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Fermions, wigs, and attractors

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ABSTRACT

We compute the modifications to the attractor mechanism due to fermionic corrections. In $\mathcal{N}=2$, $D=4$ supergravity, at the fourth order, we find terms giving rise to new contributions to the horizon values of the scalar fields of the vector multiplets.

1. Introduction

The remarkable Schwarzschild solution to Einstein equations is the first example of exact solution in general relativity. Since then, several interesting solutions have been constructed with different properties, and a number of theorems for black hole geometries have been proved. The search for new solutions lived a new *Renaissance* with the discovery of supergravity: within this theory, Einstein equations are just a sector of a broader framework, containing fermions and new matter fields. The latter are sources of the gravitational field, but they are not generic since their interactions are controlled by *supersymmetry*. Consequently, for such matter-gravity systems, new (BPS) solutions can be constructed, since second-order partial–differential Einstein equations are replaced by first-order ones, thus easier to solve. In that context, the solution to supergravity equations of motion is generically constructed *by setting to zero all fermions*, while the bosonic fields acquire non-vanishing v.e.v.'s.

For extremal black hole solutions, the *attractor mechanism* [1] has been discovered; essentially, it states that the solution computed at the horizon depends only upon the conserved charges of the system, and it is independent of the value of the matter fields at infinity. This is related to the *no-hair theorem*, under which, for example, a BPS black hole solution depends only upon its mass, its angular momentum and other conserved charges. At the dawn of these studies, some authors [2] posed the question whether the attractor mechanism has to be modified in the presence of fermions. Their conclusion was that, at the level of approximation of their computations, in the case of double-extremal BPS solutions, the mechanism is unchanged. At the same time, [3] investigated a similar issue for $\mathcal{N}=2$, $D=5$ AdS-black holes, and they found that the solution, as well as its asymptotic charges, is modified at the first order due to fermionic contributions (even though they did not study the attractor mechanism nor its possible modifications).

All these studies followed the seminal paper by Aichelburg and Embacher [4], in which they started from a $\mathcal{N}=2$, $D=4$ asymptotically flat black hole solution and computed iteratively the supersymmetric variations of the background in terms of the flat-space Killing spinors. Due to the Grassmannian nature of the fermions, this procedure ends up after a finite number of iterations, and the complete solution can be constructed. In terms of the latter, the modifications to the asymptotic charges were computed in [4]. However, once again, the attractor mechanism was not investigated.

Recently, some of the authors of the present investigation addressed the same question starting from a different perspective, namely the AdS/CFT correspondence between AdS black holes and strongly-interacting fluids on the AdS boundary. They provided the complete fermionic solution (*wig*) to non-extremal black holes in several dimensions [5].

Here, we present a complete computation of the fermionic corrections to static, spherically symmetric, asymptotically flat, dyonic, BPS double-extremal black holes of $\mathcal{N}=2$, $D=4$ supergravity. Differently from [2], we find that the scalar fields acquire a non-trivial

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contribution at the fourth order of the fermionic expansion, leading to a non-trivial modification of the attractor mechanism.

We would like to point out that we compute the wigging by performing a perturbation of the unwigged purely bosonic (double) extremal BPS extremal black hole solution; thus, within this approximation, we consider quantities like the radius of the event horizon unchanged. The complete analysis, including the study of the fully-backreacted wigged black hole metric, will be presented elsewhere [6].

The plan of the paper is as follows.

In Section 2 we introduce the simplest class of models of $\mathcal{N}=2$, $D=4$ Einstein ungauged supergravity coupled to Abelian vector multiplets, namely the so-called *minimally coupled* class.

The wigging correction of all fields in the gravity and vector multiplets is then computed in Section 3, and the modification of the attractor mechanism at the event horizon of the BPS double-extremal black hole solution is derived in Section 4.

Within the aforementioned approximation (i.e., disregarding the backreaction), the simplest example, namely the axion-dilaton model and its wigging, is studied in some detail in Section 5.

The final Section 6 gives an outlook and mentions various further future developments.

2. *Minimally Coupled* Maxwell–Einstein $\mathcal{N}=2$ supergravity

Namely, we consider n Abelian vector multiplets *minimally coupled* to the $\mathcal{N}=2$, $D=4$ gravity multiplet [7], in absence of gauging and hypermultiplets. The complex scalar fields from the vector multiplets coordinatize a class of symmetric special Kähler manifolds, namely the non-compact complex projective spaces $\overline{\mathbb{CP}}^n$, characterized by the vanishing of the so-called C-tensor C_{ijk} of special Kähler geometry (cf. e.g. [8], as well as [9], and references therein). In turn, this implies the Riemann tensor to enjoy the following expression in terms of the metric of the non-linear sigma model ($i=1,\dots,n$):

$$C_{ijk}=0 \quad \Rightarrow \quad R_{i\bar{j}k\bar{l}}=-g_{i\bar{j}}g_{k\bar{l}}-g_{i\bar{l}}g_{k\bar{j}}. \tag{1}$$

At least among the cases with symmetric scalar manifolds, minimally coupled models are the only ones that admit “pure” supergravity by simply setting $n=0$.

