \alpha \text{ and } T_l + T_r \text{ strongly commute and } \hat{\alpha} = \exp\!\left(-s\left(T_l^* + T_r^*\right)\right)\alpha; \\ 2. & \beta \text{ and } T_l - T_r \text{ strongly commute and } \hat{\beta} = \exp\!\left(s\left(T_l^* - T_r^*\right)\right)\beta; \\ 3. & \gamma \text{ and } T_l - T_r \text{ strongly commute and } \hat{\gamma} = \exp\!\left(s\left(T_r^* - T_l^*\right)\right)\gamma; \\ 4. & \delta \text{ and } T_l + T_r \text{ strongly commute and } \hat{\delta} = \exp\!\left(s\left(T_l^* + T_r^*\right)\right)\delta. \end{cases} \qquad (68)$$

Proof. The fact that $T_l + T_r$ and α strongly commute follows from the identity

$$\exp\!\left(\mathrm{i}\,\mathrm{Im}\!\left(z(T_l + T_r)\right)\right)\alpha \exp\!\left(-\mathrm{i}\,\mathrm{Im}\!\left(z(T_l + T_r)\right)\right) = \alpha.$$

We check it below:

$$\exp\!\left(\mathrm{i}\,\mathrm{Im}\!\left(z(T_l + T_r)\right)\right)\alpha \exp\!\left(-\mathrm{i}\,\mathrm{Im}\!\left(z(T_l + T_r)\right)\right) = \lambda_{z,z}\alpha\lambda_{z,z}^* = \exp(-z + z)\alpha = \alpha.$$

To prove the equality $\hat{\alpha} = \exp(-s(T_l^* + T_r^*))\alpha$ note that

$$\begin{aligned} \hat{\alpha} &= \exp\!\left(\mathrm{i}s\,\mathrm{Im}\!\left(T_r^* T_l\right)\right)\alpha \exp\!\left(-\mathrm{i}s\,\mathrm{Im}\!\left(T_r^* T_l\right)\right) \\ &= \exp\!\left(\mathrm{i}s\,\mathrm{Im}\!\left((T_l + T_r)^* T_l\right)\right)\alpha \exp\!\left(-\mathrm{i}s\,\mathrm{Im}\!\left((T_l + T_r)^* T_l\right)\right), \end{aligned} \qquad (69)$$

where we used the fact that $\exp(\mathrm{i}s\,\mathrm{Im}(T_l^* T_l)) = 1$. Using the strong commutativity of $T_l + T_r$ and α and the following identity:

$$\exp\!\left(\mathrm{i}s\,\mathrm{Im}(wT_l)\right)\alpha \exp\!\left(-\mathrm{i}s\,\mathrm{Im}(wT_l)\right) = \exp(-sw)\alpha,$$

we get $\hat{\alpha} = \exp(-s(T_l^* + T_r^*))\alpha$. This ends the proof of point 1 of (68). Using the same techniques we prove points 2, 3, 4. $\square$

Our objective is to show that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are generators of C*-algebra A. In particular we have to show that they are affiliated with A. The following proposition is the first step toward the proof of this fact.

Proposition 5.2. *Let* $(C_\infty(G) \rtimes_\rho \mathbb{C}^2, \lambda, \hat{\rho}^{\widetilde{\Psi}\otimes\Psi})$ *be the deformed $\mathbb{C}^2$-product, A its Landstad algebra and* $\hat{\alpha} \in (C_\infty(G) \rtimes_\rho \mathbb{C}^2)^\eta$ *the normal element defined in* (67). *Then* $f(\hat{\alpha}) \in M(A)$ *for any* $f \in C_\infty(\mathbb{C})$.

