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Characterizing Finite Subspaces

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In this paper we study generalizations of the following question: Is a subspace of a projective or affine space characterized by the cardinalities of intersections with all hyperplanes? In several cases the answer is affirmative.

1. INTRODUCTION

A result which is due to Jessie MacWilliams [4] states that, if the set $S \subset PG(n, q)$, projective n-space over $GF(q)$, contains $(q^{k+1}-1)/(q-1)$ points and if S has the property that every hyperplane contains either all of S or $(q^k-1)/(q-1)$ points of S, then S is a k-dimensional subspace. In this paper we consider some cases of the general question of characterizing k-dimensional subspaces of an n-dimensional space by the numbers of r-dimensional subspaces they have in common with each of the $(n-j)$-dimensional subspaces. MacWilliams' result is the case $r = 0$, $j = 1$. We obtain similar results here for the cases $r = 0$ and j arbitrary and $j = 1$, r arbitrary. We consider this problem both for projective and affine spaces, using parallel, but not identical arguments, R. C. Bose and R. C. Burton [2] have results on a related problem.

NOTATION. Throughout the paper we shall use the following notation. If q is fixed we denote by A_n (respectively P_n) n-dimensional affine (projective) space over $GF(q)$, i.e., $AG(n, q)$ $(PG(n, q))$.

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$\begin{bmatrix} A_n \\ k \end{bmatrix}$ denotes the set of all k-dimensional subspaces of A_n,

$\begin{Bmatrix} P_n \\ k \end{Bmatrix}$ denotes the set of all k-dimensional subspaces of P_n.

$\begin{bmatrix} n \\ k \end{bmatrix}_q$ (respectively, $\begin{Bmatrix} n \\ k \end{Bmatrix}_q$) denotes the number of such subspaces.

When no confusion arises we shall omit the subscript q. We recall that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q^n - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)\cdots(q - 1)} q^{n-k}, & \text{if } n \geqslant k > 0, \\ q^n, & \text{if } n \geqslant k = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = \begin{cases} \dfrac{(q^{n+1} - 1)\cdots(q^{n-k+1} - 1)}{(q^{k+1} - 1)\cdots(q - 1)}, & \text{if } n \geqslant k \geqslant 0, \\ 1, & \text{if } n \geqslant k = -1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We shall refer to k-dimensional spaces and subspaces simply as k-spaces and k-subspaces.

DEFINITION. *A subset*

$$S \subseteq \begin{bmatrix} A_n \\ r \end{bmatrix}$$

has property $A(n, q; k, r, j)$ if the following holds:

(i) $r \leqslant k \leqslant n$ *and* $1 \leqslant j \leqslant n - 1$,

(ii) $|S| = \begin{bmatrix} k \\ r \end{bmatrix}_q$, (3)

(iii) $\left|S \cap \begin{bmatrix} A_{n-j} \\ r \end{bmatrix}\right| \in \left\{0, \begin{bmatrix} k \\ r \end{bmatrix}_q, \begin{bmatrix} k-1 \\ r \end{bmatrix}_q, \ldots, \begin{bmatrix} k-j \\ r \end{bmatrix}_q\right\}$ *for every* $A_{n-j} \subset A_n$.

Analogously we have

DEFINITION. *A subset*

$$S \subseteq \begin{Bmatrix} P_n \\ r \end{Bmatrix}$$

has property $P(n, q; k, r, j)$ if the following holds:

(i) $r \leqslant k \leqslant n$ *and* $1 \leqslant j \leqslant n-1$,

(ii) $|S| = \begin{Bmatrix} k \\ r \end{Bmatrix}_q$, (4)

(iii) $\left| S \cap \begin{Bmatrix} P_{n-j} \\ r \end{Bmatrix} \right| \in \left\{ \begin{Bmatrix} k \\ r \end{Bmatrix}_q, \ldots, \begin{Bmatrix} k-j \\ r \end{Bmatrix}_q \right\}$ *for euery* $P_{n-j} \subset P_n$.