By virtue of (1), minimally coupled models exhibit simple properties, allowing for an explicit study of various solutions to the equations of motion.[1]

This class of models can be seen as describing a *multi-dilaton system* [20]; note, however, that they cannot be uplifted to $D=5$ (see e.g. [11]), nor they can be obtained by standard Calabi–Yau compactifications.

The case of only one vector multiplet ($n=1$) corresponds indeed to the so-called *axion-dilaton* system of $\mathcal{N}=2$ supergravity. This will be treated in some detail in Section 5.

Within this class, remarkable simplifications take place in the supersymmetry transformations, which are reported below; the treatment of more general models will be presented elsewhere [6].

3. The wigging

As mentioned above, we consider $\mathcal{N}=2, D=4$ Poincaré supergravity minimally coupled to n Abelian vector multiplets; as notation and conventions, we adopt the ones of [8]. The supersymmetry transformations for fermionic fields are

$$\begin{aligned}
\delta\psi_{A\mu} &= \nabla_\mu\epsilon_A-\frac{1}{4}\left(\partial_i K\bar{\lambda}^{iB}\epsilon_B-\bar{\partial}_{\bar{i}}K\bar{\lambda}^{\bar{i}}_B\epsilon^B\right)\psi_{A\mu}\\
&\quad+\left(A_A{}^{\nu B}g_{\mu\nu}+A'_A{}^{\nu B}\gamma_{\mu\nu}\right)\epsilon_B\\
&\quad+\varepsilon_{AB}T^-_{\mu\nu}\gamma^\nu\epsilon^B,\\
\delta\lambda^{iA} &= \frac{1}{4}\left(\partial_j K\bar{\lambda}^{jB}\epsilon_B-\bar{\partial}_{\bar{j}}K\bar{\lambda}^{\bar{j}}_B\epsilon^B\right)\lambda^{iA}\\
&\quad-\Gamma^i{}_{jk}\bar{\lambda}^{kB}\epsilon_B\lambda^{jA}+i\left(\partial_\mu z^i-\bar{\lambda}^{iB}\psi_{B\mu}\right)\gamma^\mu\epsilon^A\\
&\quad+G^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\varepsilon^{AB}+D^{iAB}\epsilon_B,
\end{aligned} \tag{2}$$

while bosonic fields transform as

$$\begin{aligned}
\delta e^a_\mu &= -i\bar{\psi}_{A\mu}\gamma^a\epsilon^A-i\bar{\psi}^A{}_\mu\gamma^a\epsilon_A,\\
\delta A^\Lambda_\mu &= 2\bar{L}^\Lambda\bar{\psi}_{A\mu}\epsilon_B\varepsilon^{AB}+2L^\Lambda\bar{\psi}^A{}_\mu\epsilon^B\varepsilon_{AB}\\
&\quad+i\left(f^\Lambda_i\bar{\lambda}^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\bar{f}^\Lambda_{\bar{i}}\bar{\lambda}^{\bar{i}}_A\gamma_\mu\epsilon_B\varepsilon^{AB}\right),\\
\delta z^i &= \bar{\lambda}^{iA}\epsilon_A,
\end{aligned} \tag{3}$$

where the auxiliary fields $A_A{}^{\mu B}$, $A'_A{}^{\mu B}$ are defined as

$$\begin{aligned}
A^{\mu}{}_A^B &:= -\frac{i}{4}g_{\bar{k}l}\left(\bar{\lambda}^{\bar{k}}_A\gamma^\mu\lambda^{lB}-\delta^B_A\bar{\lambda}^{\bar{k}}_C\gamma^\mu\lambda^{lC}\right),\\
A'^{\mu}{}_A^B &:= \frac{i}{4}g_{\bar{k}l}\left(\bar{\lambda}^{\bar{k}}_A\gamma^\mu\lambda^{lB}-\frac{1}{2}\delta^B_A\bar{\lambda}^{\bar{k}}_C\gamma^\mu\lambda^{lC}\right),
\end{aligned} \tag{4}$$

and the supercovariant field strength as

$$\widetilde{F}^\Lambda_{\mu\nu}:=\mathcal{F}^\Lambda_{\mu\nu}+L^\Lambda\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB}-if^\Lambda_i\bar{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]}\varepsilon_{AB}+\text{h.c.} \tag{5}$$