Proof. Let us first prove the invariance of $f(\hat{\alpha})$ under the action $\hat{\rho}^{\widetilde{\Psi} \otimes \Psi}$. It is enough to check that $\hat{\alpha}$ is invariant. In order to do that we calculate

$$\lambda_{z_1,z_2} \alpha \lambda_{z_1,z_2}^* = \rho_{z_1,z_2}(\alpha) = \exp(-z_1 + z_2)\alpha. \tag{70}$$

Furthermore

$$\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\hat{\alpha}) = \hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(U \alpha U^*)$$
$$= \hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(U) \hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\alpha) \hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(U)^*.$$

We compute $\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(U)$ and $\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\alpha)$ separately:

$$\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\alpha) = \lambda_{-s\bar{z}_1,s\bar{z}_2} \hat{\rho}_{z_1,z_2}(\alpha) \lambda_{-s\bar{z}_1,s\bar{z}_2}^*$$
$$= \lambda_{-s\bar{z}_1,s\bar{z}_2} \alpha \lambda_{-s\bar{z}_1,s\bar{z}_2}^*$$
$$= \exp(s\bar{z}_1 + s\bar{z}_2)\alpha,$$
$$\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(U) = \hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\exp(is \operatorname{Im}(T_r^* T_l)))$$
$$= \hat{\rho}_{z_1,z_2}(\exp(is \operatorname{Im}(T_r^* T_l)))$$
$$= \exp(is \operatorname{Im}((T_r^* + \bar{z}_2)(T_l + z_1)))$$
$$= U \lambda_{s\bar{z}_2,-s\bar{z}_1} \Psi(z_1, z_2).$$

Using (70) we get

$$\hat{\rho}_{z_1,z_2}^{\widetilde{\Psi} \otimes \Psi}(\hat{\alpha}) = \exp(s\bar{z}_1 + s\bar{z}_2) U \lambda_{s\bar{z}_2,-s\bar{z}_1} \alpha \lambda_{-s\bar{z}_2,s\bar{z}_1}^* U^*$$
$$= \exp(s\bar{z}_1 + s\bar{z}_2) \exp(-s\bar{z}_1 - s\bar{z}_2) U \alpha U^* = \hat{\alpha}.$$

Let us now check that the map

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto \lambda_{z_1,z_2} f(\hat{\alpha}) \lambda_{z_1,z_2}^* \in \mathrm{M}(\mathrm{C}_\infty(G) \rtimes \mathbb{C}^2) \tag{71}$$

is norm continuous. For this note that:

$$\lambda_{z_1,z_2} f(\hat{\alpha}) \lambda_{z_1,z_2}^* = U \lambda_{z_1,z_2} f(\alpha) \lambda_{z_1,z_2}^* U^*$$
$$= U f(e^{-z_1+z_2} \alpha) U^*.$$

Function f is continuous and vanishes at infinity, hence we get norm continuity (71). This shows that $f(\hat{\alpha})$ satisfies the first and second Landstad condition of (3) which is enough to be an element of $\mathrm{M}(A)$. $\quad\square$

To prove that $\hat{\alpha}$ is affiliated to A we need one more

Proposition 5.3. *The set*

$$\mathcal{I} = \big\{ f(\hat{\alpha})A \colon f \in \mathrm{C}_\infty(\mathbb{C}) \big\}$$

is linearly dense in A.

Proof. Recall that $\rho^{\widetilde{\Psi} \otimes \Psi}$ is the action of $\mathbb{C}^2$ on A implemented by unitary elements λ_{z_1, z_2}. It is easy to see that $\mathcal{I}$ is invariant under $\rho^{\widetilde{\Psi} \otimes \Psi}$. Let $g \in \mathrm{C}_\infty(\mathbb{C})$ be a function given by the formula $g(z) = (1 + \bar{z}z)^{-1}$. Then $g(\hat{\alpha}) = U(1 + \alpha^*\alpha)U^*$ and we have:

$$\big(\mathrm{C}^*(\mathbb{C}^2)g(\hat{\alpha})A\mathrm{C}^*(\mathbb{C}^2)\big)^{\mathrm{cls}} = \big(\mathrm{C}^*(\mathbb{C}^2)\big(1 + \alpha^*\alpha\big)^{-1}U^*A\mathrm{C}^*(\mathbb{C}^2)\big)^{\mathrm{cls}}$$
$$\subset \big(\mathrm{C}^*(\mathbb{C}^2)\mathcal{I}\mathrm{C}^*(\mathbb{C}^2)\big)^{\mathrm{cls}} \tag{72}$$

