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A Linear Bound for $\rho(n)$*

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I. INTRODUCTION

This paper continues the examination of the maximal number $\rho(n)$ of different primes that may occur in the order of a finite solvable group G whose elements all have orders divisible by at most n different primes. We define

$$\sigma(n) := \text{number of different primes dividing } n \qquad (n \in \mathbb{N}),$$
$$\sigma(G) := \max\{\sigma(o(g)) \mid g \in G\},$$
$$\rho(n) := \max\{\sigma(|G|) \mid G \text{ a finite solvable group with } \sigma(G) = n\}.$$

Then by [1] and [5] we already know that

$$\rho(n) \leq \frac{n(n+3)}{2} \qquad \text{for all } n \in \mathbb{N},$$

and in [3] we have improved the bound asymptotically to

$$\rho(n) \leq Cn \ln(n) \qquad \text{for all } n > 1 \text{ and a constant } C > 0.$$

Here we want to give a linear bound for $\rho(n)$ in n; we will show

$$\rho(n) \leq C(n)n \qquad \text{for all } n \in \mathbb{N},$$

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where $C(n) > 4$ for all n and

$$\lim_{n \to \infty} C(n) = 4.$$

Clearly a linear bound is asymptotically best possible, and our upper bound is not too bad, because by [2, 1.11] we know that $\rho(n) \geq 3n$ for infinitely many $n \in \mathbb{N}$.

II. RESULTS AND PROOFS

GENERAL HYPOTHESIS. *All groups considered in this paper are finite and solvable.*

At first we remember the reader of the definition of σ-reduced groups and some of their properties.

1. DEFINITION. Let $G = P_1 \cdots P_m$ (for an $m \in \mathbb{N}$) with $P_i \in \mathrm{Syl}_{p_i} G$ ($i = 1, \ldots, m$; p_i mutually distinct prime numbers). For $i < j$ assume that $P_i \leq N_G(P_j)$ and that each P_i is isomorphic to a principal factor of G. Then we let $M_G := \{P_1, \ldots, P_m\}$ and call G σ-reduced.

2. DEFINITION. Let G be a group all Sylow subgroups of which are elementary abelian. Then we define

$$e(G) := \min\{\dim_{\mathrm{GF}(p)} P \mid p \text{ prime number and } 1 < P \in \mathrm{Syl}_p G\}.$$

3. LEMMA.

(a) *Every G possesses a σ-reduced subgroup U with $\sigma(|U|) = \sigma(|G|)$.*

(b) *If G is σ-reduced and $P, Q \in M_G$ with $P \leq N_G(Q)$, then P acts either faithfully or trivially on Q.*

(c) *Let G be σ-reduced. Then*

$$G^{(i)}/G^{(i+1)} \cong P_{i,1} \times \cdots \times P_{i,n_i} \qquad \text{for } i = 0, \ldots, dl(G) - 1,$$

where $P_{i,l} \in M_G$ for all i, l. Furthermore, if $j, k \in \{0, \ldots, dl(G) - 1\}$ with $j < k$, and if $P \in M_G$ with $P \lneq G^{(k)}/G^{(k+1)}$, then there exists a $Q \lneq G^{(j)}/G^{(j+1)}$ (in M_G) such that Q acts faithfully on P.

(d) *Let $G = AH$ be a Frobenius group with cyclic complement $A = \langle a \rangle$ and elementary abelian kernel H. Moreover, let V be a faithful G-module over a field of a characteristic not dividing $|G|$. Then $e(V) \geq |A| = o(a)$.*

(e) *Let P be an elementary abelian p-group (p a prime) which acts faithfully on a vector space V over $\mathrm{GF}(q)$ for a prime power q. Then $e(P) \leq \dim_{\mathrm{GF}(q)} V$.*

Proof. See [2, 3, and 4]. ∎

Now we turn towards our theorem, but we want to describe the methods for finding elements with many prime divisors in their order a little more generally than is needed to obtain our main result, because they may be of interest.

