A LIOUVILLE THEOREM FOR THE CRITICAL GENERALIZED KORTEweg–DE VRIES EQUATION

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ABSTRACT. – We prove in this paper a rigidity theorem on the flow of the critical generalized Korteweg–de Vries equation close to a soliton up to scaling and translation. To prove this result we introduce new tools to understand nonlinear phenomenon. This will give a result of asymptotic completeness. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

We consider in this paper

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u_{xx} + u^2)_x &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

for \(u_0 \in H^1(\mathbb{R})\). This model is called the critical generalized Korteweg–de Vries equation.

Indeed, let us consider the generalized KdV equation, for any integer \(p > 1\),

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u_{xx} + u^p)_x &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

This kind of problem appears in Physics, for example in the study of waves on shallow water (see Korteweg and de Vries [12]).

Local existence in time of solutions of (2) is now well understood: see Kato [10], Ginibre and Tsutsumi [8], for the \(H^s\) theory (\(s > 3/2\)), Kenig, Ponce and Vega [11], for the \(L^2\) theory in the case of equation (1), and sharp \(H^s\) theory for (2), and Bourgain [3], for the periodic case.

In particular, from [11], we have the following existence and uniqueness result in \(H^1(\mathbb{R})\): for \(u_0 \in H^1(\mathbb{R})\), there exists \(T > 0\) and a unique maximal solution \(u \in C([0, T), H^1(\mathbb{R}))\) of (2) on \([0, T)\). Moreover, either \(T = +\infty\), or \(T < +\infty\), and then \(|u(t)|_{H^1} \to +\infty\), as \(t \uparrow T\) (see Corollary 2.11, and Corollary 2.12 in [11] for the fact that \(|u(t)|_{H^1} \to +\infty\), as \(t \uparrow T\)). In addition, we have, for all \(t \in [0, T)\),

\[
\int u^2(t) = \int u_0^2.
\]
Note that existence of singularity in finite time for $u$ (i.e. $T < +\infty$) in the space $H^1$ is still an open problem. In fact, almost no qualitative results are available for problem (1).

In equation (2), special type of solutions, called solitons, play a crucial role. Indeed, there exist solutions of (2) of the form:

$$u(t, x) = R_c(x - ct),$$

where $c > 0$ and $R_c$ satisfies the following equation

$$R_c \in H^1(\mathbb{R}), \quad R_{cxx} + R_c^p = c R_c,$$

or equivalently, by integration

$$R_c^2 + \frac{2}{p+1} R_c^{p+1} = c R_c^2.$$ 

An explicit expression for $R_c$ is available

$$R_c(x) = \left( \frac{c(p+1)}{2c^2 \left( \frac{p-1}{2} \right)^{1/(p-1)}} \right)^{1/(p-1)}.$$

(Note that $R_c(x) > 0$, $\forall x > 0$.)

Note that the flow of equation (1) close to $R_c$ (at least generically) should make precise the flow for all initial data in $H^1$. Indeed, one can expect that in the case of a global solution $u(t)$,

$$u(t) \sim \sum_i R_c \left( x - x_i(t) \right) + u_R,$$

as $t \to +\infty$, where $u_R$ is a dispersive part, $|u_R(t)|_{L^\infty} \to 0$, as $t \to +\infty$, and $0 < C_1 < c_i < C_2$; in the case of initial data $u_0$ such that $|u_0|_{L^2}$ is of order $|R_1|_{L^2}$, we have:

$$u(t) \sim R_c(x - x(t)) + u_R.$$ 

In the case of a solution blowing up at $t = 0$ (with one blow up point), one can conjecture

$$u(t) \sim u^*(x) + R_c(t) (x - x(t)),$$

or

$$u(t) \sim u^*(x) + \frac{1}{c(t)} \left( \frac{x - x(t)}{c^{1/2}(t)} \right),$$

for $|g|_{L^2} \geq |R_1|_{L^2}$, where $c(t) \to 0$, as $t \to 0$, by scaling property. In this case, $u^* \in H^1$ and corresponds to the regular part of the solution, and $R_c(t)$ is the singular part which concentrates in one point a certain part of the $L^2$ mass. It is an open problem to understand this phenomenon.

In the so-called subcritical case: $p < 5$, the Cauchy problem for (2) is globally well posed in time and it follows for energetic arguments that the solitons are $H^1$ stable (see Cazenave and Lions [4], Weinstein [24], and Bona, Souganidis and Strauss [2]).
In the supercritical case $p > 5$, numerical simulations (see Bona et al. [1] and references therein) suggest that blow up in finite time occurs for some initial data. However, no rigorous proof of existence of such solutions exists. In [2], for $p > 5$ (the supercritical case), Bona, Souganidis, and Strauss prove, using Grillakis, Shatah, and Strauss [9] type arguments, the $H^1$ instability of solitons.

