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ASPHERICAL 2-COMPLEXES

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The purpose of this paper is to study 2-complexes, to prove a certain structure theorem for them, and to present a new proof of some results on J.H.C. Whitehead's conjecture [7] that a subcomplex of an aspherical 2-complex is itself aspherical. (Aspherical means that all higher homotopy groups vanish.) Our constructive result on the Whitehead conjecture is stronger than Cockcroft's [2] but not quite as strong as that of Adams [1] which will be presented here in a slightly different form, Theorems 1 and 2. The second, slightly weaker, version has a very different proof, depending on the structure theorem for 2-complexes and is presented in Section 4. We also prove in Section 5 a special case of Whitehead's conjecture which does not follow from [1] or [2]. In proving the structure theorem (Theorem 4) we shall present a proof in Section 2 (due to R.G. Swan) of a very useful theorem concerning free groups and we propose some conjectures on aspherical 2-complexes.

First we need to define some conditions on groups. Note that the definitions contain within them statements of potential theorems.

Definition 1. A group π satisfies *condition* (A) if either π^* is not free abelian or if the following always holds: if X is a 2-complex with $\pi_1 X \cong \pi$ and $H_2 X = 0$ then X is aspherical.

Definition 2. Let $\mathcal{P}$ be the class of CW complexes X with $H_2 X = 0$ and $H_1 X$ torsion-free. A group π is *conservative* if the following always holds: Let π act on the 2-complex X. If $X/\pi \in \mathcal{P}$ then $X \in \mathcal{P}$.

Definition 3. For a group π, let $Z\pi$ be the integral group-ring, $I = (\pi - 1)Z\pi$, the augmentation ideal. π satisfies the Nakayama Condition (NC) if given a left ideal J with $IJ = J$, then $J = 0$.

Definition 4. A group π is *transfinite metabelian* (TM) if it contains no perfect subgroup except the identity.

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Observe that condition (A) and "conservative" are not a priori checkable. NC is a bit more easy to see and TM can be checked reasonably easily.

The results on the Whitehead conjecture are the following:

Theorem 1. *Let K be an aspherical 2-complex, $L \subset K$ obtained by removing 2-cells from K. Assume* $\ker(\pi_1 L \to \pi_1 K)$ *satisfies condition* (A). *Then L is aspherical.*

Theorem 1 is very simple. The interest comes in locating groups satisfying condition (A).

Theorem 2. *If π is* TM *then π satisfies condition* (A).

Note. Since subgroups of TM groups are TM, we now have Whitehead's Conjecture holding whenever $\pi_1 L$ is TM. This takes care of the free and abelian cases of [2]. The one-relator case is in Section 5.

Although not explicitly stated, this result is implicitly proved by Adams. We shall prove it explicitly. NC is a stronger condition than TM: if $\sigma \subset \pi$ is perfect, then $J = (\sigma - 1)Z\pi$ has the property that $IJ = J$. Thus if NC holds $J = 0$ so $\sigma = \{1\}$ and TM holds. Thus

Corollary. *If π satisfies* NC *then π satisfies condition* (A).

We shall, however, give in Section 4 a completely different proof of that fact using a structure theorem for 2-complexes.

We also propose the possibility of a proof of the Whitehead conjecture along the following lines:

Conjecture 1. *If π is torsion-free then π satisfies condition* (A).

Conjecture 2. *If K is a contractible 2-complex and $L \subset K$, then $\pi_1(L)$ is torsion-free.*

These, together with Theorem 1, offer a complete proof of the Whitehead conjecture:

Let K be a contractible 2-complex and $L \subset K$ a subcomplex. Let $L' = L \cup K^1$ where K^1 is the 1-skeleton of K. By the conjectures, $\pi_1(L')$ satisfies condition (A). Since $\pi_1(L') = \ker(\pi_1 L' \to \pi_1 K)$ and $K - L' =$ 2-cells, by Theorem 1, L' is aspherical. But L' and L differ only by 1-cells hence L is aspherical. For the general case of K aspherical, lift the problem to the (contractible) universal cover as in Section 1.

Condition (A) cannot hold for arbitrary groups: Let M be Poincaré's homology 3-sphere S^3/D where S^3 is the ordinary 3-sphere and D the binary icosahedral group of order 120. Let X be the 2-skeleton of M. Thus $X = M - e^3$, e^3 a 3-cell. Since $\tilde{M}$, the universal cover of M, is S^3, $\tilde{X} = S^3 - \bigcup_{i=1}^{120} e_i^3$, e_i^3 being disjoint 3-cells.