From the Vielbein postulate, the $\mathcal{N}=2$ spin connection reads (cf. e.g. [21])

$$\omega^{ab}_\mu=\frac{1}{2}e_{c\mu}\left[\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\right]+K^a{}_\mu{}^b, \tag{6}$$

where $\Omega^{abc}:=e^{\mu a}e^{\nu b}(\partial_\mu e^c_\nu-\partial_\nu e^c_\mu)$ and $K^a{}_\mu{}^b:=-i\bar{\psi}^{[a}_A\gamma^{b]}\psi^A_\mu-i\bar{\psi}^{Aa}\gamma^b\psi_{A\mu}$. For $\overline{\mathbb{CP}}^n$ models, various quantities of special geometry [8] get simplified as follows:

$$\begin{aligned}
T^-_{\mu\nu} &:= 2i(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}L^\Sigma\widetilde{F}^{\Lambda-}_{\mu\nu},\\
T^+_{\mu\nu} &:= 2i(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\bar{L}^\Sigma\widetilde{F}^{\Lambda+}_{\mu\nu},\\
G^{i-}_{\mu\nu} &:= -g^{i\bar{j}}\bar{f}^\Gamma_{\bar{j}}(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\widetilde{F}^{\Lambda-}_{\mu\nu},\\
G^{\bar{i}+}_{\mu\nu} &:= -g^{\bar{i}j}f^\Gamma_j(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\widetilde{F}^{\Lambda+}_{\mu\nu},\\
\mathcal{F}^\Lambda_{\mu\nu} &:= \partial_{[\mu}A^\Lambda_{\nu]},\\
\nabla\epsilon_A &:= \mathrm{d}\epsilon_A-\frac{1}{4}\gamma_{ab}\omega^{ab}\wedge\epsilon_A+\frac{i}{2}Q\wedge\epsilon_A,\\
Q_\mu &:= -\frac{i}{2}\left(\partial_i K\partial_\mu z^i-\bar{\partial}_{\bar{i}}K\partial_\mu\bar{z}^{\bar{i}}\right),\\
D^{iAB} &= 0,
\end{aligned} \tag{7}$$

where ω^{ab} is the spacetime spin connection, Q is the connection of the $U(1)_R$-line bundle, $\omega_A{}^B:=\frac{i}{2}\omega^x(\sigma_x)_A{}^B$ where ω^x is the connection of the (global, in this case) $SU(2)_R$-bundle and σ_x are the $SU(2)$ Pauli matrices. Note also that $\omega^A{}_B:=\varepsilon^{AC}\varepsilon_{DB}\omega_C{}^D$. Furthermore, the (anti)self-dual supercovariant field strength is defined as

$$\mathcal{F}^{\Lambda\pm}_{\mu\nu}:=\frac{1}{2}\left(\mathcal{F}^\Lambda_{\mu\nu}\pm\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma|\Lambda}\right), \tag{8}$$

and the same holds for $\tilde{F}^{\Lambda\pm}_{\mu\nu}$. Note that g is the determinant of the spacetime metric.

[1] For a treatment of the attractor mechanism [1] and marginal stability in extremal black hole solutions of these models, see e.g. [10–12], and references therein. For an analysis of the duality orbits and related *moduli spaces*, cf. [13–15]. These models have also been treated in [16], and more recently in [17].

The following identities of the special geometry of $\overline{\mathbb{CP}}^n$ are used throughout:

$$f_i^\Lambda = \nabla_i L^\Lambda := \left(\partial_i + \frac{1}{2}\partial_i K\right) L^\Lambda,$$
$$L^\Lambda = e^{\frac{K}{2}} X^\Lambda, \qquad \nabla_i f_j^\Lambda = 0,$$
$$\nabla_i \bar{f}_{\bar{j}}^\Lambda = g_{i\bar{j}} \bar{L}^\Lambda, \qquad \bar{\nabla}_{\bar{i}} L^\Lambda = 0,$$
$$\operatorname{Im}\mathcal{N}_{\Lambda\Gamma} f_i^\Lambda L^\Gamma = \operatorname{Im}\mathcal{N}_{\Lambda\Gamma} \bar{f}_{\bar{i}}^\Lambda \bar{L}^\Gamma = 0. \tag{9}$$

We now start with a purely bosonic background: the *double-extremal* (1/2-)BPS black hole. For this solution, the near-horizon conditions [1]

$$\partial_\mu z^i = 0, \qquad G^{i-}_{\mu\nu} = 0, \tag{10}$$

actually hold *all along the scalar flow*. In particular, the scalar fields are *constant* for every value of the radial coordinate r.

In this framework, major simplifications take place in the computations. At the first order, the unique non-trivial variation is given by[2]

$$\left(\delta^{(1)}\psi_{A\mu}\right)\big|_{\text{d.e.}} = \nabla_\mu \epsilon_A + \varepsilon_{AB} T^-_{\mu\nu}\gamma^\nu \epsilon^B, \tag{11}$$

which does not vanish because ϵ_A is an *anti-Killing* spinor [4,2,5]. Moreover, the subscript "d.e." denotes the evaluation on (10), throughout. Exploiting the iteration procedure, we then find that at the next order the bosonic fields are modified as follows:

$$\left(\delta^{(2)} e^a_\mu\right)\big|_{\text{d.e.}} = -i\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\gamma^a \epsilon_A + \text{h.c.},$$
$$\left(\delta^{(2)} A^\Lambda_\mu\right)\big|_{\text{d.e.}} = 2L^\Lambda\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\epsilon^B \varepsilon_{AB} + \text{h.c.} \tag{12}$$