For $p = 5$ (called the critical case), the problem becomes degenerated and presents a lot of similarities with the critical nonlinear Schrödinger equation (NLSE):

\begin{equation}
\begin{aligned}
    iu_t &= -u_{xx} - |u|^4 u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\end{equation}

(See Merle [17], and the references therein.) In this case, from a scaling invariance, for all $c > 0$, we have $|R_c|_{L^2} = |Q|_{L^2}$, where $Q(x) = R_1(x) = 3^{1/4}/\cosh^{1/2}(2x)$ (called the ground state) is the solution of

\[ Q_{xx} + Q^5 = Q. \]

From Weinstein [24], we have the following Gagliardo–Nirenberg inequality:

\begin{equation}
\forall v \in H^1(\mathbb{R}), \quad \frac{1}{6} \int v^6 \leq \frac{1}{2} \left( \int \frac{v^2}{Q^2} \right)^2 \int v_x^2.
\end{equation}

Note that the constant is optimal in (7) since the Pohozaev identity yields:

\[ E(Q) = 0, \quad \text{where } E(v) = \frac{1}{2} \int v_x^2 - \frac{1}{6} \int v^6. \]

If $u_0$ is such that $|u_0|_{L^2} < |Q|_{L^2}$, then for all $0 \leq t < T$,

\[ \frac{1}{2} \left( 1 - \left( \frac{u_0^2}{Q^2} \right)^2 \right) \int u_x^2(t) \leq E_0, \]

where $E_0 = E(u_0)$, and then $u(t)$ is globally defined.

It is conjectured that there exist blow up solutions such that $|u_0|_{L^2} \geq |Q|_{L^2}$ (see numerical simulations in Bona et al. [1]). However, unlike NLSE, there is no conformal invariance or variance identity with constant sign which would allow us to have explicit blow up solutions. Indeed, for NLSE (6), if $u(t, x)$ is a solution then

\[ \frac{1}{t^{1/2}} e^{\frac{x^2}{4t}} \left( \frac{1}{t} \right)^\frac{x}{t} \]

is also a solution, which blows up at $t = 0$. Moreover, since $e^{ict} R_c(x)$ is a solution of (6), applying the conformal transformation,

\[ S(t, x) = \frac{1}{t^{1/2}} e^{-ic/t + ix^2/4t} R_c \left( \frac{x}{t} \right) \]

is also a solution. For more details on this approach, see for example Merle [17], and the references therein.
Note that if we set, for any $t_0 \in \mathbb{R}$, $t_0 \neq 0$:  
\[ S_{t_0}(x) = e^{1/2}(t_0)S(t_0, xc(t_0)) = e^{-ic/t_0 + inx^2/2} R_c(x), \]
with $c(t) = t$, then  
\[ S_{t_0}(t, x) = e^{1/2}(t_0)S(t_0 + t^2(t_0), xc(t_0)) \]
is the solution of (6) with $S_{t_0}$ as initial data, by scaling invariance of the equation.

Therefore, since for any $t_0 \neq 0$, $S_{t_0}(t, x)$ blows up in finite time, and since $t_0 \to 0$, $S_{t_0}(x) \to R_c(x)$ (up to phase), it follows that $R_c$ is unstable. Thus, we see on this example that the understanding of the flow close to $R_c$ in a certain sense is crucial in the understanding of the singularity formation.

Note that for the KdV equation, by energy arguments (similar to the ones for the NLSE), we already know that if blow up in $H^1$ occurs then we can at least show a result of concentration of $L^2$ norm at the blow up time:
\[ \exists x(t), \text{ such that } \forall R > 0, \lim_{t \to T} \int_{|x-x(t)| \leq R} u^2(t, x) \, dx \geq \int Q^2. \]

On the other hand, a first result in the direction of blow up for (1) has been established in Martel and Merle [14], by showing that $u(t, x) = Q(x-t)$ is unstable solution with an understanding of how and when the instability phenomenon occurs (note that the instability result of Bona et al. [2] does not apply to the critical case). This result strongly suggests the existence of blow up solutions close to $Q$.

This result was established in a qualitative way, finding the interior of a parabola as instability region. It was a consequence of a Viriel type identity, energy arguments, and a property of decay of the linearized flow around $Q$.

