Characterizing Finite Subspaces

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Communicated by A. Glenson

Received August 2, 1972

In this paper we study generalizations of the following question: Is a subspace of a projective or affine space characterized by the cardinalities of intersections with all hyperplanes? In several cases the answer is affirmative.

1. INTRODUCTION

A result which is due to Jessie MacWilliams [4] states that, if the set $S \subseteq \text{PG}(n, q)$, projective $n$-space over $\text{GF}(q)$, contains $(q^{k+1} - 1)/(q - 1)$ points and if $S$ has the property that every hyperplane contains either all of $S$ or $(q^{k} - 1)/(q - 1)$ points of $S$, then $S$ is a $k$-dimensional subspace. In this paper we consider some cases of the general question of characterizing $k$-dimensional subspaces of an $n$-dimensional space by the numbers of $r$-dimensional subspaces they have in common with each of the $(n - j)$-dimensional subspaces. MacWilliams’ result is the case $r = 0$, $j = 1$. We obtain similar results here for the cases $r = 0$ and $j$ arbitrary and $j = 1$, $r$ arbitrary. We consider this problem both for projective and affine spaces, using parallel, but not identical arguments, R. C. Bose and R. C. Burton [2] have results on a related problem.

NOTATION. Throughout the paper we shall use the following notation. If $q$ is fixed we denote by $\text{A}_n$ (respectively $\text{P}_n$) $n$-dimensional affine (projective) space over $\text{GF}(q)$, i.e., $\text{AG}(n, q)$ ($\text{PG}(n, q)$).

* Supported in part by NSF Grant GP-335804.

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\( [k] \) denotes the set of all \( k \)-dimensional subspaces of \( A_n \).

\( \{P_n\} \) denotes the set of all \( k \)-dimensional subspaces of \( P_n \).

\( \binom{n}{k}_q \) (respectively, \( \binom{n}{k} \)) denotes the number of such subspaces.

When no confusion arises we shall omit the subscript \( q \). We recall that

\[
\binom{n}{k}_q = \left\{ \begin{array}{ll}
\left( \frac{q^n}{(q^k - 1)(q - 1)} \cdots (q - 1) \right) q^{n-k}, & \text{if } n \geq k > 0, \\
q^n, & \text{if } n \geq k = 0, \\
0, & \text{otherwise},
\end{array} \right.
\]

and

\[
\binom{n}{k} = \left\{ \begin{array}{ll}
\left( \frac{q^{n+1}}{(q^{k+1} - 1)} \cdots (q^{n-k+1} - 1) \right), & \text{if } n \geq k > 0, \\
1, & \text{if } n \geq k = -1, \\
0, & \text{otherwise}.
\end{array} \right.
\]

We shall refer to \( k \)-dimensional spaces and subspaces simply as \( k \)-spaces and \( k \)-subspaces.

**Definition**. A subset

\[ S \subseteq [A_n^k] \]

has property \( A(n, q; k, r, j) \) if the following holds:

(i) \( r \leq k \leq n \) and \( 1 \leq j \leq n - 1 \).

(ii) \( 1 \leq j \leq n - 1 \).

(iii) \( S \cap [A_{n-j}^k] \subseteq \{0, [k]_q, [k-j]_q, \ldots, [k-j]_q\} \) for every \( A_{n-j} \subseteq A_n \).

Analogously we have

**Definition**. A subset

\[ S \subseteq \{P_n^k\} \]
has property \( P(n, q; k, r, j) \) if the following holds:

(i) \( r \leq k \leq n \) and \( 1 \leq j \leq n - 1 \),

(ii) \( |S| = \binom{k}{r} q \),

(iii) \( S \cap \left\{ \binom{P_{n-j}}{r} q \right\} \in \left\{ \binom{k}{r} q, \ldots, \binom{k-j}{r} q \right\} \) for every \( P_{n-j} \subseteq P_n \).