In the rest of the paper, for σ-reduced groups G we will use the following notation:

$$G^{(i)}/G^{(i+1)} \cong P_{i,1} \times \cdots \times P_{i,n_i} =: H_i \leq G^{(i)} \qquad \text{for } i \in \mathbb{N}_0,$$

where $P_{j,k} \in M_G$ for all j, k; in particular,

$$n_i = \sigma\left(G^{(i)}/G^{(i+1)}\right) \qquad \text{for } i \in \mathbb{N}_0,$$

and obviously $H_i \leq N_G(H_j)$ for $i < j$.

If there is more than one σ-reduced group involved, we write $H_i = H_i(G)$, $n_i = n_i(G)$, and $P_{i,j} = P_{i,j}(G)$.

4. LEMMA. *Let G be σ-reduced, and put $n_i := n_i(G)$ for all i. Let $s \in \{0, \ldots, dl(G) - 1\}$ be fixed, and let U be an abelian subgroup of G with $U \lneqq G/G^{(s+1)}$ and the following property:*

If P_0 is any Sylow subgroup of U and $P_0 \leq P \in M_G$, and if P acts faithfully on exactly t Sylow subgroups of $G^{(s)}$ out of M_G, then $e(P_0) \geq t + 1$. (This hypothesis is fulfilled, for example, if $e(U) \geq \sigma(|G^{(s)}|) + 1$; possibly $U = 1$). Then there exist a σ-reduced $H \leq G^{(s)}$ and an $x \in U$ with the following properties:

(1) $\sigma(o(x)) = \sigma(U)$,

(2) $H \leq C_G(x)$,

(3) $dl(H) = dl(G) - s$ *and* $n_i(H) = n_{i+s}$ *for* $i = 0, \ldots, dl(H) - 1$.

Proof. To prove the lemma we claim the following:

($*$) We put $n'_i := n_{i+s}$ for $i = 0, \ldots, dl(G) - s - 1$. Then for all $j \in \{0, \ldots, dl(G) - s - 1\}$ and for all $k \in \{0, \ldots, n'_j\}$ there exist a σ-reduced group $H_{j,k} \lneqq G^{(s)}/G^{(s+j+1)}$ and a group $U_{j,k} \leq U$ with the following properties:

(1′) $\sigma(U_{j,k}) = \sigma(U)$. If P_0 is any Sylow subgroup of $U_{j,k}$ and $P_0 \leq P \in M_G$, and if P acts faithfully on exactly t Sylow subgroups out of the set $\{Q \in M_G | Q \leq G^{(s)} \text{ and } Q \cap H_{j,k} = 1\}$, then $e(P_0) \geq t + 1$.

(2′) $[U_{j,k}, H_{j,k}] = 1$.

(3′) $H_{j,k}^{(i)}/H_{j,k}^{(i+1)} \cong Q_{i,1} \times \cdots \times Q_{i,n'_i}$ for $i = 0, \ldots, j - 1$, and finally $H_{j,k}^{(j)} = Q_{j,1} \times \cdots \times Q_{j,k}$, where $1 < Q_{i,l} \in M_{H_{j,k}}$ and $Q_{i,l} \leq P_{i+s,l}$ for all i, l.

In particular, in case of $i < j$ we have $H_{i,l_1} \le H_{j,l_2}$ for any possible l_1, l_2, and if $k_1 \le k_2$ $(\le n'_j)$, we have $H_{j,k_1} \le H_{j,k_2}$ for all j. Furthermore note that $H_{j,n'_j} = H_{j+1,0}$ for all j.

If $(*)$ is shown, then we apply it with $j = dl(G) - s - 1$, $k = n'_{dl(G)-s-1} = n_{dl(G)-1}$ and set $H := H_{dl(G)-s-1, n_{dl(G)-1}}$ and $U^* := U_{dl(G)-s-1, n_{dl(G)-1}}$. As by (1′), $\sigma(U^*) = \sigma(U)$, we find an $x \in U^*$ with $\sigma(o(x)) = \sigma(U)$. Then (1) is fulfilled, and (2′), (3′) also imply (2), (3), so we are done.