Therefore, in order to understand the blow up problem, it is crucial to understand the structure of the equation close to $Q$, and in a neighborhood of:
\[ Q_{c_1, c_2}^* = \{ c^{1/4}Q(\sqrt{c}(x-x_0)), c \in (c_1, c_2), x_0 \in \mathbb{R} \}. \]

In this paper, we establish some crucial properties in this direction, introducing new techniques, in particular, a surprising rigidity theorem of equation (1) close to $Q_{c_1, c_2}^*$. This will give a result of asymptotic stability of $Q_{c_1, c_2}^*$.

Classification of entire PDE has been established for some problems. In elliptic problems, the moving plane technique (based on maximum principle) has been applied successfully to find all positive solutions of problems of the type:
\[ \Delta u + u^p = 0, \quad u > 0, \quad x \in \mathbb{R}^n. \]
(See Gidas, Ni and Nirenberg [6], Gigas and Spruck [7].)

In the parabolic situation, for the blow up solution of
\[ u_t = \Delta u + \left| u \right|^{p-1} u, \]
where $u : \mathbb{R}^N \to \mathbb{R}^M$, a Liouville theorem (based on blow up argument) has been established by Merle and Zaag [18]. For Hamiltonian system, no result has been known, except for the KdV equation ($p = 2$), using integrability theorem (see Lax [13]).

We now claim the following:
THEOREM 1 (Liouville property close to \(Q_{c_1,c_2}\)). – Let \(u_0 \in H^1(\mathbb{R})\), and let
\[
\alpha = |u_0 - Q|_{H^1}.
\]
Suppose that the solution \(u(t)\) of (1) is defined for all time \(t \in \mathbb{R}\), and assume that for some \(c_1, c_2 > 0\),
\[
\forall t \in \mathbb{R}, \quad c_1 \leq \|u(t)\|_{H^1} \leq c_2.
\]
Suppose that there exists \(x(t)\) such that
\[
v(t, x) = u\left(t, x + x(t)\right)
\]
satisfies
\[
\forall \varepsilon_0, \exists R_0 > 0, \forall t \in \mathbb{R}, \int_{|x| > R_0} v^2(t, x) \, dx \leq \varepsilon_0.
\]
There exists \(\alpha_0 > 0\), such that if \(\alpha < \alpha_0\), then there exists \(\lambda_0, x_0\) such that:
\[
u(t, x) = \lambda_0^{1/2} Q(\lambda_0(x - x_0) - \lambda_0^3 t).
\]

Remark. – From scaling properties of the equation, it is sufficient to have \(|\lambda_1 u_0(\lambda_1(\cdot - x_1)) - Q|_{H^1}\) small, for some \(\lambda_1 > 0, x_1 \in \mathbb{R}\).
In the case \(E(u_0) < 0\), the smallness condition can be weakened. It is enough to assume \(\int |u_0 - Q|^2 < \alpha_0\) to have the conclusion.

Remark. – We expect the result to be true without the smallness condition, i.e. the only solution of (1) such that there is no dispersion are \(R_{c_0}(x - x_0 - c_0 t), x_0 \in \mathbb{R}, c_0 \in \mathbb{R}\). It is still an open problem. Of course the rigidity is a consequence of the \(L^2\) compactness assumption (9) (note that \(L^2\) is an invariant of the flow). Without assumption (9), any global bounded solution satisfies the properties.

Remark. – The tools developed to prove such a theorem are rather general and are based on a reduction of a nonlinear property to a linear problem. See [15] for application to the subcritical case (2), \(p = 2, 3, 4\).

In some sense, Theorem 1 says that if the solution is not a travelling wave then it has to disperse. In fact from this result, we derive an asymptotic stability property of \(Q\) (and of \(R_c\) by rescaling).

THEOREM 2 (Asymptotic stability of \(Q_{c_1,c_2}\)). – Let \(u_0 \in H^1\), and let
\[
\alpha = |u_0 - Q|_{H^1}.
\]
Suppose that the solution \(u(t)\) of (1) is defined for all \(t \geq 0\) and assume that for some \(c_1, c_2 > 0\),
\[
\forall t \geq 0, \quad c_1 \leq \|u(t)\|_{H^1} \leq c_2.
\]
There exists \(\alpha_1 > 0\), such that if \(\alpha < \alpha_1\), then there exists \(\lambda(t), x(t)\) such that:
\[
\lambda^{1/2}(t)u(t, \lambda(t)(x - x(t))) = Q(x) + u_R(t, x),
\]
where

\[ u_R(t) \to 0 \quad \text{in } H^1 \text{ as } t \to +\infty. \]

It is easy to see that \( \forall t \geq 0, \lambda_1 \leq \lambda(t) \leq \lambda_2 \) and \( x(t) \to +\infty \) as \( t \to +\infty \).

