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NOTE

A NOTE ON HAMILTONIAN CYCLES IN BIPARTITE PLANE CUBIC MAPS HAVING CONNECTIVITY 2

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It is shown that every bipartite plane cubic map of connectivity 2 has at least eight quadrilaterals, and those with exactly eight quadrilaterals are Hamiltonian. Also, the smallest non-Hamiltonian bipartite plane cubic map is determined.

1. Introduction

We will be dealing exclusively with bipartite plane cubic maps. Since these are 2-connected (see [2, p. 272]), each face in a bipartite plane cubic map has a cycle as its boundary. The number of vertices on the boundary cycle will be called the *degree* of the face.

It is well known (see [3, p. 112]) that if v_i is the number of vertices of degree i in a plane triangulation, then the following formula holds.

$$12 = 5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5 - v_7 - 2v_8 - 3v_9 - \cdots$$

The dual of a bipartite plane cubic map is an Euler triangulation, and since our maps are assumed to have neither multiple edges nor loops, we have the following result.

Proposition 1. *Let M be a bipartite plane cubic map and let f_i denote the number of faces of M having degree i. Then,*

$$12 = 2f_4 - 2f_8 - 4f_{10} - 6f_{12} - \cdots$$

As a consequence of Proposition 1, we see that each bipartite plane cubic map has at least six quadrilaterals and those with exactly six have no face of degree greater than six. If M has exactly seven quadrilaterals, it must contain exactly one octagon. If M has exactly eight quadrilaterals, it contains either two octagons or one decagon.

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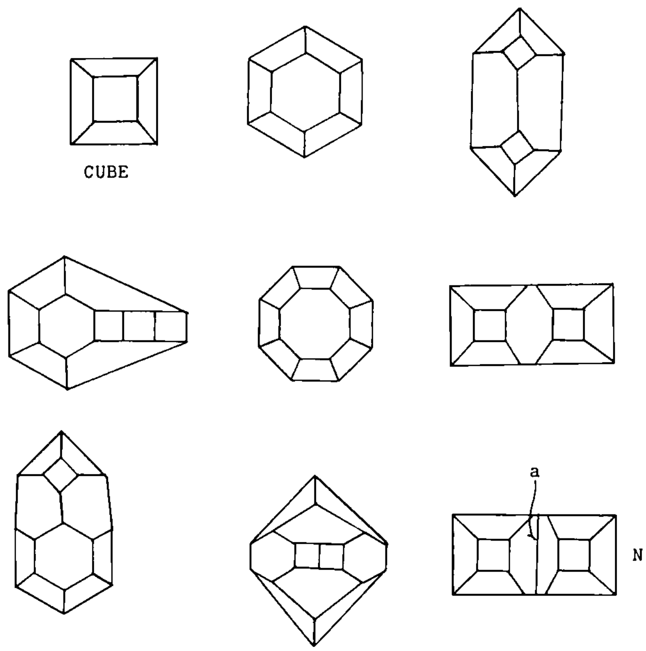

Fig. 1

Fig. 1 shows all the bipartite plane cubic maps having fewer than twenty vertices. The following proposition is easily verified.

Proposition 2. *Let M be one of the maps shown in Fig. 1. Let e be any edge in M if $M \neq N$, and any edge other than a if $M = N$. Then there is a Hamiltonian cycle in M which passes through e.*

In [1] P.R. Goodey proves the following.

Proposition 3. *Let M be a bipartite plane cubic map. Suppose e is an edge of M bordering a quadrilateral and a hexagon if M has exactly six quadrilaterals, and a quadrilateral and an octagon if M has exactly seven quadrilaterals. Then there is a Hamiltonian cycle in M passing through e.*

Corollary. *Every bipartite plane cubic map having exactly six quadrilaterals is Hamiltonian.*

Proposition 4. *Let M be a bipartite plane cubic map having exactly six quadrilaterals and an edge e which borders two quadrilaterals. Then there is a Hamiltonian cycle in M which passes through e.*

Proof. The proof of Proposition 4 is quite tedious and lengthy and hence will be omitted. It can be found in its entirety in [4] or [5]. Copies of [5] can be obtained from the author.