Thus $H_2(\tilde{X}) \cong Z^{119}$ so $\tilde{X}$ is not contractible, whence X is not aspherical. But $H_2X = H_1X = 0$. This example was pointed out by William Beckmann.

This shows that D does not satisfy condition (A), of course. But $\pi = D \oplus Z_2$ does trivially since $\pi^a \cong Z_2$.

1. Proof of Theorems 1 and 2

First for Theorem 1.

Let $p: \tilde{K} \to K$ be the universal cover of K. Let $L' = p^{-1}(L)$. Since $K - L$ is 2-cells, $\pi_1 L \to \pi_1 K$ is onto so L' is a connected cover of L and corresponds to the inclusion $\pi \lhd \pi_1 L$ where $\pi = \ker(\pi_1 L \to \pi_1 K)$. Now $\tilde{K} - L'$ is a union of 2-cells doubly indexed by $\pi_1 K$ and the cells of $K - L$. Thus $H_i(\tilde{K}, L') = 0$ for $i \neq 2$ and $H_2(\tilde{K}, L')$ is a free $Z[\pi_1 K]$-module on the cells of $K - L$. Now $\tilde{K}$ is contractible since K is aspherical, thus from the long exact homology sequence of $(\tilde{K}, L')$ we see that $H_2 L' = 0$ and $\pi^a \cong H_1 L' \cong H_2(\tilde{K}, L')$ is free abelian. Since $\pi_1 L' = \pi$ satisfies condition (A), L' is aspherical. Since L' covers L, L is aspherical. This completes the proof of Theorem 1.

We now prove Theorem 2 using the following facts proved by Adams [1]:

(A1) Torsion-free abelian groups are conservative.

(A2) An inverse limit of conservative groups is conservative.

(A3) An extension of a conservative group by a conservative group is conservative.

So now let X be a 2-complex with $\pi = \pi_1 X$ a TM group. Assume $H_1 X$ is free abelian and $H_2 X = 0$. Let $\tilde{X}$ be the universal cover of X. If we show that π is conservative then $H_2\tilde{X} = 0$. But since $\tilde{X}$ is a simply-connected 2-complex, this implies $\tilde{X}$ is contractible, whence X is aspherical and we are done.

Let Γ be the set of all normal subgroups $\sigma \lhd \pi$ such that π/σ is conservative. Let $\tau = \bigcap_{\sigma \in \Gamma} \sigma$. $\pi/\tau = \varprojlim \pi/\sigma$ so by (A2) π/τ is conservative. So it will suffice to show that τ is trivial. Since π is TM it will suffice to show that τ is perfect; i.e. that $\tau = \tau' = [\tau, \tau]$. Thus we are done if we can show that $\tau' \in \Gamma$. But by the exactness of $1 \to \tau/\tau' \to \pi/\tau' \to \pi/\tau \to 1$, (A1) and (A3) complete the proof if we prove τ/τ' is torsion-free.

Let $X' \to X$ be the cover corresponding to $\tau \lhd \pi$. Then π/τ acts on X' with quotient X. But π/τ is conservative and $X \in \mathcal{P}$. Thus $X' \in \mathcal{P}$. So $H_1 X'$ is torsion-free. But since $\pi_1 X' = \tau$, $H_1 X' = \tau/\tau'$. Thus Theorem 2 is complete.

Actually we get a bit more from the proof. Let $X \in \mathcal{P}$ with $\pi = \pi_1 X$. Let $\tau = \bigcap_{\sigma \in \Gamma} \sigma$ with Γ as above. Then corresponding to $\tau \lhd \pi$ we have a normal cover $X' \to X$ with $X' \in \mathcal{P}$. As above, τ is perfect. Thus $H_1 X' = 0$ as well as $H_2 X' = 0$. Thus every complex in $\mathcal{P}$ is covered by an acyclic complex. If we could prove that for torsion-free fundamental group, acyclic implies aspherical, we would have a proof of Conjecture 1. Thus

Proposition. *Conjecture* 1 *is implied by the following: if X is an acyclic* 2*-complex and $\pi_1 X$ is torsion-free then X is aspherical.*

Equivalent to this last statement is this: Torsion-free perfect groups satisfy condition (A).

2. Automorphisms of free groups

We present here a proof due to R.G. Swan of a result conjectured in the preliminary version of this paper. It concerns the automorphisms of a free group F and its abelianization F^a:

Lifting Theorem. *The abelianization map* $\operatorname{Aut} F \to \operatorname{Aut} F^a$ *is onto.*

Remarks. 1) This is the same as saying that any basis for F^a can be lifted to a basis for F.