**Remark.** – Note that other results on asymptotic stability for dispersion equations previously obtained rely on decay condition at infinity in \( x \) on the initial data (see [19,21] where in addition spectral assumptions on the linearized operator around \( Q \) are made in some cases). None are proved in the energy space.

**Remark.** – In [15], we prove similar results for the subcritical generalized Korteweg–de Vries equations (2) for \( p = 2, 3, 4 \).

We first use the structure of the equation around \( Q \) (or \( c_1 = 4Qc_1 = 2x \)), which allows us to do explicit calculations on the flow, following the different directions of instability using modulation theory as in [14]. Let us introduce some notation.

Recall that if we set

\[ v(t, y) = \lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) \]

and

\[ \varepsilon(t, y) = v(t, y) - Q(y) = \lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) - Q(y), \]

for \( u \) a solution of (1), and \( \lambda(t) > 0, x(t) \), two \( C^1 \) functions to be chosen later, and if we change the time variable as follows:

\[ s = \int_0^t \frac{dt'}{\lambda^3(t')}, \quad \text{or equivalently, } \frac{ds}{dt} = \frac{1}{\lambda^3}, \]

then \( \varepsilon(s) \) satisfies, for \( s \geq 0, y \in \mathbb{R}, \)

\[ \varepsilon_s = (L\varepsilon)_y + \frac{\lambda_x}{\lambda} \left( \frac{Q}{2} + yQ_y \right) + \left( \frac{x_s}{\lambda} - 1 \right)Q_y \]
\[ + \frac{\lambda_x}{\lambda} \left( \frac{\varepsilon}{2} + y\varepsilon_y \right) + \left( \frac{x_s}{\lambda} - 1 \right)\varepsilon_y - \left( 10Q^3\varepsilon^2 + 10Q^2\varepsilon^3 + 5Q^4 + \varepsilon^5 \right)_y, \]

where

\[ L\varepsilon = -\varepsilon_{xx} + \varepsilon - 5Q^4\varepsilon = -\varepsilon_{xx} + \varepsilon - \frac{15}{\cosh^2(2x)}\varepsilon. \]

(See Lemma 1 in [14].) Recall that \( x(t) \) and \( \lambda(t) \) are geometrical parameters related to the two invariances of equation (1), respectively, translation and dilatation invariances.

If, for all \( t \geq 0, u(t) \) is sufficiently close to \( Q \) in \( H^1 \), up to translation, we can define a unique \( C^1 \) function \( s \to (\lambda(s), x(s)) \) such that

\[ \forall s \geq 0, \quad \int Q^3\varepsilon(s) = \int Q_x\varepsilon(s) = 0. \]

A relation between \( \lambda, x \) and their derivatives and \( \varepsilon \) is given later on (see Lemma 3). Recall also that by the invariances of equation (1), we can assume \( \lambda(0) = 1 \) and \( x(0) = 0 \), so that \( u_0 = Q + \varepsilon(0) \) (see beginning of Section 5 in [14]).
The reason to choose such orthogonality conditions on \(v(s)\) is the fact that by Lemma 2 in [14], we have:

\[
(LQ^3, Q^3) < 0, \quad (LQ_v, Q_v) = 0, \quad \forall \epsilon \in H^1(\mathbb{R}).
\]

if \(\int Q^3 \epsilon = \int Q_v \epsilon = 0\), then \((L\epsilon, \epsilon) \geq (\epsilon, \epsilon)\).

The crucial idea is to use in various ways the fact that the Airy equation (the linear part of the generalized KdV equation) pushes the mass on the left-hand side, and that the nonlinear soliton travels to the right, which means that in some sense, linear and nonlinear effects are decoupled. This idea will appear in different contexts, at different stages of the proof.

The convergence result of Theorem 2 is a consequence of Theorem 1 which gives a classification of entire solutions close to \(Q_{c_1, c_2}\) and satisfying a compactness property in \(L^2\).

Indeed, the idea to reduce the proof of Theorem 2 to the proof of Theorem 1 is first to introduce a quantity which measures the mass of the solution at the left of the soliton. Then, we control the variation in time of this quantity by using the shape of the solution. Finally, we conclude that the asymptotic of the solution as \(t\) goes to \(+\infty\) has to remain \(L^2\) compact, which allows us to use Theorem 1.