Note that the set of \( r \)-subspaces of a \( k \)-subspace \( A_k \subseteq A \), has property \( A(n, q; k, r, j) \) for \( 1 \leq j \leq n - 1 \) (and the same holds in the projective case). The question arises whether other subsets \( S \) can have this property. If this is not the case we say that property \( A(n, q; k, r, j) \) characterizes \( k \)-subspaces of \( A_n \).

We recall the following facts from affine geometry (cf., for instance, [1]). Let \( A_l \) be an \( l \)-subspace of \( A_n \). Then there is an affine \((n-1)\) subspace of \( A_n \) such that for appropriate choices of origin in \( A \), and \( A_{n-1} \) we have \( A = A_l \oplus A_{n-1} \) (direct sum of vector spaces). If we have such a decomposition and \( a \) is a point of \( A_{n-1} \) and \( A \subseteq A_l \) then we use \( a \oplus A, = A_l \oplus a \) to denote the \( m \)-subspace \( \bigcup_{x \in A_m} (a + x) \). In the same way \( A_k \oplus A_l \) is defined for \( A_k \subseteq A_l, A_l \subseteq A_{n-1} \).

We also recall that each \( A_l \subseteq A \), has precisely \( q^{n-l} \) disjoint translates and these are just the subspaces \( A, \oplus A_l \), where \( A \), runs through \( A_{n-1} \).

We shall prove the following theorems:

**Theorem 1.** \( A(n, q; k, 0, 1) \) characterizes \( k \)-subspaces of \( A_n \).

**Theorem 2.** If \( r \geq 1 \) then \( A(n, q; k, r, 1) \) characterizes \( k \)-subspaces of \( A_n \).

**Theorem 3.** If \( r \geq -1 \) then \( P(n, q; k, r, 1) \) characterizes \( k \)-subspaces of \( P_n \).

**Theorem 4.** If \( j > 1 \) and \( (q,j) \neq (2, n-1) \) then \( A(n, q; k, 0, j) \) characterizes \( k \)-subspaces of \( A_n \).

**Theorem 5.** If \( j > 1 \) then \( P(n, q; k, 0, j) \) characterizes \( k \)-subspaces of \( P_n \).

For \( r > 0, j > 1 \) we have some partial results but no nice theorems. Exceptions and side conditions become necessary.
2. Proof of Theorem I

Let $S$ be a subset of $A$, satisfying $A(n, q; k, 0, 1)$. We must show that $S$ is a $k$-dimensional subspace, i.e.,

$$S = \begin{bmatrix} A_k \\ 0 \end{bmatrix} \text{ for some } A_k \subseteq A.$$ 

We use induction on $n$. For $n = 2$ the theorem is trivial. Now assume $n > 2$ and that the result holds for $n = 1$. We may assume $1 \leq k < n$ since the theorem is trivial for $k = 0$ and for $k = n$. If, for some $A_{n-1} \subseteq A_n$,

$$|S \cap A_{n-1}| = \begin{bmatrix} k \\ 0 \end{bmatrix},$$

we are done by induction, since we can write $A_1 = A \oplus A_{n-1}$ and then each $A_{n-2} \subseteq A_{n-1}$ must satisfy

$$A_{n-2} \cap A_{n-1} = \{ (A_{n-2} \oplus A_1) \cap S \} I \in \{ 0, \begin{bmatrix} k \\ 0 \end{bmatrix}, \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \}.$$ 

Therefore it remains to consider the possibility that for each $A_{n-1} \subseteq A_n$, we have $S \cap A_{n-1} = 0$ or $q^{k-1}$. Now, if $A_{n-1} \cap S = 0$ for some $A_{n-1}$, then let $A_{n-1}^{(1)}, \ldots, A_{n-1}^{(q-1)}$ be the other $q - 1$ translates of $A_{n-1}$ in $A$. We would have

$$q^k = S = \sum_{i=1}^{q-1} S \cap A_{n-1}^{(i)} \leq (q - 1) q^{k-1},$$

a contradiction. Hence we may assume that $S \cap A_{n-1} = q^{k-1}$ for every $A_{n-1} \subseteq A_n$. We shall show that this leads to a contradiction.