where we used the equality $\mathrm{C}^*(\mathbb{C}^2)U = \mathrm{C}^*(\mathbb{C}^2)$. Note that the set $U^*A\mathrm{C}^*(\mathbb{C}^2)$ is linearly dense in $\mathrm{C}_\infty(G) \rtimes_\rho \mathbb{C}^2$. Using the fact that α is affiliated with $\mathrm{C}_\infty(G) \rtimes_\rho \mathbb{C}^2$ we see that the set $\mathrm{C}^*(\mathbb{C}^2)(1 + \alpha^*\alpha)^{-1}U^*A\mathrm{C}^*(\mathbb{C}^2)$ is linearly dense in $\mathrm{C}_\infty(G) \rtimes_\rho \mathbb{C}^2$. Hence by (72) the set $\mathrm{C}^*(\mathbb{C}^2)\mathcal{I}\mathrm{C}^*(\mathbb{C}^2)$ is linearly dense in $\mathrm{C}_\infty(G) \rtimes_\rho \mathbb{C}^2$. Using Lemma 2.6 we get the linear density of $\mathcal{I}$ in A. $\square$

Let us define the homomorphism of C*-algebras:

$$\mathrm{C}_\infty(\mathbb{C}) \ni f \mapsto \pi(f) = f(\hat{\alpha}) \in \mathrm{M}(A).$$

Theorem 5.4. *Let π be the homomorphism defined above. π is a morphism of C*-algebras: $\pi \in \mathrm{Mor}(\mathrm{C}_\infty(\mathbb{C}); A)$. In particular $\hat{\alpha}$ is the normal element affiliated with A.*

Proof. By Proposition 5.3 we have $\overline{\pi(\mathrm{C}_\infty(\mathbb{C}))A}^{\|\cdot\|} = A$ which shows that $\pi \in \mathrm{Mor}(\mathrm{C}_\infty(\mathbb{C}); A)$. Let $\mathrm{id} \in \mathrm{C}_\infty(\mathbb{C})^\eta$ be the identity function: $\mathrm{id}(z) = z$ for all $z \in \mathbb{C}$. Applying morphism π to $\mathrm{id} \in \mathrm{C}_\infty(\mathbb{C})^\eta$ we get $\pi(\mathrm{id}) = \mathrm{id}(\hat{\alpha}) = \hat{\alpha} \in A^\eta$. $\square$

Using the same techniques we show that $\hat{\beta}, \hat{\gamma}, \hat{\delta} \, \eta \, A$. In the next theorem we prove that they are in fact generators of A.

Theorem 5.5. *Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \, \eta \, A$ be affiliated elements introduced in* (67). *Let us consider the set*:

$$\mathcal{V} = \big\{ f_1(\hat{\alpha}) f_2(\hat{\beta}) f_3(\hat{\gamma}) f_4(\hat{\delta}) \colon f_1, f_2, f_3, f_4 \in \mathrm{C}_\infty(\mathbb{C}) \big\} \subset \mathrm{M}(A).$$

Then $\mathcal{V}$ is a subset of A and $\mathcal{V}^{\mathrm{cls}} = A$. In particular A is generated by elements $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$.