2) The result is well-known in the finitely generated case [5, p. 145].

3) Throughout this discussion we will let $y, y_\alpha \in F^a$ represent the image of $x, x_\alpha \in F$.

Proof. We call $\phi \in \operatorname{Aut} F^a$ *triangular* if there is a well-ordered basis $\{x_\alpha\}$ for F such that if $\phi(y_\alpha) = \sum a_{\alpha\beta} y_\beta$ then $a_{\alpha\beta} = 0$ for $\beta > \alpha$ and $a_{\alpha\alpha} = 1$. Assume ϕ is triangular. Then define $f : F \to F$ by $f(x_\alpha) = \pi x_\beta^{a_{\alpha\beta}}$ with the product taken in the same order, so that $f(x_\alpha) = x_\alpha \prod_{\beta<\alpha} x_\beta^{a_{\alpha\beta}}$. Clearly f is an automorphism and $f^a = \phi$. Thus

Lemma 1. *If ϕ is triangular ϕ can be lifted.*

Assume we have proved the result for countably generated free groups. Let F be free on an arbitrary basis $\{x_\alpha\}_A$ and let $\phi(y_\alpha) = \sum a_{\alpha\beta} y_\beta$ be an arbitrary automorphism. Assume $\phi^{-1}(y_\alpha) = \sum a'_{\gamma\beta} y_\beta$. For each α, let $M_\alpha = \{\beta \mid a_{\alpha\beta} \neq 0 \text{ or } a'_{\alpha\beta} \neq 0\}$. Then let $M_\alpha^0 = M_\alpha$, $M_\alpha^{k+1} = \bigcup_{\beta \in M_\alpha^k} M_\beta$, and $S_\alpha = \bigcup_{k=0}^{\infty} M_\alpha^k$. Since M_α is finite, S_α is countable. We now choose any $\alpha_1 \in A$ and let $A_1 = S_{\alpha_1}$. Inductively if we have $A_\lambda \subsetneqq A$ let $A_{\lambda+1} = S_\alpha - A_\lambda$ where $\alpha \in A - A_\lambda$. If λ is a limit ordinal and $\bigcup_{\mu<\lambda} A_\mu \subsetneqq A$ let $A_\lambda = S_\alpha - \bigcup_{\mu<\lambda} A_\mu$ where $\alpha \in A - \bigcup_{\mu<\lambda} A_\mu$. Then $A = \coprod A_\lambda$ a disjoint union of countable sets. Let F_λ be the free group on the $x_\alpha, \alpha \in A_\lambda$. $F = *F_\lambda$, the free product of the F_λ and $F^a = \bigoplus_\lambda F_\lambda^a$.

Now $\phi(F_\lambda^a) \cup \phi^{-1}(F_\lambda^a) \subset \bigoplus_{\mu \leq \lambda} F_\mu^a$ by choice. Writing $\phi = ((\phi_{\lambda\mu}))$ where $\phi_{\lambda\mu} : F_\mu^a \to F_\mu^a$ we have $\phi_{\lambda\mu} = 0$ for $\lambda < \mu$. Thus $\phi_{\lambda\lambda}$ is an isomorphism. Assuming the countable case, $\phi_{\lambda\lambda}$ may be lifted to $h_\lambda \in \operatorname{Aut} F_\mu$. Define $h = *h_\lambda \in \operatorname{Aut} F$. Then

h^a agrees with ϕ on all the components except for $\mu < \lambda$. Thus $\phi(h^a)^{-1}$ is triangular, hence lifts by Lemma 1 to some $k \in \operatorname{Aut} F$. Then $(kh)^a = \phi$ so ϕ lifts to kh. Thus we need only prove the countable case. We assume now that F has $\{x_i\}_I$ as a basis, I the positive integers.

Let $E \subset \operatorname{Aut} F^a$ be the subgroup generated by all triangular automorphisms. By Lemma 1 we will be done once we prove

Lemma 2. $E = \operatorname{Aut} F^a$.

Notation. For $J \subset I$, let $\langle J \rangle$ be the subgroup of F^a generated by $\{y_i\}_J$. We need

Lemma 3. *Assume* $\phi \in \operatorname{Aut} F^a$ *is such that* $\phi(y_i) = y_i$ *for infinitely many i, then* $\phi \in E$.

We prove first the following special case:

Lemma 4. *Let* $I = I' \amalg I''$ *where* I'' *is infinite. Assume that* $\phi(\langle I' \rangle) \subset \langle I' \rangle$ *and* $\phi \mid \langle I'' \rangle$ *is the identity. Then* $\phi \in E$.