Proof of $(*)$. We proceed by induction on j:

$j = 0$: We construct the $U_{0,k}, H_{0,k}$ $(k = 0, \dots, n'_0)$ by induction on k:

$k = 0$: Take $U_{0,0} := U$ and $H_{0,0} := 1$.

$k \to k+1$: Suppose that for a $k \in \{0, \dots, n'_0 - 1\}$ we already have $U_{0,k}, H_{0,k}$.

Then we find $U_{0,k+1}, H_{0,k+1}$ as follows: Let $1 < S_{0,k+1} \le P_{s,k+1}$ be an irreducible $U_{0,k}$-module, and put

$$U_{0,k+1} := C_{U_{0,k}}(S_{0,k+1}).$$

Since $U_{0,k}/U_{0,k+1}$ is an abelian group acting faithfully and irreducibly on $S_{0,k+1}$, $U_{0,k}/U_{0,k+1}$ is cyclic; so if (for a prime p_0) $P_0 \in \mathrm{Syl}_{p_0} U_{0,k+1}$ and $P_1 \in \mathrm{Syl}_{p_0} U_{0,k}$ with $P_0 \le P_1 \le P \in M_G$, then

$$e(P_0) \begin{cases} \ge e(P_1) - 1, & \text{if } P \text{ acts faithfully on } P_{s,k+1} \\ = e(P_1), & \text{if } P \text{ acts trivially on } P_{s,k+1} \end{cases}. \qquad (+)$$

Now let $1 \ne a \in S_{0,k+1}$ and put $Q_{0,k+1} := \langle a \rangle$ and $H_{0,k+1} := H_{0,k} \times Q_{0,k+1}$. Then clearly $H_{0,k+1}$ is σ-reduced and (2′), (3′) obviously are satisfied. As (1′) holds for $U_{0,k}$ (by induction), $(+)$ immediately shows that $U_{0,k+1}$ also fulfills (1′).

$j \to j+1$: Let $j \in \{0, \dots, dl(G) - s - 2\}$ and suppose that we have already found U_{j,n'_j}, H_{j,n'_j}. Now we construct the $U_{j+1,k}, H_{j+1,k}$ $(k = 0, \dots, n'_{j+1})$ by an inductive process on k:

$k = 0$: Then simply set $U_{j+1,0} := U_{j,n'_j}$ and $H_{j+1,0} := H_{j,n'_j}$.

$k \to k+1$: Suppose that for a $k \in \{0, \dots, n'_{j+1} - 1\}$ we already have $U_{j+1,k}, H_{j+1,k}$. Then we find $U_{j+1,k+1}, H_{j+1,k+1}$ as follows.

Let

$$R_j := Q_{j,1} \times \cdots \times Q_{j,n'_j} \le H_{j,n'_j} \le H_{j+1,k} \qquad \text{(by induction)}.$$

Then we have the Zassenhaus decomposition

$$P_{j+s+1,k+1} = [R_j, P_{j+s+1,k+1}] \times C_{P_{j+s+1,k+1}}(R_j). \qquad (++)$$

For convenience we put $T := T_{j+1,k+1} := [R_j, P_{j+s+1,k+1}]$. Since $H_{j,n'_j} \leq N_G(R_j) \cap N_G(P_{j+s+1,k+1})$ and $Q_{j+1,1} \times \cdots \times Q_{j+1,k} \leq C_G(P_{j+s+1,k+1})$, surely

$$H_{j+1,k} = H_{j,n'_j}(Q_{j+1,1} \times \cdots \times Q_{j+1,k}) \leq N_G(T),$$

and $T > 1$, because else by $(++)$ and Lemma 3b $P_{j+s,1} \times \cdots \times P_{j+s,n_{j+s}}$ would centralize $P_{j+s+1,k+1}$ against Lemma 3c.