Let $\chi_A$ denote the characteristic function of $A$. Then we have

$$\sum_{A_{n-1} \subseteq A_n} |S \cap A_{n-1}|^2 = \sum_{A_{n-1} \subseteq A_n} \left( \sum_{a \in S} \chi_{A_{n-1}}(a) \right)^2$$

$$= \sum_{a \in S} \sum_{b \in S} \sum_{A_{n-1} \subseteq A_n} \chi_{A_{n-1}}(a) \chi_{A_{n-1}}(b)$$

$$= |S| \frac{q^n - 1}{q - 1} + |S|(|S| - 1) \frac{q^{n-1} - 1}{q - 1}.$$ 

Now, substituting $|S \cap A_{n-1}| = q^{k-1}$ and $S = q^k$ we find

$$(q - 1)(q^{n-1} - q^{k-1}) = 0,$$

a contradiction.
3. **Proof of Theorem 2**

Let $r \geq 1$ and let

$$S \subseteq \begin{bmatrix} A_r \end{bmatrix}$$

satisfy $A(n, q; k, r, 1)$. To prove that

$$S = \begin{bmatrix} A_k \end{bmatrix}$$

for some $A_k \subseteq A$, we use induction on $n$. The theorem is trivial for $k = r$. So we assume $n \geq k > r$, and that the result holds for $n - 1$. As in the proof of Theorem 1 we are finished if

$$S \cap \begin{bmatrix} A_{n-1} \end{bmatrix} = \begin{bmatrix} k \end{bmatrix}$$

for some $A, \ldots$.

So we may also assume that

$$|S \cap \begin{bmatrix} A_{n-1} \end{bmatrix}| = 0 \text{ or } \begin{bmatrix} k - 1 \end{bmatrix} \text{ for all subspaces } A_{n-1}.$$

We shall show that this implies $n = k$, in which case the theorem is again trivial. We consider all $A, \ldots$ in $A$, and let $a$ be the number of these for which

$$|S \cap \begin{bmatrix} A_{n-1} \end{bmatrix}| = \begin{bmatrix} k - 1 \end{bmatrix}$$

and $b$ be the number of $A_{n-1}$ for which

$$S \cap \begin{bmatrix} A_{n-1} \end{bmatrix} = 0.$$

Then

$$a + b = \begin{bmatrix} n \end{bmatrix}, \quad a \geq 0, \ b \geq 0. \quad (5)$$

We now count the number of pairs $(A, n, A_{n-1})$ with

$$A_r \in S \cap \begin{bmatrix} A_{n-1} \end{bmatrix}.$$ 

On the one hand this number is $a \begin{bmatrix} k - 1 \end{bmatrix}$ by definition of $a$. On the other hand $S$ contains $\begin{bmatrix} k \end{bmatrix}$ subspaces $A$, each of which is contained in...
\( (q^{n-r} - 1)/(q - 1) \) hyperplanes \( A_{n-1} \). Equating the two formulas we find

\[
a \begin{bmatrix} k - 1 \\ r \end{bmatrix} = \begin{bmatrix} k \end{bmatrix} \frac{q^{n-r} - 1}{q - 1}, \tag{6}
\]

From (5) and (6) we get

\[
a = \frac{q^{n-r} - 1}{q - 1} \cdot \frac{q^k - 1}{q^{k-r} - 1} q \leq \frac{q^n - 1}{q - 1} q. \tag{7}
\]

Since \( r > 0 \) and \( n \geq k \) the inequality (7) can only hold if \( n = k \). This completes the proof.

4. **Proof of Theorem 3**

Let

\[
s \subseteq \binom{P_n}{r}
\]

satisfy \( P(n, q; k, r, 1) \). To prove that

\[
s = \binom{P_k}{r}
\]

for some \( P_k \subseteq P_n \)

we again use induction on \( n \). For \( k = r, n = k \), or \( n = 2 \) the theorem is trivial. So assume \( n > k > r \) and that the theorem holds for \( n - 1 \). If

\[
s \cap \binom{P_{n-1}}{r}
\]

we are done by induction. So we assume

\[
\left| s \cap \binom{P_{n-1}}{r} \right| = \binom{k}{r}
\]

for every \( P_{n-1} \subseteq P_n \).