Lemma 4 implies Lemma 3: Let $I'' = \{i \mid \phi(y_i) = y_i\}$. We shall let 1 represent an identity automorphism wherever it appears. If I'' is infinite, let $I' = I - I''$ and write $\phi = \begin{pmatrix} g & 0 \\ l & 1 \end{pmatrix}$ as an automorphism of $\langle I' \rangle \oplus \langle I'' \rangle$. Then g is an isomorphism so $\eta = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \operatorname{Aut}(\langle I' \rangle \oplus \langle I'' \rangle)$. By Lemma 4, $\eta \in E$. But $\phi\eta^{-1}$ is triangular to $\phi \in E$.

Proof of Lemma 4. Let $M = F^a$, $N = \langle I' \rangle$. Since I'' is countably infinite $\langle I'' \rangle \cong N \oplus N \oplus \cdots$, a countable sum of copies of N (except in the trivial case where $N = 0$ and so $\phi = 1$). Define $h, k \in \operatorname{Aut} M \cong \operatorname{Aut}(N \oplus N \oplus \cdots)$ by $h = g \oplus g^{-1} \oplus g \oplus g^{-1} \oplus \cdots$, $k = 1 \oplus g \oplus g^{-1} \oplus g \oplus \cdots$ where $g \in \operatorname{Aut} N$ is $\phi \mid \langle I' \rangle$. Then $\phi = hk$. But

$$g \oplus g^{-1} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & g^{-1}-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix}.$$

Thus h and k can each be written as the product of four triangular automorphisms. So $\phi \in E$. (This proof is influenced by tricks of J.H.C. Whitehead and S. Eilenberg.) This proves Lemma 4.

Finally given an arbitrary $\phi \in \operatorname{Aut} F^a$, we filter I as $\emptyset = I_0 \subset J_0 \subset I_1 \subset J_1 \subset \cdots$ with $n \in I_n$. Each I_k, J_k will be finite initial segments; i.e. of the form $\{1, 2, \ldots, t\}$. We define them inductively: Given I_{n-1} choose J_{n-1} so that $\langle I_{n-1} \rangle \cup \phi\langle I_{n-1} \rangle \subset \langle J_{n-1} \rangle$. Consider $\phi: \langle I_{n-1} \rangle \oplus \langle I - I_{n-1} \rangle \to \langle J_{n-1} \rangle \oplus \langle I - J_{n-1} \rangle$. Choose any $i_n \in I - J_{n-1}$ and let $a_n \in \langle I - I_{n-1} \rangle$ be such that $\phi(0, a_n) = (b_n, y_{i_n})$. Write $a_n = \sum c_t y_{j_t}$, $c_t \neq 0$. Then let I_n contain n, i_n, and all the j_t.

Let $H_n = \langle I_n - I_{n-1} \rangle$. $F^a = H_1 \oplus H_2 \oplus \cdots$ and $a_n \in H_n$. Since $\phi(a_n) = b_n + y_{i_n}$ is not divisible by any integer, the same is true of a_n, hence a_n is part of a basis for H_n, so there is some automorphism h_n of H_n with $h_n(y_{i_n}) = a_n$. Then let $h =$

$\oplus h_n \in \operatorname{Aut} F^a$. Notice that $\phi h(y_{i_n}) = b_n + y_{i_n}$ where b_n is a linear combination of y_i's for $i < i_n$. Thus

$$k(y_j) = \begin{cases} y_j, & j \neq \text{any } i_n \\ b_n + y_j, & j = i_n \end{cases}$$

is triangular. Observe that $k^{-1}\phi h(y_{i_n}) = y_{i_n}$ so by Lemma 3, $k^{-1}\phi h \in E$. Thus $\phi h \in E$. But $h = h' \circ h''$ where $h' = (h_1 \oplus 1 \oplus h_3 \oplus 1 \oplus \cdots)$, $h'' = (1 \oplus h_2 \oplus 1 \oplus h_4 \oplus \cdots)$ so by Lemma 4, $h \in E$. Thus $\phi \in E$. This proves Lemma 2 which proves the Theorem.

Remark. This proof actually shows that if M is an infinitely generated free module (over any PID R) then $\operatorname{Aut} M$ is generated by triangular automorphisms (triangular with respect to a fixed basis, but any ordering). Furthermore in the countable case each automorphism can be written as a composition of 26 triangular automorphisms. In the general case, of 27 triangular automorphisms. I suspect, that both of these numbers are crude and can be reduced, but it is interesting that there is a bound to the number required. This is not true for F finitely generated.