Concerning Theorem 1, a surprising fact is that the compactness in \(L^2\) uniform in time (no loss of mass, or no dispersion) yields rigidity. In fact, we prove that this compactness property is a characterization of the soliton \(Q(x - t)\) (up to scaling and translation). Several ingredients are needed in order to prove Theorem 1.

We argue by contradiction. Assume that there exists a sequence \(u_n\) of solutions satisfying the \(H^1\) bound (8), which are \(L^2\) compact, and which satisfy \(|u_n(0) - Q|_{H^1} \to 0\) as \(n \to +\infty\).

The objective of Part A is to obtain a convergence result for the sequence of the corresponding renormalized solutions \(v_n\) of equation (13). In general, for Hamiltonian systems, no easy comparison between different norms is available, as it is the case for the nonlinear heat equation by linearizing effect. Here, we prove that the \(L^2\) compactness for all time (no dispersion), yields an equivalence between all norms (\(H^1\), \(L^{\infty}\) and \(L^2\) norms). The proof is based on a decomposition of \(v\) into a purely nonlinear part, which decays in time, and a localized part, coming from interactions on bounded sets, which decays in space on the right. The conclusion is that

\[
\forall s \in \mathbb{R}, \forall y \geq 0, \quad \left|v(s, y)\right| \leq c_1 e^{-c_2y};
\]

using this property at \(+\infty\) and \(-\infty\) in time, and the transformation \(y \to -y, s \to -s\), we obtain

\[
\forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad \left|v(s, y)\right| \leq c_1 e^{-c_2|y|},
\]

where \(c_1\) is related to the \(L^2\) norm of \(v\). Here, we use the exponential decay of the Airy function on the right (see Part A). Using this estimate, the equivalence of the norms follows from a Viriel type identity.

Then, letting \(n \to +\infty\) and using the equivalence of the norms, we see that the nonlinear Liouville theorem close to \(Q\) is equivalent to a linear Liouville theorem.

This linear Liouville theorem is proved using nonlinear tools (see Part B). There are 4 quantities crucial to understand the structure of the linear operator (one is a Viriel type identity and another is an impulsion type identity). The rigidity theorem comes from a hidden monotonicity property related to these 4 quantities. Of course, it is crucial to be able to do explicit calculations, using that the linear operator is of classical type (see [22]).

This concludes the proofs of Theorems 1 and 2.
**Part A: Nonlinear estimates**

Parts A and B are devoted to the proof of Theorem 1. We consider $u_0 \in H^1(\mathbb{R})$, such that $|u_0 - Q|_{H^1} \leq \alpha_0$, and we suppose that the solution $u(t)$ of (1) is defined in $H^1$ for all $t \in \mathbb{R}$ and satisfies

\begin{equation}
\forall t \in \mathbb{R}, \quad c_1 \leq \|u(t)\|_{H^1} \leq c_2.
\end{equation}

We suppose also that there exists $y(t)$ such that

\begin{equation}
\forall t \in \mathbb{R}, \quad v(t, x) = u(t, x + y(t))
\end{equation}

satisfies

\begin{equation}
\forall \varepsilon_0, \exists R_0 > 0, \forall t \in \mathbb{R}, \quad \int_{|x| > R_0} v^2(t, x) \, dx \leq \varepsilon_0.
\end{equation}

Throughout Part A and Part B, we consider the function $\varepsilon$ defined in the introduction:

\[ \varepsilon(t, y) = \lambda^{1/2}(t)u\left(t, \lambda(t)y - x(t)\right) - Q(y), \]

where $\lambda(t)$ and $x(t)$ are defined such that $\forall t \in \mathbb{R}$, $\int Q^3 \varepsilon(t) = \int Q_y \varepsilon(t) = 0$.

Setting

\begin{equation}
\varepsilon_s = \frac{\lambda(t)^3}{\lambda^3(t)} - \frac{1}{\lambda^3},
\end{equation}

the function $\varepsilon(s)$ satisfies, for $s \in \mathbb{R}$, and $y \in \mathbb{R}$,

\begin{align}
\varepsilon_s &= (L\varepsilon)_y + \frac{\lambda(t)^3}{\lambda(t)} \left( \frac{Q}{2} + \frac{Q_y}{y^2} \right) + \left( \frac{x_s}{\lambda(t)} - 1 \right) Q_y \\
&\quad + \frac{\lambda(t)^3}{\lambda(t)} \left( \frac{x_y}{2} + \frac{Q}{y^2} \right) + \left( \frac{x_y}{\lambda(t)} - 1 \right) Q_y - \left( 10Q^3 \varepsilon^2 + 10Q^2 \varepsilon^3 + 5Q \varepsilon^4 + \varepsilon^5 \right)_y,
\end{align}

where

\begin{equation}
L\varepsilon = -\varepsilon_{xx} + \varepsilon - 5Q^4 \varepsilon.
\end{equation}