2. Bipartite plane cubic maps having connectivity 2

Proposition 5. *Let M be a bipartite plane cubic map. M has connectivity 2 if and only if there is a face in M with two nonadjacent edges on its boundary whose removal disconnects the map.*

Proof. It is quite easy to show that a cubic map is n-connected if and only if it is n-edge-connected. Since M is 2-connected the proposition follows immediately.

Let M be a bipartite plane cubic map having connectivity 2. By Proposition 5, M has a face F with two nonadjacent edges e_0 and a_0 on its boundary whose removal disconnects M. This is depicted in Fig. 2. Straight lines in diagrams will be used to depict edges only, while curved lines may be paths of any length.

It is easily shown that each of the paths P_0 and Q_0 in the figure must contain an even number of vertices. With this in mind a decomposition process will now be described.

Erase the edges e_0 and a_0, and consider the component of the resulting map which contains the path P_0; call it L_0. If x_0 and y_0 are not connected by an edge, we can form another bipartite plane cubic map M_{L_0} by adding the edge x_0y_0 to L_0. If x_0 and y_0 are connected by an edge, we have one of the situations shown in Fig. 3. Either P_0 is an edge (left), or consists of two edges and a path P_1 (right). Dotted lines in diagrams represent portions of the map which have been erased.

Assume L_{k-1} is defined, $1 \leq k$. Let $L_k = L_{k-1} - \{x_{k-1}, y_{k-1}\}$, taking one of the forms shown in Fig. 4. If x_k and y_k are not connected by an edge, we add the edge x_ky_k to L_k forming the map M_{L_k}. If x_k and y_k are connected by an edge, the map L_k can be represented by the one diagram shown in Fig. 5, where either p_k' or p_k is an edge. Depending on which of these cases is true, L_k must take one of the forms shown in Fig. 6.

Continuing this process, there must be a positive integer k such that x_k and y_k are not connected by an edge. When this occurs we add the edge x_ky_k to L_k forming the bipartite plane cubic map M_{L_k}, ending the procedure. A similar process may be carried out beginning with the vertices w_0 and z_0 and the path Q_0.

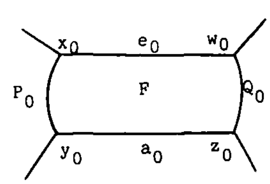

Fig. 2.

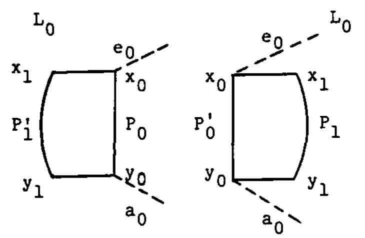

Fig. 3.

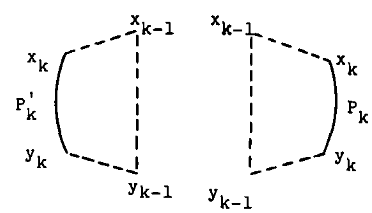

Fig. 4.

Assuming that we have a map R_j at the jth step with vertices w_j and z_j not connected by an edge, we add the edge w_jz_j forming the bipartite plane cubic map M_{R_j}. We will say that M has been *decomposed* into the maps M_{L_k} and M_{R_j}.

Proposition 6. *Each bipartite plane cubic map of connectivity* 2 *has at least eight quadrilaterals.*

Proof. Each of the maps M_{L_k} and M_{R_j} has at least six quadrilaterals. The construction of each of the maps from L_k and R_j involves the addition of two faces; therefore, each must contain four quadrilaterals of the original map. Since any map of connectivity 2 can be decomposed in this manner the proposition is established.

Corollary. *Let M be a bipartite plane cubic map with fewer than eight quadrilaterals. Then M is 3-connected.*

In light of the decomposition process the following proposition is clear.