3. Structures of 2-complexes

Notation. Throughout this section we will be using subscripts to denote the members of some unnamed indexing sets. For example, F_α is the free group on generators x_α (F_β free on $\{y_\beta\}$, etc.), W_α is the one point union (or wedge) of circles indexed by the α's. Thus $\pi_1 W_\alpha = \pi_1(W_\alpha, *)$ ($*$ the union point) can be naturally identified with F_α. Furthermore, $H_1(W_\alpha)$ is F_α^a and ξ_α may be taken as the Hurewicz map. A homomophism $F_\alpha \to F_\beta$ can be represented uniquely (up to homotopy) by a map $W_\alpha \to W_\beta$.

Let $\mathcal{P} = \{x_\alpha \mid r_\beta\}$ be a presentation of a group $G(\mathcal{P}) = G = \langle x_\alpha \mid r_\beta \rangle$. That is, $G = F_\alpha / N_\beta$ where N_β is the normal subgroup generated by the elements $r_\beta \in F_\alpha$. Corresponding to $\mathcal{P}$, there is a homomorphism $r: F_\beta \to F_\alpha$ given by $y_\beta \mapsto r_\beta$, and hence a map $r: W_\beta \to W_\alpha$ which is well-defined up to homotopy given a specific basis of F_α. If we are changing basis we will have to be more careful and state r as a word w in the specific basis. The mapping cone of r is denoted by $X(\mathcal{P})$. That is, $X(\mathcal{P})$ has one 0-cell, 1-cells indexed by the α's and 2-cells indexed by the β's and attached by $r_\beta \in F_\alpha = \pi_1(W_\alpha, *)$. $\pi_1 X(\mathcal{P}) = G(\mathcal{P})$.

Conversely given any connected 2-complex $X, X \simeq X'$ a 2-complex with a single 0-cell and for some presentation $\mathcal{P}$ of $\pi_1(X)$, $X' = X(\mathcal{P})$.

Let $\mathcal{P} = \{x_\alpha \mid r_\beta\}$ be a presentation of a group. Let θ be an automorphism of F_α. Let $x'_\alpha = \theta(x_\alpha)$. The x'_α form a new basis for F_α so each r_β which is some word $w_\beta(x_\alpha)$ is also a word $s_\beta = w'_\beta(x'_\alpha)$. As elements of F_α $r_\beta = s_\beta$ but $\theta(s_\beta) = w_\beta(x'_\alpha)$. Let $\theta(\mathcal{P}) = \{x'_\alpha \mid s_\beta\}$ clearly a different presentation of the same group. If F_γ is a free group on new generators $\{z_\gamma\}$, let $\mathcal{P} * F_\gamma = \{x_\alpha, z_\gamma \mid r_\beta, z_\gamma\}$, $G(\mathcal{P} * F_\gamma) \cong G(\mathcal{P})$.

Finally if ϕ is an automorphism of F_β then if $\{b\} = \{\beta\}$, $\phi(y_\beta)$ is some word $T_\beta(y_b)$ so $t_\beta = T_\beta(r_b)$ are new words in the x_α generating the same normal subgroup as the r_β so $\mathcal{P}\phi = \{x_\alpha \mid t_\beta\}$ is a new presentation of $G(\mathcal{P})$.

Definition. *Equivalence* of presentations is the equivalence relation $\sim$ generated by $\mathcal{P} \sim \theta\mathcal{P}$, $\mathcal{P} \sim \mathcal{P}\phi$, $\mathcal{P} \sim \mathcal{P} * F_\gamma$.

Notation. $\simeq_s$ will represent simple homotopy equivalence. Cf. [0] for a full discussion.

Proposition. *If $\mathcal{P} \sim \mathcal{P}'$ then $X(\mathcal{P}) \simeq_s X(\mathcal{P}')$.*

Proof. We shall prove $X(\mathcal{P}) \simeq_s X(\theta\mathcal{P}) \simeq_s X(\mathcal{P}\phi) \simeq_s X(\mathcal{P} * F_\gamma)$.