Each of the \( \binom{k}{r} \) elements of \( s \) is in \( \binom{n-r-1}{r} \) hyperplanes of \( P_n \). On the other hand, each of the \( \binom{n}{n-1} \) hyperplanes contains \( \binom{k-1}{r} \) elements of \( s \). Hence

\[
\binom{k-1}{r} n \left| (n - 1) = \binom{k}{r} (n - r - 2) \right.,
\]

i.e., \( (q^n - q^k)(q = q^{-r}) = 0 \). This can only hold if \( n = k \) or \( r = -1 \) and in either case the theorem is trivial.
5. Proof of Theorem 4

Let \( S \subseteq A \), satisfy \( A(n, q; k, 0, j) \) where \( j \geq 1 \) and let \( (q, j) \neq (2, n - 1) \). We shall prove that \( S \) is a \( k \)-subspace. By Theorems 1 and 2 the theorem holds for \( j = 1 \) and hence for \( n = 2 \). So assume \( n > 2 \), \( j \geq 2 \), and that the theorem holds for \( n - 1 \). We use induction on \( n \). As in the previous proofs we have:

If \( S \subseteq A_{n-1} \) for some \( A_{n-1} \subseteq A \), then we are finished. \((8)\)

Hence, from now on we may assume \( S \) is not contained in any \( A_{n-1} \). \((8')\)

As the next step we prove the following assertion:

For all \( A_{n-j-1} \subseteq A \), if \( |S \cap A_{n-j-1}| > q^{k-i} \) and \( 2 \leq i \leq j \), then \( S \cap A_{n-j-1} \supseteq q^{k-i+1} \). \((9)\)

To prove this assume \( S \cap A_{n-j-1} = q^{k-i} + x \), \( x > 0 \), \( 2 \leq i \leq j \). Choose \( A_{i+1} \) in such a way that \( A_{n-j-1} \oplus A_{i+1} = A \). Consider all \((n-j)\)-subspaces \( A_{n-j-1} \oplus A \), with \( A, C A_{j+1} \). There are \( (q^{i+1}-1)/(q-1) \) of these which contain \( A_{n-j-1} \). Since these are \((n-j)\)-subspaces they contain \( q^{k-i+1}, q^{k-i+2}, \ldots, q^{k-1} \) or \( q^k \) points of \( S \) by property \( A(n, q; k, 0, j) \). By (8’) we may assume none contains \( q^k \) points of \( S \). For \( 1 \leq m \leq i - 1 \) let \( x_m \) be the number of these \((n-j)\)-subspaces which contain \( q^k \) points of \( S \).

We have

\[
x_1 + x_2 + \cdots + x_{i-1} = (q^{i+1} - 1)/(q - 1).
\]

Now we count the elements of \( S \) as follows. Each of the subspaces \( A_{n-j-1} \oplus A \), counted by \( x_m \) has \( (q^{k-m} - q^{k-i} - x) \) elements of \( S \) which are not in \( A_{n-j-1} \). Since these \((n-j)\)-subspaces pairwise have only \( A_{n-j-1} \) as intersection we get

\[
\sum_{m=1}^{i-1} x_m(q^{k-m} - q^{k-i} - x) = q^k - q^{k-i} - x.
\]

By considering equation (11) mod \( q^{k-i+1} \) and using (10) we find

\[
xq(q^{l+1} + q^{l-2} + \cdots + 1) \equiv 0 \pmod{q^{k-i+1}},
\]

i.e., \( q^{k-i} + x = \lambda q^{k-i} \). We substitute this in (11) and then we get (again using (10)):

\[
\lambda = \frac{\sum_{m=1}^{l-1} x_m q^{l-m} - q^l}{q^{l+1} + \cdots + q} \geq \frac{q(q^{l+1} + \cdots + 1) - q^l}{q^{l+1} + \cdots + q} = q - \frac{q^{l-1} - 1}{q^{l+1} + \cdots + 1}.
\]
Since \( \lambda \) is an integer we find that \( \lambda \geq q \), thus proving (9). (For a similar argument cf. [3, Lemma A. 1.31.])