Proof. Let us start with a proof that $\mathcal{V} \subset A$. Mimicking the proof of Theorem 5.2 we show that elements of $\mathcal{V}$ satisfy the first and the second Landstad condition (3). To check that they also satisfy the third one, we need to show that

$$x f_1(\hat{\alpha}) f_2(\hat{\beta}) f_3(\hat{\gamma}) f_4(\hat{\delta}) y \in \mathrm{C}_\infty(G) \rtimes_\rho \mathbb{C}^2 \tag{73}$$

for any $x, y \in C^*(\mathbb{C}^2)$. Let us consider the set

$$\mathcal{W} = \left\{ x f_1(\hat{\alpha}) f_2(\hat{\beta}) f_3(\hat{\gamma}) f_4(\hat{\delta}) y \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

Note that $\mathcal{W} = (C^*(\mathbb{C}^2) \mathcal{V} C^*(\mathbb{C}^2))^{\text{cls}}$. We will show that:

$$\mathcal{W} = C_\infty(G) \rtimes_\rho \mathbb{C}^2$$

which is a stronger property than (73). Using (67) we get

$$\mathcal{W} = \left\{ x U f_1(\alpha) U^{*2} f_2(\beta) f_3(\gamma) U^2 f_4(\delta) U^* y \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

By unitarity of U we can substitute x with $x U^*$ and y with $U y$ not changing $\mathcal{W}$:

$$\mathcal{W} = \left\{ x f_1(\alpha) U^{*2} f_2(\beta) f_3(\gamma) U^2 f_4(\delta) y \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

The map

$$\mathbb{C}^2 \ni (z_1, z_2) \mapsto \rho_{z_1, z_2}\big(f(\alpha) \big) = f\big(\exp(-z_1 + z_2) \alpha \big)$$

is norm continuous, hence:

$$\left\{ f(\alpha) x \colon f \in C_\infty(\mathbb{C}), x \in C^*(\mathbb{C}^2) \right\}^{\text{cls}} = \left\{ x f(\alpha) \colon f \in C_\infty(\mathbb{C}), x \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

In particular

$$\mathcal{W} = \left\{ f_1(\alpha) x U^{*2} f_2(\beta) f_3(\gamma) U^2 f_4(\delta) y \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

Similarly, we commute $f_4(\delta)$ and y:

$$\mathcal{W} = \left\{ f_1(\alpha) x U^{*2} f_2(\beta) f_3(\gamma) U^2 y f_4(\delta) \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

Substituting x with $x U^2$ and y with $U^{*2} y$ we get

$$\mathcal{W} = \left\{ f_1(\alpha) x f_2(\beta) f_3(\gamma) y f_4(\delta) \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

Commuting back $f_1(\alpha)$ ($f_4(\delta)$ resp.) and x (y resp.) we obtain

$$\mathcal{W} = \left\{ x f_1(\alpha) f_2(\beta) f_3(\gamma) f_4(\delta) y \colon f_1, f_2, f_3, f_4 \in C_\infty(\mathbb{C}), \ x, y \in C^*(\mathbb{C}^2) \right\}^{\text{cls}}.$$

The last set is obviously the whole $C_\infty(G) \rtimes_\rho \mathbb{C}^2$. Therefore we conclude that elements of $\mathcal{V}$ satisfies the Landstad conditions and $\mathcal{V} \subset A$. Moreover $\mathcal{V}$ is $\rho^{\widetilde{\Psi} \otimes \Psi}$-invariant and the set $C^*(\mathbb{C}^2) \mathcal{V} C^*(\mathbb{C}^2)$ is linearly dense in $C_\infty(G) \rtimes_\rho \mathbb{C}^2$. Using Lemma 2.6 we see that $\mathcal{V}^{\text{cls}} = A$.

In particular $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ separate representations of A and

$$\left(1 + \hat{\alpha}^*\hat{\alpha}\right)^{-1}\left(1 + \hat{\beta}^*\hat{\beta}\right)^{-1}\left(1 + \hat{\gamma}^*\hat{\gamma}\right)^{-1}\left(1 + \hat{\delta}^*\hat{\delta}\right)^{-1} \in A.$$

By Theorem 3.3 of [18] we see that A is generated by $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$. $\quad\square$

5.2. Commutation relations

The aim of this section is to show that generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ satisfy relations (62). Note that in general it is impossible to multiply affiliated elements, so we have to give a precise meaning to (62). We start with considering a more general type of relations. Let p, q be real, strictly positive numbers and (R, S) a pair of normal operators acting on H. The precise meaning of the relations

$$RS = pSR,$$
$$RS^* = qS^*R$$

was given in [16]:

Definition 5.6. Let (R, S) be a pair of normal operators acting on a Hilbert space H. We say that (R, S) is a (p, q)-commuting pair if:

1. $|R|$ and $|S|$ strongly commute.
2. $(\text{Phase } R)(\text{Phase } S) = (\text{Phase } S)(\text{Phase } R)$.
3. On $\ker R^\perp$ we have

$$(\text{Phase } R)|S|(\text{Phase } R)^* = \sqrt{pq}\,|S|.$$

4. On $\ker S^\perp$ we have

$$(\text{Phase } S)|R|(\text{Phase } S)^* = \sqrt{q/p}\,|R|.$$

The set of all (p, q)-commuting pairs of normal operators acting on a Hilbert space H is denoted by $D_{p,q}(H)$. Note that $(1, 1)$-commuting pair of normal operators is just a strongly commuting pair of operators.

We need a version of the above definition which is suitable for a pair of normal elements affiliated with a C*-algebra. In what follows we shall use the symbol $z(T)$ to denote the z-transform of an element T: $z(T) = T(1 + T^*T)^{-\frac{1}{2}}$.

Definition 5.7. Let A be a C*-algebra and (R, S) a pair of normal elements affiliated with A. We say that (R, S) is a (p, q)-commuting pair if

1. $z(R)z(S^*) = z(\sqrt{pq}\,S^*)z(\sqrt{q/p}\,R)$,
2. $z(\sqrt{q/p}\,R)z(S) = z(\sqrt{pq}\,S)z(R)$.

The set of all (p, q)-commuting pairs of normal elements affiliated with a C*-algebra A is denoted by $D_{p,q}(A)$.

It turns out that Definitions 5.6 and 5.7 are in a sense equivalent. Namely we have:

Proposition 5.8. *Let (R, S) be a pair of normal operators acting on H. It is a (p, q)-commuting pair in the sense of Definition 5.6 if and only if*

$$\begin{cases} z(R)z(S^*) = z(\sqrt{pq}\, S^*)z(\sqrt{q/p}\, R), \\ z(\sqrt{q/p}\, R)z(S) = z(\sqrt{pq}\, S)z(R). \end{cases} \tag{74}$$

Proof. It is easy to see that a pair (R, S) of (p, q)-commuting operators satisfies (74). We will prove the opposite implication. Using (74) we get:

$$\begin{aligned} z(\sqrt{q/p}\, R)z(S)z(S)^* &= z(\sqrt{pq}\, S)z(R)z(S)^* \\ &= z(\sqrt{pq}\, S)z(\sqrt{pq}\, S)^*z(\sqrt{q/p}\, R). \end{aligned} \tag{75}$$

Hence

$$\begin{aligned} &z(\sqrt{q/p}\, R)^*z(\sqrt{q/p}\, R)z(S)z(S)^* \\ &= z(\sqrt{q/p}\, R)^*z(\sqrt{pq}\, S)z(\sqrt{pq}\, S)^*z(\sqrt{q/p}\, R) \\ &= z(S)z(S)^*z(\sqrt{q/p}\, R)^*z(\sqrt{q/p}\, R). \end{aligned} \tag{76}$$

T is a normal operator, hence $z_T^* z_T = z_T z_T^* = z_{|T|}^2$ and we get

$$\big(z(\sqrt{q/p}\, |R|)\big)^2 z(|S|)^2 = z(|S|)^2 \big(z(\sqrt{q/p}\, |R|)\big)^2.$$

This shows that $(|R|, |S|)$ is a pair of strongly commuting operators.