$X(\theta\mathcal{P})$ is the mapping cone of a map corresponding to $w' : F_\beta \to F_\alpha$ where $w'(y_\beta) = w'_\beta(x_\alpha)$, whereas $X(\mathcal{P})$ is the mapping cone corresponding to $w : F_\beta \to F_\alpha$ where $w(y_\beta) = w_\beta(x_\alpha)$. But $\theta w'_\beta(x_\alpha) = w'_\beta(x'_\alpha) = s_\beta = r_\beta = w_\beta(x_\alpha)$. Thus $w \sim \theta w'$ where $\theta : W_\alpha \to W_\alpha$ is the simple homotopy equivalence corresponding to θ on F_α. Thus the cones on w and w' are simple homotopy equivalent, $X(\mathcal{P}) \simeq_s X(\theta\mathcal{P})$. Similarly we get $X(\mathcal{P}) \simeq_s X(\mathcal{P}\phi)$. $X(\mathcal{P} * F_\gamma)$ is the mapping cone of $w \vee \mu : W_\beta \vee W_\gamma \to W_\alpha \vee W_\gamma$ where w is as above and μ is the identity. Thus $X(\mathcal{P} * F_\gamma) = X(\mathcal{P}) \vee C$ where C is the cone on μ, but the cone on an identity is collapsible so $X(\mathcal{P} * F_\gamma) \simeq_s X(\mathcal{P})$.

$\theta : W_\alpha \to W_\alpha$ is a simple homotopy equivalence because $\pi_1(W_\alpha)$ is free.

Definition. Given a presentation $\mathcal{P} = \{x_\alpha \mid r_\beta\}$ the map $r : F_\beta \to F_\alpha$ induces $r^a : F^a_\beta \to F^a_\alpha$. Let $H_1\mathcal{P} = \operatorname{cok} r^a$ and $H_2\mathcal{P} = \ker r^a$.

Since r^a is the induced map $H_1 W_\beta \to H_1 W_\alpha$, the long exact mapping cone sequence yields $H_i(\mathcal{P}) = H_i X(\mathcal{P})$, $i = 1, 2$. In particular, $H_1\mathcal{P} = G(\mathcal{P})^a$.

Theorem 3. *Let $\mathcal{P}$ be a presentation with $H_1\mathcal{P}$ a direct sum of cyclic groups. Then $\mathcal{P} \sim \mathcal{P}' = \{x_\alpha, y_\beta \mid y_\beta^{n_\beta} s_\beta, t_\gamma\}$ where the s_β and t_γ are commutators and thus $H_1\mathcal{P} \cong [\bar{x}_\alpha, \bar{y}_\beta \mid n_\beta \bar{y}_\beta]$, $H_2\mathcal{P} \cong [\bar{t}_\gamma]$.*

Remarks. 1) By commutators we mean that s_β and t_γ are in the commutator subgroup of $\langle x_\alpha, y_\beta \rangle$.

2) We use the notation $[g_\alpha \mid h_\beta]$ to mean the quotient of the free abelian group on symbols g_α modulo the subgroup generated by the words h_β in the g_α.

Proof. Let $\mathcal{P} = \{u_\delta \mid w_\varepsilon\}$. Letting I be the image of $w^a : F^a_\varepsilon \to F^a_\delta$ we get $0 \to I \to F^a_\delta \to H_1\mathcal{P} \to 0$ exact. By the stacked bases theorem [3], we can find a basis $\{\bar{x}_\alpha, \bar{y}_\beta\}$ for F^a_δ and a basis $\{n_\beta \bar{y}_\beta\}$ for I where the n_β are integers.

Since I is free abelian $F_\varepsilon^a \to I$ splits so we can find a basis $\{\bar{r}_\beta, \bar{q}_\gamma\}$ with $w^a(\bar{r}_\beta) = n_\beta \bar{y}_\beta$. By the lifting theorem these new bases can be lifted to bases $\{x_\alpha, y_\beta\}$ for F_δ and $\{r_\beta, q_\gamma\}$ for F_ε. Let $\mathcal{P}' = \theta\mathcal{P}\phi = \{x_\alpha, y_\beta \mid w(r_\beta), w(q_\gamma)\}$. $\mathcal{P} \sim \mathcal{P}'$ and we note that $w(r_\beta) = y_\beta^{n_\beta} s_\beta$, $w(q_\gamma) = t_\gamma$ where s_β and t_γ are commutators. Clearly $H_1\mathcal{P}' \cong [\bar{x}_\alpha, \bar{y}_\beta \mid n_\beta \bar{y}_\beta]$ and $H_2\mathcal{P}' \cong [\bar{t}_\alpha]$ hence the full result. (θ and ϕ above are the automorphisms of F_δ and F_ε corresponding to these newly found bases.)