If for some \( A_{n-j-1} \) we would have \( S \cap A_{n-j-1} \geq q^{k-1} \), then any \( A_{n-j} \) containing \( A_{n-j-1} \) and any other point of \( S \) would contain all of \( S \), contradicting \((8')\).

Thus far we have established that in our proof we may use: For each \( A_{n-j-1} \subset S \)

\[
S \cap A_{n-j-1} = \begin{cases} q^{k-i} & \text{with } 2 \leq i \leq j \\ < q^{k-j} & \end{cases}
\]  
(12)

We now consider those \( A_{n-j-1} \), if any, for which \( S \cap A_{n-j-1} = q^{k-j+y} \)
with \( 0 \leq y \leq j \) and we choose one for which \( y \) is maximal. Again we write \( A \) as \( A_{n-j-1} \oplus A_{j+1} \). Since \( A_{n-j-1} \) does not contain all of \( S \) there is a point \( a \in A_{j+1} \), not the origin of \( A_{j+1} \), such that \( S \cap (A_{n-j-1} \oplus a) \neq \emptyset \).

Let \( A \) be the line through the origin of \( A_{j+1} \) and the point \( a \). Since \( A_{n-j-1} \oplus A \) is the disjoint union of \( q \) (parallel) subspaces of dimension \( n-j \), and since \( y \) was maximal, we have

\[
q^{k-j+y} < S \cap (A_{n-j-1} \oplus A) \leq q^{k-j+y+1}.
\]  
(13)

By \( A(q, n; k, 0, j) \) we must have equality on the right in (13) and then by the maximality of \( y \) and (12) we find that the following statement holds: All translates of \( A_{n-j-1} \) having any points of \( S \) must have exactly \( q^{k-j+y} \)
points of \( S \) and furthermore, if \( A \) is a line in \( A_{j+1} \) containing two points \( a \) and \( b \) for which \( S \cap (A_{n-j-1} \oplus a) = S \cap (A_{n-j-1} \oplus b) = q^{k-j+y} \), then this equality holds for all points of \( A \). We refer to this statement as (14).

In order to complete the proof we must now consider the following cases:

**Case I.** \( S \cap A_{n-j-1} < q^{k-j} \) for every \( A_{n-j-1} \subset S \).

**Case IIa.** (14) holds for some \( A_{n-j-1} \) specified by maximality of \( y, q > 2 \).

**Case IIb.** as IIa but with \( q = 2, j \neq n-1 \).

We complete the proof as follows:

**Case I.** Since \( S \cap A_{n-j-1} < q^{k-j} \) for every \( A_{n-j-1} \) it follows from \( A(n, q; k, 0, j) \) that \( S \cap A_{n-j} = 0 \) or \( q^{k-j} \) for every \( A_{n-j} \subset S \).

But for every \( A_{n-j} \) we can write \( A_{n-j} \oplus A_{j} \) which can be interpreted as splitting \( A_{n-j} \) into \( q^j \) parallel \((n-j)\)-spaces. This shows that
S \cap A_{n-j} = q^{k-j}$ for every $A_{n-j}$ and hence $S \cap A_{n-1} = q^{k-1}$ for every $A_{n-1}$ C $A$. Now the theorem follows from Theorem 1.

**Case IIa.** Now (14) holds and $q > 2$. Let $T$ be the set of points $t$ in $A_{j+1}$ for which $S \cap (A_{n-j-1} \oplus t) = q^{k-j+y}$. We saw above that $S \subset A_{n-1} \oplus T$ and that if $a, b$ are two points of $T$ then the line through $a$ and $b$ is in $T$. This implies that $T$ is a subspace (of dimension $\neg y$) in $A_{j+1}$. Hence $S$ is contained in a subspace of dimension $n \neg 1$, contradicting ($8'$).