Note that the set of r-subspaces of a k-subspace $A_k \subseteq A$, has property $A(n, q; k, r, j)$ for $1 \leqslant j \leqslant n-1$ (and the same holds in the projective case). The question rises whether other subsets S can have this property. If this is not the case we say that property $A(n, q; k, r, j)$ *characterizes* k-subspaces of A_n.

We recall the following facts from affine geometry (cf., for instance, [1]). Let A_l be an l-subspace of A_n. Then there is an affine $(n-1)$ subspace of A_n such that for appropriate choices of origin in A_n and A_{n-l} we have $A_n = A_l \oplus A_{n-l}$ (direct sum of vector spaces). If we have such a decomposition and a is a point of A_{n-l} and $A_m \subseteq A_l$ then we use $a \oplus A_m = A_m \oplus a$ to denote the m-subspace $\bigcup_{b \in A_m} (a+b)$. In the same way $A_k \oplus A_j$ is defined for $A_k \subseteq A_l$, $A_j \subseteq A_{n-l}$.

We also recall that each $A_l \subseteq A_n$ has precisely q^{n-l} disjoint translates and these are just the subspaces $A_m \oplus A_l$ where A_m runs through A_{n-l}.

We shall prove the following theorems:

THEOREM 1. *$A(n, q; k, 0, 1)$ characterizes k-subspaces of A_n.*

THEOREM 2. *If $r \geqslant 1$ then $A(n, q; k, r, 1)$ characterizes k-subspaces of A_n.*

THEOREM 3. *If $r \geqslant -1$ then $P(n, q; k, r, 1)$ characterizes k-subspaces of P_n.*

THEOREM 4. *If $j > 1$ and $(q,j) \neq (2, n-1)$ then $A(n, q; k, 0, j)$ characterizes k-subspaces of A_n.*

THEOREM 5. *If $j > 1$ then $P(n, q; k, 0, j)$ characterizes k-subspaces of P_n.*

For $r > 0$, $j > 1$ we have some partial results but no nice theorems. Exceptions and side conditions become necessary.

2. Proof of Theorem I

Let S be a subset of A, satisfying $A(n, q; k, 0, 1)$. We must show that S is a k-dimensional subspace, i.e.,

$$S = \begin{bmatrix} A_k \\ 0 \end{bmatrix} \text{ for some } A_k \subseteq A,.$$

We use induction on n. For $n = 2$ the theorem is trivial. Now assume $n > 2$ and that the result holds for $n - 1$. We may assume $1 \leqslant k < n$ since the theorem is trivial for $k = 0$ and for $k = n$. If, for some $A_{n-1} \subset A_n$,

$$| S \cap A_{n-1} | = \begin{bmatrix} k \\ 0 \end{bmatrix},$$

we are done by induction, since we can write $A, = A, \oplus A_{n-1}$ and then each $A_{n-2} \subset A_{n-1}$ must satisfy

$$A,-, \cap S \; I = |(A_{n-2} \oplus A_1) \cap S \; I \in \left\{0, \begin{bmatrix} k \\ 0 \end{bmatrix}, \begin{bmatrix} k-1 \\ 0 \end{bmatrix}\right\}.$$

Therefore it remains to consider the possibility that for each $A_{n-1} \subset A$, we have $S \cap A_{n-1} = 0$ or q^{k-1}. Now, if $A_{n-1} \cap S = 0$ for some A_{n-1}, then let $A_{n-1}^{(1)}, \ldots, A_{n-1}^{(q-1)}$ be the other $q - 1$ translates of A_{n-1} in A,. We would have

$$q^k = S = \sum_{i=1}^{q-1} S \cap A_{n-1}^{(i)} \leqslant (q-1)\, q^{k-1},$$

a contradiction. Hence we may assume that $S \cap A_{n-1} = q^{k-1}$ for every $A_{n-1} \subset A$,. We shall show that this leads to a contradiction.