As $U_{j+1,k} \leq C_G(R_j) \cap N_G(P_{j+s+1,k+1})$, we have

$$U_{j+1,k} \leq N_G(T).$$

Now let $1 < S_{j+1,k+1} \leq T$ be an irreducible $U_{j+1,k}$-module, and put

$$U_{j+1,k+1} := C_{U_{j+1,k}}(S_{j+1,k+1}).$$

Since $U_{j+1,k}/U_{j+1,k+1}$ is an abelian group acting faithfully and irreducibly on $S_{j+1,k+1}$,

$$U_{j+1,k}/U_{j+1,k+1} \text{ is cyclic.} \qquad (+++)$$

Obviously $1 < S_{j+1,k+1} \leq C_T(U_{j+1,k+1})$, and as $H_{j+1,k} \leq C_G(U_{j+1,k}) \leq C_G(U_{j+1,k+1})$, $H_{j+1,k}$ normalizes $C_T(U_{j+1,k+1}) > 1$. Hence let $1 < Q_{j+1,k+1} \leq C_T(U_{j+1,k+1})$ be an irreducible $H_{j+1,k}$-module, and we build the semidirect product

$$H_{j+1,k+1} := H_{j+1,k}Q_{j+1,k+1}.$$

Then clearly $H_{j+1,k+1}$ is σ-reduced, and as in the case $j = 0$ with the help of $(+++)$ we see that $U_{j+1,k+1}$ fulfills (1'). Moreover, $H_{j+1,k+1}$ and $U_{j+1,k+1}$ fulfill (2'). As to (3'), obviously it suffices to show that $Q_{j+1,k+1} \leq H^{(j+1)}_{j+1,k+1}$, and for this we have to show that there exists an $l \in \{1, \dots, n'_j\}$ such that $Q_{j,l}$ acts faithfully on $Q_{j+1,k+1}$. But this is clear, because if no such l existed, then $Q_{j+1,k+1} \leq C_{P_{j+s+1,k+1}}(R_j)$ would follow contradicting $Q_{j+1,k+1} \leq T$ and $(++)$. Hence also (3') holds for $H_{j+1,k+1}$, and we are done. ∎

Lemma 4 is a tool for inductively finding elements with many prime divisors in their order. Now we describe in a rather general way how Lemma 4 works.

5. LEMMA.

(a) *Let G be a σ-reduced group. Suppose there are $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$ ($i = 1, \ldots, m$) with $\sum_{i=1}^{m} k_i = dl(G)$ such that we can apply Lemma 4 successively in steps of lengths ($s =$) k_i as follows: We put $G_1 := G$, and if we already have found G_j for a $j \in \{1, \ldots, m-1\}$, we find an abelian $U_j \lneq G_j'/G_j^{(k_j+1)} \leq G_j/G_j^{(k_j+1)}$ to which we may apply Lemma 4 (with $s \to k_j$) which yields an $x_j \in U_j$ and a σ-reduced $G_{j+1} \leq G_j^{(k_j)}$ with the properties described in Lemma 4. (Clearly $G_m = 1$.) Then $\sigma(o(x_1 \ldots x_m)) = \sum_{i=1}^{m} \sigma(x_i) \leq \sigma(G)$.*

(b) *Let X be a class of σ-reduced groups, $k_i \in \mathbb{N}$ ($i = 1, \ldots, m$ for an $m \in \mathbb{N}$), $h_j := \sum_{i=1}^{j} k_i$ ($j = 0, \ldots, m$) and $f_i \colon \mathbb{N}^{k_i} \to \mathbb{N}$ functions such that the following holds:*

(1) *If $G \in X$ and H is a σ-reduced subgroup of G for which there is an $N \in \{h_j | j = 0, \ldots, m\}$ such that $H \leq G^{(N)}$, $\sigma(|H|) = \sigma(|G^{(N)}|)$, and $n_i(H) = n_{i+N}(G)$ for $i = 0, \ldots, dl(H) - 1 = dl(G) - N - 1$, then $H \in X$.*