Proposition 7. *Let M be a bipartite plane cubic map of connectivity* 2. *Suppose M is decomposed into the maps M_{L_k} and M_{R_j}. Then M is Hamiltonian if and only if M_{L_k} and M_{R_j} each have a Hamiltonian cycle passing through the respective edges x_ky_k and w_jz_j.*

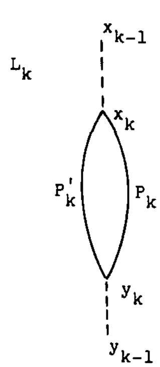

Fig. 5.

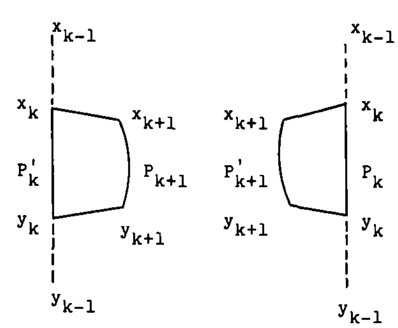

Fig. 6.

Clearly, for M to be non-Hamiltonian either there is no Hamiltonian cycle in M_{L_k} passing through $x_k y_k$, or there is no Hamiltonian cycle in M_{R_j} passing through $w_j z_j$. In light of Proposition 2, it is clear that the smallest non-Hamiltonian bipartite plane cubic map of connectivity 2 must decompose into two maps M_{L_0} and M_{R_0} where one is the map N in Fig. 1 and the other is the cube. This map is shown in Fig. 7.

Proposition 8. *Let M be a bipartite plane cubic map of connectivity 2 and having exactly eight quadrilaterals. Then M is Hamiltonian.*

Proof. Since each map in the decomposition of M contains four quadrilaterals of the original map, the two faces formed by adding either $x_k y_k$ or $w_j z_j$ must be quadrilaterals, hence leaving the maps M_{L_k} and M_{R_j} with six each. By Proposition 4 we can find Hamiltonian cycles in these two maps passing through $x_k y_k$ and $w_j z_j$. By Proposition 7 these can be extended to form a Hamiltonian cycle in M.

3. Conclusion

In working on the conjecture of Barnette (see [6]) that every 3-connected bipartite plane cubic map is Hamiltonian, one soon finds that the ability to find Hamiltonian cycles through specified edges in such maps is of primary importance. Indeed, it is fairly easy to show that the notions are equivalent (see [4, p. 217]). Also, in searching for an inductive procedure to prove the conjecture it is difficult to avoid maps having connectivity 2. In [4] the author proves that there is a Hamiltonian cycle through any specified edge in a bipartite plane cubic map having exactly six quadrilaterals. The proof depends on the decomposition process described here for maps of connectivity 2.

Finally, notice that the map in Fig. 7 has twelve quadrilaterals. Considering the nature of non-Hamiltonian plane cubic maps which have been found in the past, the following conjecture is not unreasonable.

Conjecture 1. *Every bipartite plane cubic map having connectivity 2 and fewer than twelve quadrilaterals is Hamiltonian.*

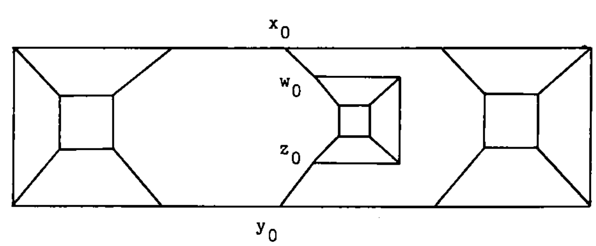

Fig. 7.

If we combine this with Barnette's conjecture, we have the following stronger conjecture.

Conjecture 2. *Every bipartite plane cubic map with fewer than twelve quadrilaterals is Hamiltonian.*

The proof of Conjecture 2 in the nine quadrilateral case depends on the ability to find a Hamiltonian cycle through any specified edge in a bipartite plane cubic map having exactly seven quadrilaterals. As yet, this is unproved.

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