Using the polar decomposition of normal operators R and S we rewrite the second equation of (74):

$$\mathrm{Phase}(R)z\big(\sqrt{q/p}\, |R|\big)z(|S|)\, \mathrm{Phase}(S) = \mathrm{Phase}(S)z\big(\sqrt{pq}\, |S|\big)z(|R|)\, \mathrm{Phase}(R).$$

Strong commutativity of $|R|$ and $|S|$ and identities

$$\mathrm{Phase}(S)\, \mathrm{Phase}(S)^*z(|S|) = z(|S|),$$
$$\mathrm{Phase}(R)\, \mathrm{Phase}(R)^*z(|R|) = z(|R|)$$

gives

$$\begin{aligned} &\mathrm{Phase}(R)\, \mathrm{Phase}(S)\, \mathrm{Phase}(S)^*z(|S|)z\big(\sqrt{q/p}\, |R|\big)\, \mathrm{Phase}(S) \\ &= \mathrm{Phase}(S)\, \mathrm{Phase}(R)\, \mathrm{Phase}(R)^*z(|R|)z\big(\sqrt{pq}\, |S|\big)\, \mathrm{Phase}(R). \end{aligned}$$

Uniqueness of the polar decomposition implies that phases of R and S commute:

$$\text{Phase}(R)\,\text{Phase}(S) = \text{Phase}(S)\,\text{Phase}(R).$$

Using Eq. (75) we get

$$\text{Phase}(R)z\big(\sqrt{q/p}\,|R|\big)z\big(|S|\big) = z\big(\sqrt{pq}\,|S|\big)\text{Phase}(R)z\big(\sqrt{q/p}\,|R|\big).$$

We already know that $|R|$ and $|S|$ strongly commute, hence

$$\text{Phase}(R)z\big(|S|\big)z\big(\sqrt{q/p}\,|R|\big) = z\big(\sqrt{pq}\,|S|\big)\text{Phase}(R)z\big(\sqrt{q/p}\,|R|\big).$$

This shows that on $\ker R^{\perp}$ we have

$$\text{Phase}(R)|S|\,\text{Phase}(R)^* = \sqrt{pq}\,|S|.$$

Similarly, one can prove that $\text{Phase}(S)|R|\,\text{Phase}(S)^* = \sqrt{q/p}\,|R|$ on $\ker S^{\perp}$. $\quad\square$

The next theorem shows that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \, \eta \, A$ satisfy relations (62) in the sense of Definition 5.7.

Theorem 5.9. *Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \, \eta \, A$ be elements given by* (67). *Then*

1. $(\hat{\alpha}, \hat{\delta}), (\hat{\beta}, \hat{\gamma}) \in D_{1,1}(A)$,
2. $(\hat{\alpha}, \hat{\beta}), (\hat{\gamma}, \hat{\delta}) \in D_{1,t}(A)$,
3. $(\hat{\alpha}, \hat{\gamma}), (\hat{\beta}, \hat{\delta}) \in D_{1,t^{-1}}(A)$,

where $t = \exp(-2s)$. Consider normal elements $\hat{\alpha}\hat{\delta}, \hat{\beta}\hat{\gamma} \, \eta \, A$ (a product of two strongly commuting normal elements is well defined). Then $(\hat{\alpha}\hat{\delta}, \hat{\beta}\hat{\gamma}) \in D_{1,1}(A)$ and

$$\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} = 1.$$

Proof. Directly from (67) it follows that $(\hat{\alpha}, \hat{\delta}) \in D_{1,1}(A)$ and $(\hat{\beta}, \hat{\gamma}) \in D_{1,1}(A)$. Note that the affiliated element $\alpha\delta \, \eta C_{\infty}(G)$ is ρ-invariant: $\rho_{z_1,z_2}(\alpha\delta) = \alpha\delta$ where ρ is the action defined by (64). Therefore, at the level of the crossed product, $\alpha\delta$ commutes with $C^*(\mathbb{C}^2)$. Using the fact that $U \in M(C^*(\mathbb{C}^2))$ we get

$$\hat{\alpha}\hat{\delta} = U\alpha\delta U^* = \alpha\delta.$$

Similar reasoning shows that $\hat{\beta}\hat{\gamma} = \beta\gamma$. Therefore $(\hat{\alpha}\hat{\delta}, \hat{\beta}\hat{\gamma}) \in D_{1,1}(A)$ and

$$\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} = \alpha\delta - \beta\gamma = 1.$$