At the third order, the only non-vanishing variations read

$$\left(\delta^{(3)}\psi_{A\mu}\right)\big|_{\text{d.e.}} = \left(\delta^{(2)}\nabla_\mu\right)\epsilon_A + \left(\delta^{(2)} T^-_{\mu\nu}\right)\gamma^\nu \epsilon^B \varepsilon_{AB},$$
$$\left(\delta^{(3)}\bar{\lambda}^{iA}\right)\big|_{\text{d.e.}} = -\left(\delta^{(2)} G^{i-}_{\mu\nu}\right)\bar{\epsilon}_B \varepsilon^{AB}\gamma^{\mu\nu}, \tag{13}$$

where

$$\left(\delta^{(2)} G^{i-}_{\mu\nu}\right)\big|_{\text{d.e.}} = -g^{i\bar{j}} \bar{f}^\Gamma_{\bar{j}} \operatorname{Im}\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(2)}\tilde{F}^{\Lambda-}_{\mu\nu}\right),$$
$$\left(\delta^{(2)}\tilde{F}^\Lambda_{\mu\nu}\right)\big|_{\text{d.e.}} = \left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right) + 2L^\Lambda\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\left(\delta^{(1)}\psi^B_\nu\right)\varepsilon_{AB} + \text{h.c.},$$
$$\left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right)\big|_{\text{d.e.}} = \nabla_{[\mu}\left(\delta^{(2)} A^\Lambda_{\nu]}\right),$$
$$\left(\delta^{(2)} T^-_{\mu\nu}\right)\big|_{\text{d.e.}} = 2i \operatorname{Im}\mathcal{N}_{\Gamma\Lambda} L^\Gamma\left(\delta^{(2)}\tilde{F}^{\Lambda-}_{\mu\nu}\right),$$
$$\begin{aligned}\left(\delta^{(2)}\mathcal{F}^{\Lambda\pm}_{\mu\nu}\right)\big|_{\text{d.e.}} = {} & \frac{1}{2}\left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right) \pm \frac{i}{4}\left(\delta^{(2)}\varepsilon_{\mu\nu\rho\sigma}\right)\mathcal{F}^{\Lambda|\rho\sigma} \\ & \pm \frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\left[g^{\alpha\rho} g^{\beta\sigma}\left(\delta^{(2)}\mathcal{F}^\Lambda_{\alpha\beta}\right)\right. \\ & \left. + 2\left(\delta^{(2)} g^{\alpha\rho}\right) g^{\beta\sigma}\mathcal{F}^\Lambda_{\alpha\beta}\right],\end{aligned} \tag{14}$$

and the same result is obtained for $\tilde{F}^{\Lambda\pm}_{\mu\nu}$.

4. Modification of the attractor mechanism

By proceeding further with the iteration, one finds that the most relevant contribution to the variation takes place at the fourth order, at which a non-vanishing contribution to the variation of the scalar fields is firstly observed. Thus, the scalar fields get affected by the wigging at the fourth order in the anti-Killing spinors, even on the simplest background, namely in the case of double-extremal BPS black hole:

$$\begin{aligned}\left(\delta^{(4)} A^\Lambda_\mu\right)\big|_{\text{d.e.}} = {} & 2\bar{L}^\Lambda\left(\delta^{(3)}\bar{\psi}_{A\mu}\right)\epsilon_B \varepsilon^{AB} \\ & + i f^\Lambda_i\left(\delta^{(3)}\bar{\lambda}^{iA}\right)\gamma_\mu \epsilon^B \varepsilon_{AB} + \text{h.c.},\end{aligned}$$
$$\left(\delta^{(4)} e^a_\mu\right)\big|_{\text{d.e.}} = -i\left(\delta^{(3)}\bar{\psi}^A_\mu\right)\gamma^a \epsilon_A + \text{h.c.} \tag{15}$$

By a long but straightforward algebra, the computation of the fourth-order variation of the scalar fields can be computed to read:

$$\left(\delta^{(4)} z^i\right)\big|_{\text{d.e.}} = \left(\delta^{(4)} z^i_\nabla\right)\big|_{\text{d.e.}} + \left(\delta^{(4)} z^i_T\right)\big|_{\text{d.e.}},$$

where we separated two contributions: the one from the spinor covariant derivative and the one from the graviphoton field-strength