Let χ_A denote the characteristic function of A. Then we have

$$\begin{aligned} \sum_{A_{n-1} \subset A_n} | S \cap A_{n-1} |^2 &= \sum_{A_{n-1} \subset A_n} \Big(\sum_{a \in S} \chi_{A_{n-1}}(a) \Big)^2 \\ &= \sum_{a \in S} \sum_{b \in S} \sum_{A_{n-1} \subset A_n} \chi_{A_{n-1}}(a) \; \chi_{A_{n-1}}(b) \\ &= | S | \frac{q^n - 1}{q - 1} + | S |(| S | - 1) \frac{q^{n-1} - 1}{q - 1}. \end{aligned}$$

Now, substituting $| S \cap A_{n-1} = q^{k-1}$ and $S = q^k$ we find

$$(q - 1)(q^{n-1} - q^{k-1}) = 0,$$

a contradiction.

3. PROOF OF THEOREM 2

Let $r \geqslant 1$ and let

$$S \subseteq \begin{bmatrix} A \\ r \end{bmatrix}$$

satisfy $A(n, q; k, r, 1)$. To prove that

$$S = \begin{bmatrix} A_k \\ r \end{bmatrix}$$

for some $A_k \subseteq A$, we use induction on n. The theorem is trivial for $k = r$. So we assume $n \geqslant k > r$, and that the result holds for $n - 1$. As in the proof of Theorem 1 we are finished if

$$S \cap \begin{bmatrix} A_{n-1} \\ r \end{bmatrix} \Big| = \begin{bmatrix} k \\ r \end{bmatrix} \text{ for some } A,\text{-}, .$$

So we may also assume that

$$\left| S \cap \begin{bmatrix} A_{n-1} \\ r \end{bmatrix} \right| = 0 \text{ or } \begin{bmatrix} k-1 \\ r \end{bmatrix} \text{ for all subspaces } A_{n-1}.$$

We shall show that this implies $n = k$, in which case the theorem is again trivial. We consider all $A,\text{-}$, in A, and let a be the number of these for which

$$\left| S \cap \begin{bmatrix} A_{n-1} \\ r \end{bmatrix} \right| = \begin{bmatrix} k-1 \\ r \end{bmatrix}$$

and b be the number of A_{n-1} for which

$$S \cap \begin{bmatrix} A_{n-1} \\ r \end{bmatrix} \Big| = 0.$$

Then

$$a + b = \begin{bmatrix} n \\ n-1 \end{bmatrix}, \qquad a \ 3 \ 0, \ b \geqslant 0. \tag{5}$$

We now count the number of pairs $(A,., A_{n-1})$ with

$$A_r \in S \cap \begin{bmatrix} A_{n-1} \\ r \end{bmatrix}.$$

On the one hand this number is $a \begin{bmatrix} k-1 \\ r \end{bmatrix}$ by definition of a. On the other hand S contains $\begin{bmatrix} k \\ r \end{bmatrix}$ subspaces A, each of which is contained in

$(q^{n-r}-1)/(q-1)$ hyperplanes A_{n-1}. Equating the two formulas we find

$$a\begin{bmatrix} k-1 \\ r \end{bmatrix} = \begin{bmatrix} k \\ r \end{bmatrix} \frac{q^{n-r}-1}{q-1}. \tag{6}$$

From (5) and (6) we get

$$a = \frac{q^{n-r}-1}{q-1} \cdot \frac{q^k-1}{q^{k-r}-1} q \leqslant \frac{q^n-1}{q-1} q. \tag{7}$$

Since $r > 0$ and $n \geqslant k$ the inequality (7) can only hold if $n = k$. This completes the proof.

4. Proof of Theorem 3

Let

$$S \subseteq \begin{Bmatrix} P_n \\ r \end{Bmatrix}$$

satisfy $P(n, q; k, r, 1)$. To prove that

$$S = \begin{Bmatrix} P_k \\ r \end{Bmatrix} \text{ for some } P_k \subseteq P_n$$

we again use induction on n. For $k = r$, $n = k$, or $n = 2$ the theorem is trivial. So assume $n > k > r$ and that the theorem holds for n 1. If

$$S \cap \begin{Bmatrix} P_{n-1} \\ r \end{Bmatrix} \Bigg| \text{ for some } P_{n-1} \subset P_n$$

we are done by induction. So we assume

$$\left| S \cap \begin{Bmatrix} P_{n-1} \\ r \end{Bmatrix} \right| = \begin{Bmatrix} k \\ r \end{Bmatrix} \text{ for every } P_{n-1} \subset P_n .$$