(2) *If $G \in X$, $i \in \{1, \ldots, m\}$, and $dl(G) \geq k_i + 1$, then G possesses an abelian subgroup U (depending on G) with $U \lneq G'/G^{(k_i+1)}$ such that $\sigma(U) = f_i(n_0(G), \ldots, n_{k_i}(G))$ and such that if P_0 is a Sylow subgroup of U with $P_0 \leq P \in M_G$, then*

$$e(P_0)\begin{cases} \geq \sigma(|G^{(k_i)}|) + 1, & \text{if } P \lneq G/G^{(k_i)} \\ \geq \sigma(|G^{(k_i+1)}|) + 1, & \text{if } P \lneq G^{(k_i)}/G^{(k_i+1)} \cong H_{k_i} \end{cases}. \qquad (+)$$

Now let $G \in X$ with $dl(G) \geq 1 + \sum_{i=1}^{m} k_i$. Then G' contains an element y with

$$\sigma(o(y)) = \sum_{i=1}^{m} f_i\big(n_{h_{i-1}}(G), \ldots, n_{h_i}(G)\big) \leq \sigma(G),$$

(and if the condition

(*) *"One already knows the primes occurring in the order of all such subgroups U as described above"*

is fulfilled, one also knows the primes occurring in the order of y).

Proof.

(a) is trivial.

(b) By replacing G by $G/G^{(dl(G)-h_m-1)}$ we may assume that $dl(G) = 1 + \sum_{i=1}^{m} k_i$, and we proceed by induction on m. By the hypotheses there exists an abelian subgroup $U \lneq G'/G^{(k_1+1)}$ of G with $\sigma(U) = f_1(n_0(G),\ldots,n_{k_1}(G))$ and the property $(+)$ with $i = 1$. If $m = 1$, we are done. So let $m > 1$. Now we apply Lemma 4 (with $s \to k_1$) which yields an $x \in U$ and a σ-reduced $H \leq G^{(k_1)}$ as described there. Our hypothesis (1) allows us to apply induction to H which yields an $y_0 \in H' \leq G^{(k_1+1)}$ (the primes in the order of which are known in the case of $(*)$) with

$$\sigma(o(y_0)) = \sum_{i=1}^{m-1} f_{i+1}\big(n_{h'_{i-1}}(H),\ldots,n_{h'_i}(H)\big) \leq \sigma(H') \leq \sigma(G) - \sigma(U),$$

where $h'_j := \sum_{i=1}^{j} k_{i+1}$ for $j = 0,\ldots,m-1$; and by Lemma 4 (2) we have $xy_0 = y_0x$. Put $y := xy_0$. (So if condition $(*)$ holds, we know the primes occurring in $o(y)$.) Together with Lemma 4 (3) follows

$$\begin{aligned}
\sigma(G) \geq \sigma(o(y)) &= \sigma(o(x)) + \sigma(o(y_0))\\
&= f_1\big(n_0(G),\ldots,n_{k_1}(G)\big) + \sum_{i=1}^{m-1} f_{i+1}\big(n_{h'_{i-1}}(H),\ldots,n_{h'_i}(H)\big)\\
&= f_1\big(n_0(G),\ldots,n_{k_1}(G)\big) + \sum_{i=1}^{m-1} f_{i+1}\big(n_{h'_{i-1}+k_1}(G),\ldots,n_{h'_i+k_1}(G)\big)\\
&= f_1\big(n_0(G),\ldots,n_{k_1}(G)\big) + \sum_{i=1}^{m-1} f_{i+1}\big(n_{h_i}(G),\ldots,n_{h_{i+1}}(G)\big)\\
&= f_1\big(n_{h_0}(G),\ldots,n_{h_1}(G)\big) + \sum_{i=2}^{m} f_i\big(n_{h_{i-1}}(G),\ldots,n_{h_i}(G)\big)
\end{aligned}$$

which is the assertion. ∎

We give an application of Lemma 5. Note that $[x] = \max\{n \in \mathbb{N} \mid n \leq x\}$ for $x \in \mathbb{R}$.