$$\begin{aligned}\left(\delta^{(4)} z^i_\nabla\right)\big|_{\text{d.e.}} := {} & g^{i\bar{j}} \bar{f}^\Gamma_{\bar{j}} \operatorname{Im}\mathcal{N}_{\Gamma\Lambda}\left(\bar{\epsilon}_C \gamma^{\mu\nu}\epsilon_D\right)\varepsilon^{DC} \\ & \times \left\{\frac{1}{4} R^-_{\mu\nu ab} L^\Lambda\left(\bar{\epsilon}^A \gamma^{ab}\epsilon^B\right)\varepsilon_{AB}\right. \\ & - \frac{1}{2} F^{\rho\sigma|\Lambda}\varepsilon_{abcd}\left[\left(\nabla_\mu \bar{\epsilon}_A \gamma^a \epsilon^A\right) e^b_\nu e^c_\rho e^d_\sigma\right. \\ & \left. + e^a_\mu e^\nu_\nu\left(\nabla_\rho \bar{\epsilon}_A \gamma^c \epsilon^A\right) e^d_\sigma + \text{h.c.}\right] \\ & \left. + F^\Lambda_{\alpha\beta}\varepsilon_{\mu\nu\rho}{}^\beta\left(\nabla_\lambda \bar{\epsilon}_A \gamma^c \epsilon^A + \text{h.c.}\right) g^{\lambda(\rho} e^{\alpha)}_c\right\},\end{aligned} \tag{16}$$

$$\begin{aligned}\left(\delta^{(4)} z^i_T\right)\big|_{\text{d.e.}} := {} & g^{i\bar{j}} \bar{f}^\Gamma_{\bar{j}} \operatorname{Im}\mathcal{N}_{\Gamma\Lambda}\left(\bar{\epsilon}_C \gamma^{\mu\nu}\epsilon_D \varepsilon^{DC}\right) \\ & \times \left\{2L^\Lambda\left[T^-_{\rho[\nu}\left(\nabla_{\mu]}\bar{\epsilon}_A \gamma^\rho \epsilon^A\right) + T^-_{\rho[\nu}\left(\nabla_{\mu]}\bar{\epsilon}^A \gamma^\rho \epsilon_A\right)\right.\right. \\ & \left. + \varepsilon^{AB} T^-_{\rho[\mu} T^-_{\nu]\sigma}\left(\bar{\epsilon}_A \gamma^{\rho\sigma}\epsilon_B\right)\right]^- \\ & - \frac{1}{2} F^{\rho\sigma|\Lambda}\varepsilon_{\rho\nu\omega\sigma} T^-_{\lambda\mu}\left[\varepsilon_{AB}\left(\bar{\epsilon}^A \gamma^{\lambda\omega}\epsilon^B\right) + \text{h.c.}\right] \\ & \left. + \frac{1}{2} F^\Lambda_{\rho\sigma}\varepsilon_{\mu\nu}{}^{\sigma\lambda} T^-_{\lambda\omega}\left[\varepsilon_{AB}\left(\bar{\epsilon}^A \gamma^{\rho\omega}\epsilon^B\right) + \text{h.c.}\right]\right\},\end{aligned} \tag{17}$$

where we defined

$$R^-_{\mu\nu ab} := \frac{1}{2}\left(R_{\mu\nu ab} - \frac{i}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma ab}\right). \tag{18}$$

Since

$$\left(\delta^{(1)} z^i\right)\big|_{\text{d.e.}} = \left(\delta^{(2)} z^i\right)\big|_{\text{d.e.}} = \left(\delta^{(3)} z^i\right)\big|_{\text{d.e.}} = 0, \tag{19}$$

it thus follows that the complete fermionic wig of the n complex scalar fields z^i in the background of a double-extremal 1/2-BPS black hole in $\mathcal{N}=2$, $D=4$ *minimally coupled* supergravity reads (in absence of gauging and hypermultiplets):

$$z^i_{WIG}\big|_{\text{d.e.}} := z^i_{(0)}\big|_{\text{d.e.}} + \frac{1}{4!}\left(\delta^{(4)} z^i\right)\big|_{\text{d.e.}} \neq z^i_{(0)}\big|_{\text{d.e.}}, \tag{20}$$

where $z^i_{(0)}|_{\text{d.e.}}$ denotes the "*unwigged*", near-horizon value of the scalar fields; according to the attractor mechanism [1], the latter depends only on the electric and magnetic charges of the black hole (for a detailed treatment, see [11], and references therein).

5. Axion-dilaton model

As an illustrative example, we analyze the simplest case within *minimally coupled* $\mathcal{N}=2$ supergravity, namely the $\overline{\mathbb{CP}}^1$ model, with only one vector multiplet (containing one complex scalar field z) coupled to the gravity multiplet.