Note that from Lemma 2 in [14], $L$ satisfies a convexity property, i.e.

\begin{equation}
\text{if } (\varepsilon, Q^3) = (\varepsilon, Q_y) = 0 \text{ then } (L\varepsilon, \varepsilon) \geq (\varepsilon, \varepsilon).
\end{equation}

2. **Pointwise exponential decay**

It follows from the definition of $\varepsilon$ and the assumptions on $u$ that $\varepsilon$ satisfies the following properties:

(H1) **Orthogonality conditions**: 

\[ \forall s \in \mathbb{R}, \quad (\varepsilon(s), Q^3) = (\varepsilon(s), Q_y) = 0. \]
(H2) $H^1$ bounds: There exists $\lambda_1, \lambda_2 > 0$ such that:

$$\forall s \in \mathbb{R}, \quad \lambda_1 \leq \lambda(s) \leq \lambda_2.$$ 

(H3) $L^2$ compactness: $\exists s_0 > 0$, $\exists R_0(s_0) > 0$, such that:

$$\forall s \in \mathbb{R}, \quad |\varepsilon(s)|_{L^2(|y| > R_0)} \leq \varepsilon_0.$$ 

(H4) Uniform bound on $\varepsilon$ in $H^1$:

$$\forall s \in \mathbb{R}, \quad |\varepsilon(s)|_{H^1} \leq C \sqrt{u_0},$$

where $|u_0 - Q|_{H^1} \leq u_0$.

Indeed, (H1) follows directly from the definition of $\lambda(s)$ and $x(s)$. Next, (H2) follows from the $H^1$ bounds on $u(t)$, see (15). (H3) follows from (16) and the fact that $\forall s \in \mathbb{R}, |y(s) - x(s)| \leq C$ (this follows from elementary arguments).

Finally, (H4) is a consequence of assumptions (H1), (H2), the smallness condition on the initial data and energy arguments.

**Proof of (H4).** – It is a consequence of the conservation laws and the spectral properties of the operator $L$.

For all $s \geq 0$, from Lemma 3(i)–(iii) in [14], we have:

\(\int Q \varepsilon(s) + \frac{1}{2} \int \varepsilon^2(s) = \int Q \varepsilon(0) + \frac{1}{2} \int \varepsilon^2(0) = M_0,\) (21)

\(E(Q + \varepsilon(s)) = \lambda^2(s) E_0,\) (22)

\(\left| E(Q + \varepsilon(s)) + M_0 - \frac{1}{2} \left( L \varepsilon(s), \varepsilon(s) \right) \right| \leq C |\varepsilon(s)|_{H^1} |\varepsilon(s)|_{L^2}^2,\) (23)

Recall that (23) comes from the linearization of the energy (straightforward calculation)

\(E(Q + \varepsilon) + \left( \int Q \varepsilon + \frac{1}{2} \int \varepsilon^2 \right) = \frac{1}{2} (L \varepsilon, \varepsilon) - \frac{1}{6} \left[ 20 \int Q^3 \varepsilon^3 + 15 \int Q^2 \varepsilon^4 + 6 \int Q \varepsilon^5 + \int \varepsilon^6 \right].\) (24)

We first have $|\lambda(0) - 1| \leq C \alpha_0$ (see Proposition 1 in [14]). Since

\(\left| \varepsilon(0) \right|_{H^1} = |\lambda^{1/2}(0) u(0, \lambda(0)) x|_{H^1} \leq C \alpha_0,\)

we have, by (21) and (23),

\(M_0 \leq C |\varepsilon(0)|_{L^2} \leq C \alpha_0,\)

\(E_0 = E(Q + \varepsilon(0)) \leq M_0 + \frac{1}{2} \left( L \varepsilon(0), \varepsilon(0) \right) + C \alpha_0^3 \leq C \alpha_0.\)

Therefore, by (H2),

\(\forall s \in \mathbb{R}, \quad E(Q + \varepsilon(s)) = \lambda^2(s) E_0 \leq C \lambda^2_0 \alpha_0.\)
By (20), since (H1) holds, we have:

\[ |\epsilon(s)|^2_{L^2} \leq (L\epsilon(s), \epsilon(s)), \]

and

\[ |\epsilon(s)|^2_{H^1} = (L\epsilon(s), \epsilon(s)) + 5 \int Q^4 \epsilon^2(s) \leq (L\epsilon(s), \epsilon(s)) + c_1 |\epsilon(s)|^2_{L^2} \leq C(L\epsilon, \epsilon). \]