**Case IIb.** Again (14) holds and now $q = 2$, $j \neq n - 1$. Choose any $A_{n-j-2}$ in $A_{n-j-1}$ and write $A_0 = A_{n-j-2} \oplus A_{j+2}$. Let $a, b, c$ be three points in $A_{j+2}$ such that $S \cap (A_{n-j-2} \oplus t) > 0$ for $t = a, b, c$. Let $d$ be the fourth point in the plane in $A_{j+2}$ determined by $a, b, c$. Consider the three subspaces $(A_{n-j-2} \oplus t)$, $t = a, b, c$. Any two form an $(n - j - 1)$-subspace. If one of these is a translate of $A_{n-j-1}$, then the other one together with $A_{n-j-2} \oplus d$ also is. All four together must thus have $2k^{j+y+1}$ points of $S$. Since any two form an $(n - j - 1)$-subspace, by maximality of $y$ no two can have more than $2k^{j+y}$ together. The only way this can occur is for all four to have exactly $2k^{j+y-1}$.

On the other hand suppose no two of $A_{n-j-2} \oplus t$ where $t = a, b, c$ form a translate of $A_{n-j-1}$. In this case let $a', b', c', d'$ be points such that $A_{n-j-2} \oplus t$ and $A_{n-j-2} \oplus t'$ form translates of $A_{n-j-1}$, $t = a, b, c, d$. Since those translates corresponding to $t = a, b, c$ and $d$ have some points of $S$ in them, all three have precisely $2k^{j+y}$. By the reasoning in the previous paragraph, this can only occur for each of the six spaces $A_{n-j-2} \oplus u$, $u = a, b, c, d$, having exactly $2k^{j+y-1}$ points of $S$ in them. Now consider the four spaces $A_{n-j-2} \oplus u, u = a, b', c', d$. Now $a, b', c', d$ form a plane in $A_{j+2}$ and $A_{n-j-2} \oplus a$ and $A_{n-j-2} \oplus b'$ together form an $(n - j - 1)$-subspace with a maximal number $2k^{j+y}$ of points of $S$. By (14) any translate of such a space has either no points of $S$ or $2k^{j+y}$ of them. Thus the translate consisting of $(A_{n-j-2} \oplus c')$ and $(A_{n-j-2} \oplus d)$ has $2k^{j+y}$ points of $S$, and hence $A_{n-j-2} \oplus d$ has $2k^{j+y-1}$ of them.

In either situation we saw that, if $a, b, c$ are three points of $A_{j+2}$ with $S \cap (A_{n-j-2} \oplus t) > 0$ for $t = a, b, c$, then $S \cap (A_{n-j-2} \oplus d) > 0$ where $d$ is the fourth point of the plane determined by $a, b, c$ and furthermore all four intersections consist of $2k^{j+y-1}$ points. It follows that, if $T$ is the set of points $t$ in $A_{j+2}$ for which $S \cap (A_{n-j-2} \oplus t) > 0$, then $T$ is a subspace of dimension $\neg y + 1$. Hence $S \subset A_{n-j-2} \oplus T$, a subspace of dimension $\leq n \neg 1$, contradicting ($8'$). Now the proof is complete.

We remark that Theorem 4 is false when $q = 2$ and $j \neq n - 1$. For then any set of $2k$ points of $A$, satisfies $A(n, 2; k, 0, n \neg 1)$. The analogous problem does not arise, however, in the proof of Theorem 5, which is the projective analog of Theorem 4.
6. Proof of Theorem 5

In the proof of Theorem 5 we need some facts about projective and affine spaces (see [1], for example).

**Lemma.** Let $P_{n-1}$ be an $(n - 1)$-subspace of $P_n$. Then we can write $P_n = P_{n-1} \cup (A_{n-1} \oplus P_{l-1})$, a disjoint union, where $A_{n-1}$ is an affine $(n - 1)$-space, $P_{l-1}$ a projective $(l - 1)$-space, and $\oplus$ indicates Cartesian product as sets. Further, let $P_s$ be an $s$-subspace of $P_n$, and assume $P_s \cap P_{n-1} = P_i$, an $i$-subspace, $1 \leq i \leq n - 1$. Then $P_s$ must be one of the $s$-subspaces $P_i \cup (A_{m} \oplus P_{s-i-1})$, where $A_{i+1}$ is an affine $(i + 1)$-subspace of $A_{n-1}$, and $P_{s-i-1}$ is a projective $(s - i - 1)$-subspace of $P_{i-1}$.