In this case, we find convenient to consider the symplectic frame specified by the holomorphic prepotential

[2] Note that from now on the r.h.s. is intended evaluated on the background (10).

$$F := -iX^0X^1, \tag{21}$$

which arises out by suitably truncating the $\mathcal{N}=4$ "pure" theory (see e.g. the discussion in [20,12]), and it determines the following Kähler potential (cf. e.g. [19]):

$$K = -\ln\left[2(z+\bar{z})\right], \tag{22}$$

from which the metric function is derived:

$$g_{1\bar{1}} = \left(g^{1\bar{1}}\right)^{-1} = \frac{1}{(z+\bar{z})^2}. \tag{23}$$

In special coordinates, after a Kähler gauge-fixing ($\Lambda = 0, 1$ throughout the present section):

$$X^\Lambda = (1, z), \tag{24}$$

and then one can derive the covariantly holomorphic symplectic sections of special geometry:

$$L^\Lambda := e^{\frac{K}{2}} X^\Lambda = \frac{1}{\sqrt{2(z+\bar{z})}}(1, z), \tag{25}$$

$$M_\Lambda := \mathcal{N}_{\Lambda\Sigma} L^\Sigma = -i\frac{1}{\sqrt{2(z+\bar{z})}}(z, 1), \tag{26}$$

and their Kähler-covariant derivatives

$$f^\Lambda := \left(\partial_z + \frac{1}{2}\partial_z K\right) L^\Lambda = \frac{1}{\sqrt{2}(z+\bar{z})^{3/2}}(-1, \bar{z}) \tag{27}$$

(note the suppression of the i-index in f_i^Λ, due to the presence of only one scalar field).

In a symplectic frame defined by a prepotential F, the symmetric complex kinetic matrix of vector fields is defined as (see for instance [18,22], and references therein)

$$\mathcal{N}_{\Lambda\Sigma} := \bar{F}_{\Lambda\Sigma} - 2i\bar{T}_\Lambda\bar{T}_\Sigma\left(L^\Gamma \operatorname{Im} F_{\Gamma\Xi} L^\Xi\right), \tag{28}$$

$$F_{\Lambda\Sigma} := \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}, \tag{29}$$

$$T_\Lambda := -i\frac{\operatorname{Im} F_{\Lambda\Xi}\bar{L}^\Xi}{\bar{L}^\Gamma \operatorname{Im} F_{\Gamma\Sigma}\bar{L}^\Sigma}. \tag{30}$$

In the case under consideration, the 2×2 kinetic vector matrix reads

$$\mathcal{N}_{\Lambda\Sigma} = -i\,\mathrm{diag}\left(z, \frac{1}{z}\right), \tag{31}$$

thus yielding

$$\operatorname{Im}\mathcal{N}_{\Lambda\Sigma} = -\frac{z+\bar{z}}{2}\,\mathrm{diag}\left(1, \frac{1}{|z|^2}\right), \tag{32}$$

$$\operatorname{Re}\mathcal{N}_{\Lambda\Sigma} = \frac{z-\bar{z}}{2i}\,\mathrm{diag}\left(1, -\frac{1}{|z|^2}\right). \tag{33}$$

5.1. Double-extremal black hole

We are now going to derive the explicit values for the various fields in our configuration. We will be dealing with an asymptotically flat, static, spherically symmetric, dyonic 1/2-BPS *double-extremal* black hole, with z constant for every value of the radial coordinate r. Following the conventions of [19], we consider a dyonic black hole metric[3]

$$ds^2 = \left(1+\frac{M}{r}\right)^{-2} dt^2 - \left(1+\frac{M}{r}\right)^2\left(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right), \tag{34}$$

with gauge field strengths given by

$$\mathcal{F}^\Lambda = \left(1+\frac{M}{r}\right)^{-2}\frac{2Q^\Lambda}{r^2}\,dt\wedge dr - 2P^\Lambda \sin\theta\,d\theta\wedge d\phi, \tag{35}$$

where Q^Λ and P^Λ are symplectic, z-dependent, real quantities, defined by [19]

$$\begin{pmatrix} P^\Lambda \\ Q^\Lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p^\Lambda \\ (\operatorname{Im}\mathcal{N}^{-1}\operatorname{Re}\mathcal{N}p)^\Lambda - (\operatorname{Im}\mathcal{N}^{-1}q)^\Lambda \end{pmatrix}. \tag{36}$$

As showed in the third reference of [1], in order to have a supersymmetric attractor solution one must require that $G^{i-}_{\mu\nu} = 0$ on the horizon; such a requirement constrains the scalar z to be a function only of the electric (q_Λ) and magnetic (p^Λ) charges of the black hole. Starting from (22), it turns out that the value of the constant scalar is fixed to be

$$z^{(0)}\Big|_{\text{d.e.}} = \frac{q_0 - ip^1}{q_1 - ip^0} = \frac{Q^1 - iP^1}{Q^0 - iP^0}\bigg|_{\text{d.e.}}, \tag{37}$$

where in the last step the inverse of (36), namely [19]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \begin{pmatrix} 2P^\Lambda \\ 2\operatorname{Re}\mathcal{N}_{\Lambda\Sigma}P^\Sigma - 2\operatorname{Im}\mathcal{N}_{\Lambda\Sigma}Q^\Sigma \end{pmatrix}, \tag{38}$$

has been exploited. Note that $z^{(0)}|_{\text{d.e.}}$ given by (37) expresses the value of the scalar field z at the *zeroth* order.