Therefore, by (23), we obtain:

\[ |\epsilon(s)|^2_{H^1} \leq C(L\epsilon(s), \epsilon(s)) \leq C(M_0 + E(Q + \epsilon(s))) + C|\epsilon(s)|^3_{H^1} \leq C_\alpha_0 + C|\epsilon(s)|^3_{H^1}. \]

Since \( |\epsilon(0)|^2_{H^1} \leq C_\alpha_0^2 \), for \( \alpha_0 \) small enough, we obtain:

\[ \forall s \in \mathbb{R}, \quad |\epsilon(s)|^2_{H^1} \leq C\alpha_0, \]

which concludes the proof of (H4). \( \square \)

Note that if \( E_0 < 0 \), then it is enough to assume \( \int \epsilon^2(0) \) small. Indeed, by the Gagliardo–Nirenberg inequality (7), if \( |\epsilon(0)|_{L^2} \leq \alpha_0 \leq 1 \), we have from the linearization of the energy (24),

\[
\frac{1}{2} \left( 1 - \left( \frac{\alpha_0^2}{Q^2} \right)^2 \right) \int \epsilon_s^2(0) \leq \frac{1}{2} \int \epsilon_s^2(0) - \frac{1}{6} \int \epsilon^6(0) \\
\leq E(Q + \epsilon(0)) + \int Q\epsilon(0) + \frac{1}{6} \left[ 15 \int Q^4 \epsilon^2(0) + 20 \int Q^3 \epsilon^3(0) + 15 \int Q^2 \epsilon^4(0) + 6 \int Q \epsilon^5(0) \right].
\]

From \( E(Q + \epsilon(0)) < 0 \) and \( \alpha_0 \) small, there exists a function \( G \in L^2(\mathbb{R}) \) such that:

\[
\int \epsilon_s^2(0) \leq \int G(|\epsilon(0)| + |\epsilon(0)|^5) \leq C\alpha_0 (1 + |\epsilon(0)|_{L^\infty}^4) \leq C\alpha_0 (1 + \alpha_0^2 \int \epsilon_s^2(0)).
\]

Therefore, by choosing \( \alpha_0 \) small enough, we obtain \( \int \epsilon_s^2(0) \leq C\alpha_0 \), which reduces us to the previous case.

We prove the following proposition, which is a crucial result in the proof of the theorem.

PROPOSITION 1 (Uniform exponential decay). – Let \( \epsilon \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) be a solution of equation (18) satisfying (H1), (H2) and (H3). Let \( a \) and \( b \) be defined by

\[
a = \sup_{\mathbb{R}} |\epsilon(s)|_{H^1}, \quad b = \sup_{\mathbb{R}} |\epsilon(s)|_{L^2},
\]

Then, there exists \( a_0 > 0 \) and two constants \( \theta_1, \theta_2 > 0 \), such that if \( a < a_0 \), we have:

\[
\forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |\epsilon(s, y)| \leq \theta_1 \sqrt{ab} e^{-\theta_2 |y|}.
\]
Remark. – Assumption (H3) concerning $L^2$ compacteness is crucial in Proposition 1. Indeed, for any solution $u$ of the critical KdV equation (1) close to $Q$ for all time, we can define a solution $\varepsilon$ of (18) satisfying (H1) and (H2), see §1 in [14]. However, (26) is not true in general.

Remark. – This type of result is quite surprising. It says that for an entire solution of equation (18), which is of KdV type (with oscillatory behavior), an $L^2$ information on the solution gives uniform exponential decay at infinity (which is more a property of elliptic problems).

Proof. –

Step 1. Reduction of the problem.

We set

$$f_1(s, y) = \frac{\lambda s}{\lambda} \left( \frac{Q}{2} + y Q_y \right),$$

$$f_2(s, y) = \left( \frac{x_t}{\lambda} - 1 \right) Q - 5Q^4 \varepsilon - 10Q^3 \varepsilon^2 - 10Q^2 \varepsilon^3 - 5Q \varepsilon^4,$$

so that equation (18) can be rewritten

$$\varepsilon_s + \varepsilon_{yyy} - \frac{x_t}{\lambda} \varepsilon_y = \frac{\lambda s}{\lambda} \left( \frac{\varepsilon}{2} + y \varepsilon_y \right) + f_1 + f_2 y - (\varepsilon^5)_y,$$

(see also Lemma 1 in [14]).