6. LEMMA. *Let $l \geq 2$ ($l \in \mathbb{N}$), $k \in \mathbb{N}_0$, and let $X(l,k)$ be the class of σ-reduced groups G with the property that for every prime p with $p \mid |G|$ the following holds:*
If $P \in \mathrm{Syl}_p\, G$ and $P \lneq G^{(j)}/G^{(j+1)}$ for a $j = j(p)$ with $j \geq k$ and $l \mid (j-k)$, then

$$p \geq \sigma(|G^{(j+l+1)}|) + 1.$$

Let $G \in X(l,k)$ and $h := [(dl(G) - k - 1)/l]$, and put

$$T := \prod_{i=1}^{h} H_{li+k}.$$

Then $\sigma(T) = \sigma(|T|)$; *in particular,*

$$\sum_{i=1}^{h} n_{li+k} \leq \sigma(G).$$

Proof. Let l and k be fixed and $X := X(l, k)$.

Applying Lemma 4 with $s \to k$ and $U \to 1$ shows that without loss of generality we may assume $k = 0$. At first we show:

> (+) *Let* H *be* σ*-reduced with* $dl(H) = l + 1$, *and let* q_0 *be the smallest prime dividing* $|H/H'|$. *Let* $U := H_l(H)$. *Then* $\sigma(U) = n_l(H)$ *and* $e(U) \geq q_0$.

To see this we only have to show that $e(P_{l,j}(H)) \geq q_0$ for $j = 1, \ldots, n_l(H)$. So let j be fixed. By Lemma 3(c) there is a $Q_2 \in M_H$ (depending on j) with $Q_2 \lneq H'/H''$ acting faithfully on $P_{l,j}(H)$, and (again by Lemma 3(c)) there is a $Q_1 \in M_H$ with $Q_1 \lneq H/H'$ which acts faithfully (and thus fixed point freely) on Q_2. Thus Q_1Q_2 is a Frobenius group acting faithfully on $P_{l,j}(H)$ whence by Lemma 3(d) $e(P_{l,j}) \geq |Q_1| \geq q_0$. This is (+).

Now we define $f\colon \mathbb{N}^{l+1} \to \mathbb{N}$ by $f(x_0, \ldots, x_l) := x_l$ ($x_i \in \mathbb{N}$ for all i).

Let $G \in X$ with $dl(G) \geq l + 1$. By (+) for $V := H_l(G)$ we have $\sigma(V) = n_l(G)$ and $e(V) \geq q_1$, where q_1 is the smallest prime dividing $|G/G'|$. Now $G \in X$ implies $q_1 \geq \sigma(|G^{(l+1)}|) + 1$. So we have

$$\sigma(V) = f(n_0(G), \ldots, n_l(G))$$

and

$$e(V) \geq \sigma(|G^{(l+1)}|) + 1.$$

So we may apply Lemma 5(b) with $X \to X$, $m \to h$, $k_i \to l$, $f_i \to f$, and $h_j \to lj$, and the assertion follows. ∎

As in [3] let for $k \in \mathbb{N}$

$$\rho(n, k) := \max\{\sigma(|G|) \mid G \text{ a finite solvable group with } \sigma(G) = n \text{ and no prime dividing } |G| \text{ is less than } k\}.$$

7. THEOREM.

(a) *Let* G *be a* σ*-reduced group such that all primes dividing* $|G|$ *are greater than* $\sigma(|G'''|)$. *Then*

$$\sigma(|G|) \leq 4\sigma(G).$$

In particular, $\rho(n, \rho(n)) \leq 4n$ *for all* $n \in \mathbb{N}$.

(b) *There is a* $C\colon \mathbb{N} \to \mathbb{R}$ *with* $C(n) > 4$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} C(n) = 4$ *such that*

$$\rho(n) \le C(n)n \qquad \text{for all } n \in \mathbb{N}.$$

Proof.