By setting $z = z^{(0)}|_{\text{d.e.}}$, one gets $G^{i-}_{\mu\nu} = 0$, and the BPS bound is saturated [19]:

$$M^2 = -2\left[\operatorname{Im}\mathcal{N}_{\Gamma\Lambda}\left(Q^\Gamma Q^\Lambda + P^\Gamma P^\Lambda\right)\right]_{\text{d.e.}} = |Z|^2_{\text{d.e.}} = q_0q_1 + p^0p^1 = \frac{A_{H(0)}}{4} = \frac{S_{BH(0)}}{\pi}, \tag{39}$$

where Z is the $\mathcal{N}=2$ central charge function:

$$Z := L^\Lambda q_\Lambda - M_\Lambda p^\Lambda, \tag{40}$$

and $A_{H(0)}$ and $S_{BH(0)}$ respectively denote the horizon area and the Bekenstein–Hawking entropy of the black hole at the *zeroth order* (recall the comment at the end of Section 1).

In order for the scalar to be fixed at order n (in particular $n = 4$), one has to require the vanishing of the supersymmetry variation $\delta^{(n-2)}G^{i-}_{\mu\nu}$. As shown below, due to the presence of a gauge field variation, this is true only up to $n = 3$.

5.2. Fourth order scalar variation

We now proceed to computing the fourth order variation of the scalar field z in the double-extremal BPS axion-dilaton background specified by (34), (35) and (37), as described in Section 5.1.

We start and recall the Minkowski–Killing spinors in spherical coordinates:

$$\epsilon_A = \left[\cos\frac{\theta}{2}\left(\sin\frac{\phi}{2}\mathbb{1}_4 + \cos\frac{\phi}{2}\gamma_{23}\right) + \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2}\gamma_{13} - \sin\frac{\phi}{2}\gamma_{12}\right)\right]\zeta_A, \tag{41}$$

$$\epsilon^A = \left[\cos\frac{\theta}{2}\left(\sin\frac{\phi}{2}\mathbb{1}_4 + \cos\frac{\phi}{2}\gamma_{23}\right) + \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2}\gamma_{13} - \sin\frac{\phi}{2}\gamma_{12}\right)\right]\zeta^A, \tag{42}$$

[3] This metric is of Papetrou–Majumdar form, thus the radius of the event horizon is located at $r = r_H = 0$.

where $\zeta_1 = \frac{(1+\gamma_5)}{2}\mathbf{1}$, $\zeta_2 = \frac{(1+\gamma_5)}{2}\mathbf{2}$, $\zeta^1 = \frac{(1-\gamma_5)}{2}\mathbf{1}$, $\zeta^2 = \frac{(1-\gamma_5)}{2}\mathbf{2}$ and $\mathbf{1}, \mathbf{2}$ are Majorana spinors defined as

$$\mathbf{1} = \{a_1, a_2, -a_2^*, -a_1^*\}, \qquad \mathbf{2} = \{b_1, b_2, -b_2^*, -b_1^*\}, \tag{43}$$

with a, b denoting *constant* complex Grassmannian numbers.

As mentioned above, the non-vanishing variation for the scalar field z is induced by the correction that $G^{i-}_{\mu\nu}$ *does acquire* at the second order. In fact, one achieves the following result:

$$\left(\delta^{(4)} z^i\right)\big|_{\text{d.e.}} = \left(\delta^{(3)} \lambda^{\bar{i}A}\right)\big|_{\text{d.e.}} \epsilon_A = -\left(\delta^{(2)} G^{i-}_{\mu\nu}\right) \gamma^{\mu\nu}\big|_{\text{d.e.}} \bar{\epsilon}_B \epsilon_A \varepsilon^{AB}, \tag{44}$$

(cf. Section 3 for the explicit variation of the fields); we should note that we exploited the special geometry identity (see e.g. [8])

$$\operatorname{Im} \mathcal{N}_{\Lambda\Sigma} \bar{f}^{\Lambda}_{\bar{j}} \bar{L}^{\Sigma} = 0. \tag{45}$$

5.3. Final result

By recalling the results (16) and (17), the value of the scalar field at the fourth order (37) yields

$$\left(\delta^{(4)} z\right)\big|_{\text{d.e.}} = \left(\delta^{(4)} z_T\right)\big|_{\text{d.e.}}. \tag{46}$$

It should be stressed that, upon acting with all vacuum super-isometries as supersymmetry parameters, $(\delta^{(4)} z)|_{\text{d.e.}}$ acquires a dependence also on the *unbroken* super-isometries. This redundance can be eliminated by a gauge choice on the gravitino field, in order to work with a "pure" anti-Killing spinor with 4 (complex) degrees of freedom. In order to highlight their contribution, we redefine the *constant* Minkowski–Killing spinor zero modes as follows:

$$\begin{aligned} A &:= a_1 + i b_1, \qquad B := b_2 - i a_2, \\ C &:= a_1^* + i b_1^*, \qquad D := b_2^* - i a_2^*. \end{aligned} \tag{47}$$