Now let us prove that $(\hat{\alpha}, \hat{\beta}) \in D_{1,t}(A)$. Using the faithful representation π^{can} of A on $L^2(G)$ we can treat generators $\hat{\alpha}, \hat{\beta}$ as normal operators acting on $L^2(G)$. We will show that $(\hat{\alpha}, \hat{\beta}) \in D_{1,t}(L^2(G))$ which by Proposition 5.8 is equivalent with the containment $(\hat{\alpha}, \hat{\beta}) \in D_{1,t}(A)$. Using Lemma 5.1 we get:

$$\begin{cases} \text{Phase}(\hat{\alpha}) = \exp\big(is\,\text{Im}(T_l + T_r)\big)\,\text{Phase}(\alpha), \\ |\hat{\alpha}| = \exp\big(-s\,\text{Re}(T_l + T_r)\big)|\alpha|, \\ \text{Phase}(\hat{\beta}) = \exp\big(-is\,\text{Im}(T_l - T_r)\big)\,\text{Phase}(\beta), \\ |\hat{\beta}| = \exp\big(s\,\text{Re}(T_l - T_r)\big)|\beta|. \end{cases} \tag{77}$$

Moreover, it is easy to check that

$$\begin{cases} \exp\big(is\,\text{Im}(T_l + T_r)\big)\beta \exp\big(-is\,\text{Im}(T_l + T_r)\big) = \exp(-2s)\beta, \\ \exp\big(is\,\text{Im}(T_l - T_r)\big)\alpha \exp\big(-is\,\text{Im}(T_l - T_r)\big) = \exp(-2s)\alpha, \\ \exp\big(-is\,\text{Re}(T_l + T_r)\big)|\beta| \exp\big(is\,\text{Re}(T_l + T_r)\big) = \big|\exp(2is)\beta\big| = |\beta|, \\ \exp\big(-is\,\text{Re}(T_l - T_r)\big)|\alpha| \exp\big(is\,\text{Re}(T_l - T_r)\big) = \big|\exp(2is)\alpha\big| = |\alpha|. \end{cases} \tag{78}$$

Eqs. (77) and (78) show together that:

1. $\text{Phase}(\hat{\alpha})\,\text{Phase}(\hat{\beta}) = \text{Phase}(\hat{\beta})\,\text{Phase}(\hat{\alpha})$,

2. $\text{Phase}(\hat{\alpha})|\hat{\beta}|\,\text{Phase}(\hat{\alpha})^* = \exp(-2s)|\hat{\beta}|$,

3. $\text{Phase}(\hat{\beta})|\hat{\alpha}|\,\text{Phase}(\hat{\beta})^* = \exp(-2s)|\hat{\alpha}|$,

4. $|\hat{\alpha}|$ and $|\hat{\beta}|$ strongly commute.

Note that $\ker\hat{\alpha} = \ker\hat{\beta} = \{0\}$ hence $(\hat{\alpha}, \hat{\beta}) \in D_{1,t}(L^2(G))$. Using the same techniques we prove all other assertions of our theorem. $\square$

5.3. Comultiplication

Let $\Delta^{\Psi} \in \text{Mor}(A; A \otimes A)$ be the comultiplication on A. As was shown in Theorem 4.11, it is given by:

$$\Delta^{\Psi}(a) = \Upsilon \Delta(a) \Upsilon^*, \tag{79}$$

where $\Delta \in \text{Mor}(C_\infty(G) \rtimes \mathbb{C}^2; C_\infty(G) \rtimes \mathbb{C}^2 \otimes C_\infty(G) \rtimes \mathbb{C}^2)$ is uniquely characterized by two properties:

- $\Delta(T_l) = T_l \otimes I$, $\Delta(T_r) = I \otimes T_r$;
- Δ restricted to $C_\infty(G)$ coincides with the comultiplication on $C_\infty(G)$.