Each of the $\{{k \atop r}\}$ elements of S is in $\{{n-r-1 \atop n-r-2}\}$ hyperplanes of P_n. On the other hand, each of the $\{{n \atop n-1}\}$ hyperplanes contains $\{{k-1 \atop r}\}$ elements of S. Hence

$$\begin{Bmatrix} k-1 \\ r \end{Bmatrix}\begin{Bmatrix} n \\ n-1 \end{Bmatrix} = \begin{Bmatrix} k \\ r \end{Bmatrix}\begin{Bmatrix} n-r-1 \\ n-r-2 \end{Bmatrix},$$

i.e., $(q^n - q^k)(q - q^{-r}) = 0$. This can only hold if $n = k$ or $r = -1$ and in either case the theorem is trivial.

5. Proof of Theorem 4

Let S $\subseteq$ A, satisfy A $(n, q; k, 0, j)$ where j $\geqslant$ 1 and let $(q, j) \neq (2, n-1)$. We shall prove that S is a k-subspace. By Theorems 1 and 2 the theorem holds for j = 1 and hence for $n = 2$. So assume $n > 2$, j $\geqslant$ 2, and that the theorem holds for $n - 1$. We use induction on n. As in the previous proofs we have:

If S C A_{n-1} for some A_{n-1} C A,, , then we are finished. (8)

Hence, from now on we may assume S is not contained in any A_{n-1}. (8′)

As the next step we prove the following assertion:

For all A_{n-j-1} C A, if $|S \cap A_{n-j-1}| > q^{k-i}$ and $2 \leqslant i \leqslant j$, then $S \cap A_{n-j-1} \geqslant q^{k-i+1}$. (9)

To prove this assume $S \cap A_{n-j-1} = q^{k-i} + x$, $x > 0$, $2 \leqslant i \leqslant j$. Choose A_{j+1} in such a way that $A_{n-j-1} \oplus A_{j+1} = A$, . Consider all $(n-j)$-subspaces $A_{n-j-1} \oplus A$, with A, C A_{j+1}. There are $(q^{j+1}-1)/(q \quad 1)$ of these which contain A_{n-j-1}. Since these are $(n \quad j)$-subspaces they contain $q^{k-i+1}, q^{k-i+2}, \ldots, q^{k-1}$ or q^k points of S by property $A(n, q; k, 0, j)$. By (8') we may assume none contains q^k points of S. For $1 \leqslant m \leqslant i-1$ let x_m be the number of these $(n$ -j)-subspaces which contain q^{k-m} points of s.

We have

$$x_1 + x_2 + \cdots + x_{i-1} = (q^{j+1}-1)/(q-1). \tag{10}$$

Now we count the elements of S as follows. Each of the subspaces $A_{n-j-1} \oplus A$, counted by x_m has $(q^{k-m} - q^{k-i} - x)$ elements of S which are not in A_{n-j-1}. Since these $(n-j)$-subspaces pairwise have only A_{n-j-1} as intersection we get

$$\sum_{m=1}^{i-1} x_m(q^{k-m} - q^{k-i} - x) = q^k - q^{k-i} - x. \tag{11}$$

By considering equation (11) mod q^{k-i+1} and using (10) we find

$$xq(q^{j-1} + q^{j-2} + \cdots + 1) \equiv 0 \pmod{q^{k-i+1}},$$

i.e., $q^{k-i} + x = \lambda q^{k-i}$. We substitute this in (11) and then we get (again using (10)):

$$\lambda = \frac{\sum_{m=1}^{i-1} x_m q^{i-m} - q^i}{q^j + \cdots + q} \geqslant \frac{q(q^j + \cdots + 1) - q^i}{q^j + \cdots + q} = q - \frac{q^{i-1} - 1}{q^{j-1} + \cdots + 1}.$$

Since λ is an integer we find that $\lambda \geqslant q$, thus proving (9). (For a similar argument cf. [3, Lemma A. 1.31.)