We shall prove Theorem 5 by induction on $n$, the cases $n = 2, n = k$, $k = 0$ being trivial. Let $S \subseteq P_n$ satisfy $P(n, q; k, 0, j)$. We may assume $n > k > 0$ and that the theorem holds for $n - 1$. We also assume $j \geq 2$ since $j = 1$ is the case covered by Theorem 3. We claim the following holds for every $P_{n-j-1}$ in $P_n$:

Either $S \cap P_{n-j-1} = \{k - j + y\}$ for some $0 \leq y \leq j$, (15)

or else $S \cap P_{n-j-1} \subset \{k \cdot j\}$.

The proof of (15) is similar to our proof of (9). It is sufficient to show that, if

$$S \cap P_{n-j-1} > \{k - j + y\},$$

then

$$|S \cap P_{n-j-1}| \geq \{k - j + y + 1\}.$$ 

If $y = j - 1$, this follows from $P(n, q; k, 0, j)$, since then every $P_{n-j}$ containing $P_{n-j-1}$ must contain all of $S$. So we may assume $y < j - 1$.

Let

$$|S \cap P_{n-j-1}| = \{k - j + y\} + x, \quad x > 0, 0 \leq y < j - 1.$$

Consider all $P_{n-j}$ containing $P_{n-j-1}$. By $P(n, q; k, 0, j)$, these must have $S \cap P_{n-j}$ equal to one of $\{k-j+y+2\}, \ldots, \{k\}$. Let $x_m$ be the number of these
(n - j)-subspaces which contain \( \{k-j+y+m\} \) points of \( S \), \( 1 \leq m \leq j - y \). Then we have

\[
\sum_{m=1}^{3^v} x_m = \binom{j}{0}.
\]

(16)

By counting the elements of \( S \) in \( P_{n-j} \setminus P_{n-j-1} \) for each \( P_{n-j} \) and adding, we get

\[
\sum_{m=1}^{j-y} x_m \left[ \binom{k-j+y+m}{0} - \binom{k-j+y}{0} - x \right]
\]

\[
= \binom{k}{0} - \binom{k-j+y}{0} - x.
\]

(17)

By reducing (17) mod \( q^{k-j+y+1} \) we find

\[-x \sum_{m=1}^{j-y} x_m \equiv -x \pmod{q^{k-j+y+1}},\]

and using (16) this yields

\[q(q^{i-1} + \cdots + 1)x \equiv 0 \pmod{q^{k-j+y+1}}.\]

Hence we can write \( x = \lambda q^{k-j+y}, \lambda > 0 \). By substituting this in (17), dividing by \( q^{k-j+y} \), and using (16) we find

\[
\sum_{m=1}^{j-v} x_m(q^m + q^{m-1} + \cdots + q) = (q^{i-v} + \cdots + q) = \lambda(q^i + \cdots + q).
\]

Therefore

\[
\lambda \geq \left[ \sum_{m=1}^{j-v} x_m \right] (q^{i-v-1} + \cdots + 1) (q^{i-1} + \cdots + q)^{-1}
\]

\[
= q - \frac{q^{i-v-1} + \cdots + q}{q^{i-1} + \cdots + 1} > q - 1.
\]

As \( \lambda \) is an integer, we get \( \lambda \geq q \) and

\[
|S \cap P_{n-j-1}| \geq \binom{k-j+y}{0} + q^{k-j+y+1} = \binom{k-j+y+1}{0}.
\]