In our case the unitary element Υ is of the following form:

$$\Upsilon = \exp\big(is\,\text{Im}\big(T_r^* \otimes T_l\big)\big). \tag{80}$$

Theorem 5.10. *Let (A, Δ^{Ψ}) be the quantum group considered above and let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ be the generators of A given by (67). Comultiplication Δ^{Ψ} acts on generators in the standard way:*

$$\begin{aligned} \Delta^{\Psi}(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha} + \hat{\beta} \otimes \hat{\gamma}, && \Delta^{\Psi}(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\delta}, \\ \Delta^{\Psi}(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\alpha} + \hat{\delta} \otimes \hat{\gamma}, && \Delta^{\Psi}(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta} + \hat{\gamma} \otimes \hat{\beta}. \end{aligned} \tag{81}$$

Remark 5.11. The action of Δ^Ψ in the formula above is given by the sum of affiliated elements. In general it is not a well-defined operation. But in our case (as will be shown) this is a sum of two normal strongly commuting elements of $(A \otimes A)^\eta$. This operation is well defined and gives a normal element affiliated with $(A \otimes A)^\eta$.

Proof. Applying morphism Δ to $U \in \mathrm{M}(\mathrm{C}^*(\Gamma^2))$ (see (66)) we get:

$$\Delta(U) = \exp\bigl(is\,\mathrm{Im}\bigl(T_l \otimes T_r^*\bigr)\bigr). \tag{82}$$

Let R be a unitary element given by the formula:

$$R = \exp\bigl(is\,\mathrm{Im}\bigl(T_r^* \otimes T_l\bigr)\bigr) \exp\bigl(is\,\mathrm{Im}\bigl(T_l \otimes T_r^*\bigr)\bigr).$$

Using (79), (80) and (82) we get:

$$\begin{aligned}
\Delta^\Psi(\hat\alpha) &= R(\alpha \otimes \alpha + \beta \otimes \gamma)R^* \\
&= R(\alpha \otimes \alpha)R^* + R(\beta \otimes \gamma)R^*.
\end{aligned} \tag{83}$$

Note that

$$\bigl(U^* \otimes U^*\bigr)R = \exp\bigl(is\,\mathrm{Im}\bigl(T_r^* \otimes I - I \otimes T_r^*\bigr)(I \otimes T_l - T_l \otimes I)\bigr). \tag{84}$$

It is easy to check that elements $(T_r \otimes I - I \otimes T_r)$ and $(I \otimes T_l - T_l \otimes I)$ strongly commute with $\alpha \otimes \alpha$. Hence by identity (84), $(U^* \otimes U^*)R$ commutes with $\alpha \otimes \alpha$. Similarly, we check that the unitary element $(U \otimes U)R$ commutes with $\beta \otimes \gamma$. Using these two facts we get

$$\begin{aligned}
R(\alpha \otimes \alpha)R^* &= (U \otimes U)\bigl(U^* \otimes U^*\bigr)R(\alpha \otimes \alpha)R^*(U \otimes U)\bigl(U^* \otimes U^*\bigr) \\
&= (U \otimes U)(\alpha \otimes \alpha)\bigl(U^* \otimes U^*\bigr) = \hat\alpha \otimes \hat\alpha
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
R(\beta \otimes \gamma)R^* &= \bigl(U^* \otimes U^*\bigr)(U \otimes U)R(\beta \otimes \gamma)R^*\bigl(U^* \otimes U^*\bigr)(U \otimes U) \\
&= \bigl(U^* \otimes U^*\bigr)(\beta \otimes \gamma)(U \otimes U) = \hat\beta \otimes \hat\gamma.
\end{aligned} \tag{86}$$

Eqs. (83), (85), (86) give:

$$\Delta^\Psi(\hat\alpha) = \hat\alpha \otimes \hat\alpha + \hat\beta \otimes \hat\gamma.$$

All other assertions of our theorem are proven using the same techniques. $\square$

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