If for some A_{n-j-1} we would have $S \cap A_{n-j-1} \geqslant q^{k-1}$, then any A_{n-j} containing A_{n-j-1} and any other point of S would contain all of S, contradicting (8').

Thus far we have established that in our proof we may use: For each $A_{n-j-1} C S$

$$S \cap A_{n-j-1} = \begin{cases} q^{k-i} & \text{with } 2 \leqslant i \leqslant j \quad \text{or} \\ < q^{k-j} \end{cases} \tag{12}$$

We now consider those A_{n-j-1}, if any, for which $S \cap A_{n-j-1} = q^{k-j+y}$ with $0 \leqslant y \leqslant j \quad 2$ and we choose one for which y is maximal. Again we write A, as $A_{n-j-1} \oplus A_{j+1}$. Since A_{n-j-1} does not contain all of S there is a point $a \in A_{j+1}$, not the origin of A_{j+1}, such that $S \cap (A_{n-j-1} \oplus a) \neq \varnothing$. Let A, be the line through the origin of A_{j+1} and the point a. Since $A_{n-j-1} \oplus A$, is the disjoint union of q (parallel) subspaces of dimension $n - j \quad 1$, and since y was maximal, we have

$$q^{k-j+y} < S \cap (A_{n-j-1} \oplus A_1)| \leqslant q^{k-j+y+1}. \tag{13}$$

By $A(q, n; k, 0, j)$ we must have equality on the right in (13) and then by the maximality of y and (12) we find that the following statement holds: All translates of A_{n-j-1} having any points of S must have exactly q^{k-j+y} points of S and furthermore, if A, is a line in A_{j+1} containing two points a and b for which $S \cap (S_{n-j-1} \oplus a)| = S \cap (A_{n-j-1} \oplus b)| = q^{k-j+y}$, then this equality holds for all points of A,. We refer to this statement as (14).

In order to complete the proof we must now consider the following cases:

Case I. $S \cap A_{n-j-1} < q^{k-j}$ for every $A_{n-j-1} C A$, .

Case IIa. (14) holds for some A_{n-j-1} specified by maximality of y, $q > 2$.

Case IIb. as IIa but with $q = 2$, $j \neq n - 1$.

We complete the proof as follows:

Case I. Since $S \cap A_{n-j-1} < q^{k-j}$ for every A_{n-j-1} it follows from $A(n, q; k, 0, j)$ that $S \cap A_{n-j} = 0$ or q^{k-j} for every A_{n-j} C A, . But for every A_{n-j} we can write A, $= A_{n-j} \oplus A_j$ which can be interpreted as splitting A, into q^j parallel $(n - j)$-spaces. This shows that

$S \cap A_{n-j} = q^{k-j}$ for every A_{n-j} and hence $S \cap A_{n-1} = q^{k-1}$ for every $A_{n-1} \subset A$, . Now the theorem follows from Theorem 1.

Case IIa. Now (14) holds and $q > 2$. Let T be the set of points t in A_{j+1} for which $S \cap (A_{n-j-1} \oplus t)| = q^{k-j+y}$. We saw above that $S \subset A_{n-j-1} \oplus T$ and that if a, b are two points of T then the line through a and b is in T. This implies that T is a subspace (of dimension $j - y$) in A_{j+1}. Hence S is contained in a subspace of dimension $n - 1$, contradicting (8').

Case IIb. Again (14) holds and now $q = 2$, $j \neq n - 1$. Choose any A_{n-j-2} in A_{n-j-1} and write $A, = A_{n-j-2} \oplus A_{j+2}$. Let a, b, c be three points in A_{j+2} such that $S \cap (A_{n-j-2} \oplus t)| > 0$ for t = a, b, c. Let d be the fourth point in the plane in A_{j+2} determined by a, b, c. Consider the three subspaces $(A_{n-j-2} \oplus t)$, t = a, b, c. Any two form an $(n - j - 1)$-subspace. If one of these is a translate of A_{n-j-1}, then the other one together with $A_{n-j-2} \oplus d$ also is. All four together must thus have $2^{k-j+y+1}$ points of S. Since any two form an $(n - j - 1)$-subspace, by maximality of y no two can have more than 2^{k-j+y} together. The only way this can occur is for all four to have exactly $2^{k-j+y-1}$.