This establishes (15).
As the next step we show that it is impossible that the second possibility in (15),

\[ |S \cap P_{n-j-1}| < \binom{k-j}{0}, \]

holds for all \( P_{n-j-1} \) in \( P_n \). Suppose, on the contrary, that this were true. Consider any \( P_{n-j} \). Since each of the \( \binom{n-j}{k-j} \)-subspaces of \( P_{n-j} \) has fewer than \( \binom{k-j}{n-j-1} \) points of \( S \), and each point of \( S \) in \( P_{n-j} \) is in \( \binom{n-j-1}{0} \) \( (n-j-1) \)-subspaces of \( P_{n-j} \), we have

\[ |S \cap P_{n-j}| < \binom{n-j}{0} \binom{k-j}{0} \binom{n-j-1}{0} \leq \binom{k-j+1}{0}. \]  

By \( P(n, q; k, 0, j) \), this implies that now

\[ |S \cap P_{n-j}| = \binom{k-j}{0} \]

for every \( (n-j) \)-subspace of \( P_n \). Each of the \( \binom{k}{0} \) points of \( S \) is in \( \binom{n-j-1}{0} \) \( (n-j) \)-subspaces of \( P_n \). It follows that

\[ \binom{n}{k-j} \binom{n-j}{0} = \binom{k}{n-1} \binom{n-1}{0} \binom{n-j-1}{0}, \]

i.e., \( n = k \), a contradiction.

Thus, for some \( P_{n-j-1} \) in \( P_n \) and some \( y, 0 \leq y < j - 1 \), we must have

\[ |S \cap P_{n-j-1}| = \binom{k-j+y}{0}. \]

Choose \( P'_{n-j-1} \) in \( P_n \) so that \( y \) is maximal. Consider a \( P_{n-j} \) in \( P_n \). By the same reasoning as used for (18) we find

\[ |S \cap P_{n-j}| < \binom{n-j}{0} \binom{k-j+y}{0} \binom{n-j-1}{0} \leq \binom{k-j+y+1}{0}. \]  

We apply the argument used for (18) and (19) to all the \( (n-j-2)\)-
subspaces contained in $P'_{n-j-1}$. It follows that at least one of these, say $P'_{n-j-2}$, has
\[
| S \cap P'_{n-j-2} | = \frac{(n-j-2)(k-j+y)}{\begin{vmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{vmatrix}} \geq \{ k-j+y-1 \}.
\] (20)

At this point we use the lemma. Write
\[ P_n = P'_{n-j-2} \cup (A_{n-j-1} \oplus P_{j+1}). \]
Let $T$ be the set of points $t$ in $P_{j+1}$ such that $S \cap \{ t \oplus A_{n-j-1} \} > 0$. If
\[ | S \cap P'_{n-j-2} | = \alpha > \{ k-j+y-1 \}, \]
then, by maximality of $\alpha$, $T = P_{j+1}$ and
\[ S \cap (t \oplus A_{n-j-1}) = \{ k-j+y \} \ominus \alpha \text{ for every } t \in T. \]
This implies
\[ \alpha + \{ j + 1 \} \ominus \{ k-j+y \} \ominus \alpha = \{ k \}, \]
i.e.,
\[ \alpha = \{ k-j+y \} - q^{k-j+y} \frac{q^{j+y} - 1}{q^{j+1} - 1}, \]
which is not an integer! Hence
\[ \alpha = \{ k-j+y-1 \}. \]

Again, by maximality of $\alpha$, it follows that
\[ | T | = \{ j-y \}. \]

If $a \in T$, $b \in T$, and $P_1$ is the line through $a$ and $b$, then $P'_{n-j-2} \cup (P_1 \oplus A_{n-j-1})$ is an $(n-j)$-subspace which contains more than $\{ k-j+y \}$ points of $S$. By (19) this implies that this $(n-j)$-subspace contains exactly $\{ k-j+y+1 \}$ points of $S$ which is possible only if every point of $P_1$
is in $T$. Hence $T$ is a $(j - y)$-subspace of $P_{j+1}$ and therefore $S \subseteq P_{n-j-2} \cup (T \oplus A_{n-j-1})$, which by the lemma is a subspace of dimension $n - y - 1 \leq n - 1$. Now the theorem follows by induction.

References