(a) Let $X(\cdot,\cdot)$ be as in Lemma 6. Then obviously $G \in X(2,0) \cap X(2,1)$, so that by Lemma 6—applied with $l = 2$, $k = 0, 1$—we have

$$\begin{aligned}\sigma(|G|) &= \sum_{i=0}^{dl(G)-1} n_i \\ &= n_0 + n_1 + \sum_{i=1}^{[(dl(G)-1)/2]} n_{2i} + \sum_{i=1}^{[(dl(G)-2)/2]} n_{2i+1} \\ &\le \sigma(G) + \sigma(G) + \sigma(G) + \sigma(G) = 4\sigma(G).\end{aligned}$$

As to $\rho(n, \rho(n))$, note that if H is a group with $\sigma(H) = n$, $\sigma(|H|) = \rho(n, \rho(n))$ such that $p \ge \rho(n)$ for all primes p with $p \mid |H|$, then by Lemma 3(a) without loss of generality we may assume that H is σ-reduced.

(b) For $m \in \mathbb{N}$ let $\pi(m)$ be the number of primes not exceeding m. Let G be a group with $\sigma(G) = n$ and $\sigma(|G|) = \rho(n)$; without loss of generality let G be σ-reduced. Now let τ be the set of prime numbers not exceeding $\rho(n)$, and take $H_1 \in \mathrm{Hall}_\tau G$ and $H_2 \in \mathrm{Hall}_{\tau'} G$. Thus clearly

$$\rho(n) = \sigma(|G|) = \sigma(|H_1|) + \sigma(|H_2|) \le \pi(\rho(n)) + \rho(n, \rho(n)),$$

and so by (a)

$$\rho(n) \le \pi(\rho(n)) + 4n.$$

As $\pi(m) \le 6(m/\ln(m))$ for all $m \ge 2$ (and as $\rho(m) \ge 2$ for all $m \in \mathbb{N}$), it follows

$$\rho(n) \le 6\frac{\rho(n)}{\ln(\rho(n))} + 4n \qquad \text{for all } n \in \mathbb{N}$$

which is equivalent to

$$\rho(n)\left(1 - \frac{6}{\ln(\rho(n))}\right) \le 4n \qquad \text{for all } n \in \mathbb{N}. \qquad (*)$$

Since $\lim_{n\to\infty}(1 - 6/\ln(\rho(n))) = 1$, for any $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$

such that we have

$$\rho(n) \le (4+\epsilon)n \qquad \text{for all } n \ge N_\epsilon$$

which is the assertion. ∎

As $\rho(n) \ge 2n$ for all $n \in \mathbb{N}$ (see [2, 1.11]), by the formula $(*)$ in the proof of Theorem 7 we have

$$\rho(n) \le 4n \bigg/ \left(1 - \frac{6}{\ln(2n)}\right) \qquad \text{for all } n \ge \frac{e^6}{2}.$$

So for any $K \in \mathbb{R}$ with $K > 4$ we have

$$\rho(n) \le Kn \qquad \text{for all } n \ge \tfrac{1}{2}e^{6K/(K-4)}.$$

We finish by mentioning some open questions on the subject:

8. PROBLEM. As now we have a linear bound for $\rho(n)$ in n, the question of the behaviour of the derived length of groups with $\sigma(|G|) = \rho(\sigma(G))$ arises. For the moment, let us call such groups σ-maximal. Until now, by [3, Lemma 3(c)] we only know $dl(G) \le 2\sigma(G)$, if all Sylow subgroups of G are elementary abelian.

Does there exist a universal constant C such that $dl(G) \le C$ for all σ-maximal groups, or at least for all σ-reduced σ-maximal groups?

Which are the minimal, resp. maximal, functions f, resp. g, such that there are infinitely many $n \in \mathbb{N}$ and σ-maximal groups G_n with $\sigma(G_n) = n$ and $dl(G_n) \le f(n)$ resp. $dl(G_n) \ge g(n)$? What about these functions, if only σ-reduced σ-maximal groups are regarded?

9. CONJECTURE. $\rho(n) = 3n$ for all $n \ge 4$.

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