Such a redefinition allow us to work only with A, B, C and D, since these are the only generators for the black hole wig itself (their complex conjugates are the zero modes for the black hole Killing spinors). Using these variables, we finally achieve the result

$$\begin{aligned} \left(\delta^{(4)} z\right)\big|_{\text{d.e.}} = {} & \frac{M^4}{4(M+r)^4} \\ & \times \left[\frac{P^0 Q^1 - P^1 Q^0}{(P^0 + iQ^0)^2 (Q^0 + iP^0)(P^1 - iQ^1)}\right]_{\text{d.e.}} \\ & \times \mathcal{Q} \sin^2\phi \sin^2\theta \end{aligned} \tag{48}$$

$$\begin{aligned} = {} & \frac{M^4}{(M+r)^4} \frac{p^0 q_0 - p^1 q_1}{(p^0 + iq_1)^2 (p^0 - iq_1)(q_0 + ip^1)} \\ & \times \mathcal{Q} \sin^2\phi \sin^2\theta, \end{aligned} \tag{49}$$

within the constraint $q_0 q_1 + p^0 p^1 > 0$ imposed by the saturation of the BPS bound (39). Note that we have introduced the "*quadri-linear*" $\mathcal{Q} := ABCD$, and Eqs. (36) and (37) have been used. Also, for $M = 0$ the result (49) vanishes, as expected.

By evaluating the expression (49) on the event horizon $r = r_H = 0$ of the bosonic solution (34) (denoted by the subscript "d.e.h."; recall the comment at the end of Section 1), one obtains

$$\left(\delta^{(4)} z\right)\big|_{\text{d.e.h.}} = \frac{p^0 q_0 - p^1 q_1}{(p^0 + iq_1)^2 (p^0 - iq_1)(q_0 + ip^1)} \mathcal{Q} \sin^2\phi \sin^2\theta. \tag{50}$$

As resulting from (50) and (20), in the *near-horizon* background of a double-extremal BPS axion-dilaton black hole, upon performing (the near-horizon limit of) a finite supersymmetry transformation, the axion-dilaton z *is not constant any more*, but acquires a *dependence on the angles* ϕ and θ.

Nevertheless, for $M \neq 0$, one can single out *at least* three peculiar charge configurations in which z *does* remain fixed, and given by (37), i.e. in which[4] $(\delta^{(4)} z)|_{\text{d.e.}} = 0 = (\delta^{(4)} z)|_{\text{d.e.h.}}$:

$$\left.\begin{array}{l} \textbf{I}.\ p^0 = p^1 = 0 \Longrightarrow z^{(0)}\big|_{\text{d.e.}} = q_0/q_1; \\ \textbf{II}.\ q_0 = q_1 = 0 \Longrightarrow z^{(0)}\big|_{\text{d.e.}} = p^1/p^0; \\ \textbf{III}.\ p^1/p^0 = q_1/q_0. \end{array}\right\}$$
$$\Rightarrow \quad z_{WIG}\big|_{\text{d.e.}} = z_{WIG}\big|_{\text{d.e.h.}} = z^{(0)}\big|_{\text{d.e.h.}} = z^{(0)}\big|_{\text{d.e.}} \tag{51}$$

6. Conclusions

Eq. (20), with $(\delta^{(4)} z^i)|_{\text{d.e.}}$ given by the results (48)–(49) and (17), expresses how the value of the axion-dilaton gets *modified by the fermionic wig along the radial flow* in the background of a bosonic BPS double extremal black hole of $\mathcal{N} = 2$ supergravity.

In particular, its near-horizon limit, in which the expressions (48)–(49) are replaced by (50), yields that the *attractor mechanism gets modified by the fermionic wig*. It is therefore the first evidence – in the simplest case provided by the (double extremal) axion-dilaton black hole, of what we dub the "*fermionic-wigged*" *attractor mechanism*: the *fermionic-wigged* value, depending on the "quadri-linear" $\mathcal{Q}$ as well as on the angles ϕ and θ, of the scalar fields in the near-horizon geometry of the double-extremal 1/2-BPS black hole is *different* from the corresponding, purely charge–dependent, horizon attractor value at the *zeroth* order.

We would like to stress once again that we adopted the approximation of computing the fermionic wig by performing a perturbation of the unwigged, purely bosonic (double) extremal BPS extremal black hole solution; thus, within this approximation, we consider quantities like the radius of the event horizon unchanged.

We leave to further future work [6] the complete analysis of the fully-backreacted wigged black hole solution, also including the study of its thermodynamical properties, and the computation of its Bekenstein–Hawking entropy; this may be done also in the non-supersymmetric (non-BPS) case.

Our analysis may also be applied to higher dimensions, as well as to extended supergravities.

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[4] Note that (49) and its horizon limit (50) only differ by the r-dependent prefactor $M^4/(M+r)^4$. Furthermore, $z^{(0)}|_{\text{d.e.h.}} = z^{(0)}|_{\text{d.e.}}$, because we are considering a *double-extremal* bosonic solution (34).

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