On the other hand suppose no two of $A_{n-j-2} \oplus t$ where $t = a, b,$ c form a translate of A_{n-j-1}. In this case let a', b', c', d' be points such that $A_{n-j-2} \oplus t$ and $A_{n-j-2} \oplus t'$ form translates of A_{n-j-1}, $t = a, b, c, d$. Since those translates corresponding to t = a, b, and c have some points of S in them, all three have precisely 2^{k-j+y}. By the reasoning in the previous paragraph, this can only occur for each of the six spaces $A_{n-j-2} \oplus u$, $u = a, b, c, a'$ b' c' having exactly $2^{k-j+y-1}$ points of S in them. Now consider the four spaces $A_{n-j-2} \oplus u$, $u = a, b', c', d$. Now a, b', c', d form a plane in A_{j+2} and $A_{n-j-2} \oplus$ a and $A_{n-j-2} \oplus b'$ together form an $(n - j - 1)$-subspace with a maximal number 2^{k-j+y} of points of S. By (14) any translate of such a space has either no points of S or 2^{k-j+y} of them. Thus the translate consisting of $(A_{n-j-2} \oplus$ c') and $(A_{n-j-2} \oplus d)$ has 2^{k-j+y} points of S, and hence $A_{n-j-2} \oplus d$ has $2^{k-j+y-1}$ of them.

In either situation we saw that, if $a, b,$ c are three points of A_{j+2} with $S \cap (A_{n-j-2} \oplus t)| > 0$ for t = a, b, c, then $S \cap (A_{n-j-2} \oplus d)| > 0$ where d is the fourth point of the plane determined by $a, b,$ and c and furthermore all four intersections consist of $2^{k-j+y-1}$ points. It follows that, if T is the set of points t in A_{j+2} for which $S \cap (A_{n-j-2} \oplus t)| > 0$, then T is a subspace of dimension $j - y + 1$. Hence $S \subseteq A_{n-j-2} \oplus T$, a subspace of dimension $\leqslant n - 1$, contradicting (8'). Now the proof is complete.

We remark that Theorem 4 is false when $q = 2$ and $j = n - 1$. For then any set of 2^k points of A, satisfies $A(n, 2; k, 0, n - 1)$. The analogous problem does not arise, however, in the proof of Theorem 5, which is the projective analog of Theorem 4.

6. Proof of Theorem 5

In the proof of Theorem 5 we need some facts about projective and affine spaces (see [1], for example).

LEMMA. *Let P_{n-l} be an $(n-l)$-subspace of P_n. Then we can write $P_n = P_{n-l} \cup (A_{n-l+1} \oplus P_{l-1})$, a disjoint union, where A_{n-l+1} is an affine $(n-l+1)$-space, P_{l-1} aprojective $(l-1)$-space, and $\oplus$ indicates Cartesian product as sets. Further, let P_s be an s-subspace of P_n, and assume $P_s \cap P_{n-l} = P_i$, an i-subspace, $-1 \leqslant i \leqslant n-1$. Then P_s must be one of the s-subspaces $P_i \cup (A_{i+1} \oplus P_{s-i-1})$, where A_{i+1} is an affine $(i+1)$-subspace of A_{n-l+1}, and P_{s-i-1} is a projective $(s-i-1)$-subspace of P_{l-1}.*

We shall prove Theorem 5 by induction on n, the cases $n = 2$, $n = k$, $k = 0$ being trivial. Let $S \subset P_n$ satisfy $P(n, q; k, 0, j)$. We may assume $n > k > 0$ and that the theorem holds for $n-1$. We also assume $j \geqslant 2$ since $j = 1$ is the case covered by Theorem 3. We claim the following holds for every P_{n-j-1} in P_n:

$$\begin{aligned} &\text{Either } S \cap P_{n-j-1} = \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix} \text{ forsomey,} 0 \leqslant y \leqslant j, \\ &\text{or else } S \cap P_{n-j-1} I < \begin{Bmatrix} k-j \\ 0 \end{Bmatrix}. \end{aligned} \tag{15}$$

The proof of (15) is similar to our proof of (9). It is sufficient to show that, if

$$s \cap P_{n-j-1} > \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix},$$

then

$$|S \cap P_{n-j-1}| \geqslant \begin{Bmatrix} k-j+y+1 \\ 0 \end{Bmatrix}.$$

If $y = j-1$, this follows from $P(n, q; k, 0, j)$, since then every P_{n-j} containing P_{n-j-1} must contain all of S. So we may assume $y < j-1$.