As an immediate corollary, we get the following:

Theorem 4. *Let X be a connected 2-complex with H_1X a direct sum of cyclics. Then $X \simeq_s X(\mathcal{P})$ where $\mathcal{P} = \{x_\alpha, y_\beta \mid y_\beta^{n_\beta} s_\beta, t_\gamma\}$ with s_β and t_γ commutators so that $H_1X = [\bar{x}_\alpha, \bar{y}_\beta \mid n_\beta \bar{y}_\beta]$ and $H_2X = [\bar{t}_\gamma]$.*

4. Using the Structure Theorem to study asphericity

For the statement and use of the proposition below, we need to recall the Fox derivative [4]: a derivation $\partial : G \to M$ is a function from a group G to a left G-module M such that $\partial(xy) = \partial x + x\partial y$. If F is free on a basis $\{x_i\}$ there is a unique derivation $\partial_i : F \to ZF$ such that $\partial_i(x_j) = \delta_{ij}$, the Kronecker delta. For the induced map we also write $\partial_i : F \to ZG$, if G is a quotient of F. From the derivation property, it follows immediately that $\partial 1 = 0$, $\partial x^{-1} = -x^{-1}\partial x$, and $\partial[G, G] \subseteq IM$ where I is the augmentation ideal of ZG, ∂ any derivation.

Recall that a 2-complex X is aspherical if and only if $\pi_2 X = 0$. Thus it is important to understand how to calculate $\pi_2 X$ from information about $\mathcal{P}$ where $X \simeq X(\mathcal{P})$. Let $\mathcal{P} = \{x_\alpha \mid r_\beta\}$ be a presentation of $G = G(\mathcal{P})$. For each α and β consider $\partial_\alpha r_\beta$ as an element of ZG. Then the matrix $((\partial_\alpha r_\beta))$ may be considered a ZG-morphism $\partial : \bigoplus_\beta ZG \to \bigoplus_\alpha ZG$.

Proposition. $\pi_2 X \cong \ker \partial$.

Proof. We look at $\tilde{X}$, the universal cover of X. Since $\tilde{X}$ is 1-connected $\pi_2\tilde{X} \cong H_2\tilde{X}$, but $\pi_2\tilde{X} \cong \pi_2 X$. Thus it is sufficient to prove that $H_2\tilde{X} \cong \ker \partial$. This is immediate because $C_*\tilde{X}$, the CW-chains of $\tilde{X}$, have the following form: $C_0\tilde{X} = ZG$ (since $C_0X = Z$), $C_1\tilde{X} = \bigoplus_\alpha ZG$ ($C_1X = \bigoplus_\alpha Z$) and $C_2\tilde{X} = \bigoplus_\beta ZG$ ($C_2X = \bigoplus_\beta Z$) where d_1 sends the αth generator to $x_\alpha - 1$ and d_2 is precisely ∂ as above. Since $C_3\tilde{X} = 0$, $H_2\tilde{X} = \ker d_2 = \ker \partial$.

We now show how to use Theorem 3 to prove that NC groups satisfy condition (A). Of course, this fact follows from Theorem 2, but the method of proof may shed some light on how one could further study the problem.

Assume H_1X is free abelian and $H_2X = 0$. Then $X \simeq_s X(\mathcal{P})$ where by Theorem 4, $\mathcal{P} = \{x_\alpha, y_\beta \mid y_\beta s_\beta\}$ with the s_β commutators. Now if s is a commutator $ds \in I$, the augmentation ideal of ZG for *any* derivation d. Thus letting $\{\gamma\} = \{\alpha\} \cup \{\beta\}$, $\partial = ((\partial_\gamma(y_\beta s_\beta)))$. $\partial_\gamma(y_\beta s_\beta) = \delta_{\gamma\beta} + y_\beta \partial_\gamma s_\beta$.

Now $\pi_2 X = \ker \partial$. Assume $\partial\lambda = 0$, $\lambda = (\lambda_\beta)$. Then for each γ, $0 = \sum_\beta \partial_\gamma(y_\beta s_\beta)\lambda_\beta = \sum_\beta \delta_{\gamma\beta}\lambda_\beta + \sum_\beta (y_\beta \partial_\gamma s_\beta)\lambda_\beta$. In particular, then, if γ is one of the β's, this says $\lambda_\gamma = -\sum_\beta y_\beta(\partial_\gamma s_\beta)\lambda_\beta$. Thus if J is the left ideal of ZG generated by the λ_β, we get $J = IJ$. If G satisfies the Nakayama condition, then $J = 0$ whence $\lambda = 0$, $\pi_2 X = 0$ and X is aspherical.

This gives an entirely different proof of the Corollary to Theorem 2. The hope is that the tools used in this method can be further exploited in the study of 2-complexes.