Let

$$|S \cap P_{n-j-1}| = \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix} + x, \qquad x > 0, 0 \leqslant y < j-1.$$

Consider all P_{n-j} containing P_{n-j-1}. By $P(n, q; k, 0, j)$, these must have $S \cap P_{n-j}$ equal to one of $\{{k-j+y+1 \atop 0}\}, \ldots, \{{k \atop 0}\}$. Let x_m be the number of these

$(n-j)$-subspaces which contain $\left\{{k-j+y+m \atop 0}\right\}$ points of S, $1 \leqslant m \leqslant j-y$. Then we have

$$\sum_{m=1}^{j-y} x_m = \left\{{j \atop 0}\right\}. \tag{16}$$

By counting the elements of S in $P_{n-j} \backslash P_{n-j-1}$ for each P_{n-j} and adding, we get

$$\sum_{m=1}^{j-y} x_m \left[\left\{{k-j+y+m \atop 0}\right\} - \left\{{k-j+y \atop 0}\right\} - x\right]$$
$$= \left\{{k \atop 0}\right\} \quad \left\{{k-j+y \atop 0}\right\} - x. \tag{17}$$

By reducing (17) mod $q^{k-j+y+1}$ we find

$$-x \sum_{m=1}^{j-y} x_m \equiv -x \pmod{q^{k-j+y+1}},$$

and using (16) this yields

$$q(q^{j-1} + \cdots + 1)x \equiv 0 \pmod{q^{k-j+y+1}}.$$

Hence we can write $x = \lambda q^{k-j+y}$, $\lambda > 0$. By substituting this in (17), dividing by q^{k-j+y}, and using (16) we find

$$\sum_{m=1}^{j-y} x_m(q^m + q^{m-1} + \cdots + q) - (q^{j-y} + \cdots + q) = \lambda(q^j + \cdots + q).$$

Therefore

$$\lambda \geqslant \left[\sum_{m=1}^{j-y} x_m \quad (q^{j-y-1} + \cdots + 1)\right](q^{j-1} + \cdots + q)^{-1}$$
$$= q - \frac{q^{j-y-1} + \cdots + q}{q^{j-1} + \cdots + 1} > q - 1.$$

As λ is an integer, we get $\lambda \geqslant q$ and

$$|S \cap P_{n-j-1}| \geqslant \left\{{k-j+y \atop 0}\right\} + q^{k-j+y+1} = \left\{{k-j+y+1 \atop 0}\right\}.$$

This establishes (15).

As the next step we show that it is impossible that the second possibility in (15),

$$|S \cap P_{n-j-1}| < \begin{Bmatrix} k-j \\ 0 \end{Bmatrix},$$

holds for all P_{n-j-1} in P_n. Suppose, on the contrary, that this were true. Consider any P_{n-j}. Since each of the $\{{n-j \atop 0}\}$ $(n-j-1)$-subspaces of P_{n-j} has fewer than $\{{k-j \atop 0}\}$ points of S, and each point of S in P_{n-j} is in $\{{n-j-1 \atop 0}\}$ $(n-j-1)$-subspaces of P_{n-j}, we have

$$|S \cap P_{n-j}| < \frac{\begin{Bmatrix} n-j \\ 0 \end{Bmatrix}\begin{Bmatrix} k-j \\ 0 \end{Bmatrix}}{\begin{Bmatrix} n-j-1 \\ 0 \end{Bmatrix}} \leqslant \begin{Bmatrix} k-j+1 \\ 0 \end{Bmatrix}. \tag{18}$$

By $P(n, q; k, 0, j)$, this implies that now

$$|S \cap P_{n-j}| = \begin{Bmatrix} k-j \\ 0 \end{Bmatrix}$$

for every $(n-j)$-subspace of P_n. Each of the $\{{k \atop 0}\}$ points of S is in $\{{n-1 \atop n-j-1}\}$ $(n-j)$-subspaces of P_n. It follows that

$$\begin{Bmatrix} n \\ n-j \end{Bmatrix}\begin{Bmatrix} k-j \\ 0 \end{Bmatrix} = \begin{Bmatrix} k \\ 0 \end{Bmatrix}\begin{Bmatrix} n-1 \\ n-j-1 \end{Bmatrix},$$

i.e., $n = k$, a contradiction.

Thus, for some P_{n-j-1} in P_n and some y, $0 \leqslant y < j - 1$, we must have

$$|S \cap P_{n-j-1}| = \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix}.$$

Choose P'_{n-j-1} in P_n so that y is maximal. Consider a P_{n-j} in P_n. By the same reasoning as used for (18) we find

$$|S \cap P_{n-j}| < \frac{\begin{Bmatrix} n-j \\ 0 \end{Bmatrix}\begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix}}{\begin{Bmatrix} n-j-1 \\ 0 \end{Bmatrix}} \leqslant \begin{Bmatrix} k-j+y+1 \\ 0 \end{Bmatrix}. \tag{19}$$

We apply the argument used for (18) and (19) to all the $(n-j-2)$-

subspaces contained in P'_{n-j-1}. It follows that at least one of these, say P'_{n-j-2}, has

$$|S \cap P'_{n-j-2}| \geqslant \frac{\begin{Bmatrix} n-j-2 \\ 0 \end{Bmatrix}\begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix}}{\begin{Bmatrix} n-j-1 \\ 0 \end{Bmatrix}} \geqslant \begin{Bmatrix} k-j+y-1 \\ 0 \end{Bmatrix}. \quad (20)$$

At this point we use the lemma. Write

$$P_n = P'_{n-j-2} \cup (A_{n-j-1} \oplus P_{j+1}).$$

Let T be the set of points t in P_{j+1} such that $|S \cap (t \oplus A_{n-j-1})| > 0$. If

$$|S \cap P'_{n-j-2}| = \alpha > \begin{Bmatrix} k-j+y-1 \\ 0 \end{Bmatrix},$$

then, by maximality of y, $T = P_{j+1}$ and

$$|S \cap (t \oplus A_{n-j-1})| = \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix} - \alpha \text{ for every } t \in T.$$

This implies

$$\alpha + \begin{Bmatrix} j+1 \\ 0 \end{Bmatrix}\left[\begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix} - \alpha\right] = \begin{Bmatrix} k \\ 0 \end{Bmatrix},$$

i.e.,

$$\alpha = \begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix} - q^{k-j+y}\,\frac{q^{j-y}-1}{q^{j+1}-1},$$

which is not an integer! Hence

$$\alpha = \begin{Bmatrix} k-j+y-1 \\ 0 \end{Bmatrix}.$$

Again, by maximality of y, it follows that

$$|T| = \begin{Bmatrix} j-y \\ 0 \end{Bmatrix}.$$

If $a \in T$, $b \in T$, and P_1 is the line through a and b, then $P'_{n-j-2} \cup (P_1 \oplus A_{n-j-1})$ is an $(n-j)$-subspace which contains more than $\begin{Bmatrix} k-j+y \\ 0 \end{Bmatrix}$ points of S. By (19) this implies that this $(n-j)$-subspace contains exactly $\begin{Bmatrix} k-j+y+1 \\ 0 \end{Bmatrix}$ points of S which is possible only if every point of P_1

is in T. Hence T is a $(j - y)$-subspace of P_{j+1} and therefore $S \subseteq P'_{n-j-2} \cup (T \oplus A_{n-j-1})$, which by the lemma is a subspace of dimension $n - y - 1 \leqslant n - 1$. Now the theorem follows by induction.

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