5. A special case of Whitehead's conjecture

Theorem 3. *Let K be a finite aspherical 2-complex and $L \subset K$. Assume that there is a finite aspherical 3-complex X with $\pi_1 X \cong \pi_1 L = \pi$ and $H_3 X = 0$. Then L is aspherical.*

Note. In particular this covers the cases of π finitely generated free (X a 1-complex) or π a torsion-free finitely one relator group (X a 2-complex) which are in [2]. Thus of Cockcroft's results we omit only the case of a one relator group with torsion.

Proof. First we need to prove the following lemma which appears implicitly in [2]. The result is due independently to several authors, the first probably H. Hopf.

Lemma 3. *Let K be an aspherical 2-complex and $L \subset K$. Then $H_2 L \cong H_2(\pi_1 L)$.*

Proof. Add n-cells to L for $n \geq 3$ to form M an aspherical space. Then $H_2(M) \cong H_2(\pi_1 L)$. Look at the exact diagram

$$\begin{array}{ccccccccccc}
\cdots \to & \pi_3 M & \to & \pi_3(M,L) & \xrightarrow{\partial} & \pi_2 L & \to & \pi_2 M & \to & \pi_2(M,L) & \to \cdots \\
& \downarrow & & \Big\| a & & \Big\| b & & \downarrow & & \downarrow & \\
\cdots \to & H_3 M & \to & H_3(M,L) & \xrightarrow{\partial'} & H_2 L & \xrightarrow{\alpha} & H_2 M & \to & H_2(M,L) & \to \cdots
\end{array}$$

where the arrows down are the Hurewicz maps.

By the construction of M, $H_2(M,L) = 0$ so α is onto. By the Hurewicz theorem a is an epimorphism. Since $\pi_3 M = \pi_2 M = 0$, ∂ is an isomorphism. Thus $\operatorname{im} \partial' = \operatorname{im} b$, so to prove α an isomorphism it will suffice to show that $b = 0$.

Look at the following exact diagram:

$$\begin{array}{ccccc}
\pi_3(K,L) & \longrightarrow & \pi_2 L & \longrightarrow & \pi_2 K \\
\downarrow & & \downarrow b & & \downarrow \\
H_3(K,L) & \longrightarrow & H_2 L & \xrightarrow{\beta} & H_2 K.
\end{array}$$

Now $H_3(K, L) = 0$ so β is 1–1. $\pi_2 K = 0$ so $\beta b = 0$. Thus $b = 0$, and the lemma is proved.

Now look at the following exact sequences

$$0 \to C_3\tilde{X} \to C_2\tilde{X} \to C_1\tilde{X} \to C_0\tilde{X} \to Z \to 0 \tag{1}$$

$$0 \to \pi_2 L \to C_2\tilde{L} \to C_1\tilde{L} \to C_0\tilde{L} \to Z \to 0. \tag{2}$$

These are exact since $\tilde{X}$ is contractible and $H_2\tilde{L} \cong \pi_2 L$. Now, by the Schanuel Lemma, since $C_*\tilde{L}$ and $C_*\tilde{X}$ are all free $Z\pi$-modules, we get

$$C_3\tilde{X} \oplus C_2\tilde{L} \oplus C_1\tilde{X} \oplus C_0\tilde{L} \cong \pi_2 L \oplus C_2\tilde{X} \oplus C_1\tilde{L} \oplus C_0\tilde{X}. \tag{3}$$

The ranks of $C_*\tilde{X}$ and $C_*\tilde{L}$ as free $Z\pi$-modules are the same as $C_*\tilde{X} \otimes_{Z\pi} Z = C_*X$ and $C_*\tilde{L} \otimes_{Z\pi} Z = C_*L$ as free abelian groups. Now $H_0 L \cong Z \cong H_0 X$, $H_1 L \cong \pi^a \cong H_1 X$, $H_2 L \cong H_2 X$ by Lemma 3, and $H_3 L \cong H_3 X$ since they are both 0. Since $H_*L \cong H_*X$ the alternating sums of the ranks over Z of C_*L and C_*X are the same. Thus the same is true of $C_*\tilde{L}$ and $C_*\tilde{X}$ over $Z\pi$. Thus $C_3\tilde{X} \oplus C_2\tilde{L} \oplus C_1\tilde{X} \oplus C_0\tilde{L} \cong C_2\tilde{X} \oplus C_1\tilde{L} \oplus C_0\tilde{X} = M$, some finitely generated free $Z\pi$-module. Then (3) becomes $M \cong \pi_2 L \oplus M$. By Kaplansky's Theorem [6], $\pi_2 L = 0$ so L is aspherical.

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