We introduce:

$$\eta(s, x) = \frac{1}{\lambda^{1/2}(s)} \varepsilon \left( \frac{1}{\lambda} (s - \lambda^{-1}(s)) x \right),$$

to take care of the term $\frac{\lambda}{\lambda^2} (\varepsilon/2 + y \varepsilon_y)$ in the estimates.

We verify that

$$\lambda^{1/2} \eta_x + \lambda^{7/2} \eta_{xxx} - \lambda^{1/2} x_t \eta_x = f_1(s, \lambda^{-1} x) + f_2 y(s, \lambda^{-1} x) - \lambda^{7/2} \left( \eta^5 \right)_x.$$

Changing the time variable $s \to t$ by the formula

$$s = \int_0^t \frac{dt'}{\lambda^3(t')}$$

or equivalently,

$$\frac{ds}{dt} = \frac{1}{\lambda^3},$$

we obtain

$$\eta_t + \eta_{xxx} - x_t \eta_x = g_1 + g_2 y - (\eta^5)_x,$$

where

$$g_1(t, x) = \lambda^{-7/2} f_1(t, \lambda^{-1} x),$$

$$g_2(t, x) = \lambda^{-5/2} \left( \frac{x_t}{\lambda} - 1 \right) Q(\lambda^{-1} x) - 5\lambda^{-2} Q^4(\lambda^{-1} x) \eta$$

$$- 10\lambda^{-3/2} Q^3(\lambda^{-1} x) \eta^2 - 10\lambda^{-1} Q^2(\lambda^{-1} x) \eta^3 - 5\lambda^{-1/2} Q(\lambda^{-1} x) \eta^4.$$

Let $(t_n)$ be a sequence such that $t_n \to -\infty$. For $n \in \mathbb{N}$, define $\eta_n$ by:

$$\eta_n(t, x) = \eta(t + t_n, x).$$
Then $\eta_n$ satisfies

$$
(\eta_n)_t + (\eta_n)_{xxx} - x_t(t + t_n)(\eta_n)_x = g_1(t + t_n) + g_2(x(t + t_n) - (\eta_n)_x), \quad (s, t) \in \mathbb{R} \times \mathbb{R},
$$

and

$$
\eta_n(0, x) = \eta(t_n, x), \quad x \in \mathbb{R}.
$$

Now, we split $\eta_n$ in two parts, doing a "nonlinear decomposition of $\eta". We set

$$
\eta_n(t, x) = \eta_{I,n}(t, x) + \eta_{II,n}(t, x),
$$

where $\eta_{I,n}$ is solution of

$$
(\eta_{I,n})_t + (\eta_{I,n})_{xxx} - x_t(t + t_n)(\eta_{I,n})_x = -((\eta_{I,n})^5)_x, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},
$$

$$
\eta_{I,n}(0, x) = \eta(t_n, x), \quad x \in \mathbb{R},
$$

(32)

(we will see in Step 2 that for $a$ small enough, such a solution $\eta_{I,n}$ of (33) is well-defined globally in time in $H^1(\mathbb{R})$) and $\eta_{II,n}$ is solution of

$$
(\eta_{II,n})_t + (\eta_{II,n})_{xxx} - x_t(t + t_n)(\eta_{II,n})_x = g_1(t + t_n) + g_2(x(t + t_n) - (\eta_{I,n} - (\eta_{II,n})^5)_x, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},
$$

$$
\eta_{II,n}(0, x) = 0, \quad x \in \mathbb{R}.
$$

(34)

Note that $\eta_{I,n}$ represents the purely nonlinear part of the function $\eta_n$ and $\eta_{II,n}$ represents the interacting part of the solution (with local in space interactions).

The idea of the rest of the proof is first to show the following properties of $\eta_{I,n}, \eta_{II,n}$:

In Step 3, we show that for $t \in \mathbb{R}$, we have

$$
\eta_{I,n}(t - t_n) \to 0 \quad \text{in } L^\infty(\mathbb{R}) \text{ as } n \to +\infty.
$$

This is done by using some results of Kenig, Ponce and Vega [11], about well-posedness and scattering for the generalized KdV equation with critical exponent and for small initial data in $L^2(\mathbb{R})$ and the compactness assumptions on $\varepsilon$.

In Step 4, we prove that $\eta_{II,n}$ satisfies

$$
\forall t \in \mathbb{R}, \forall x \geq 0, \quad |\eta_{II,n}(t, x)| \leq \theta_1 \sqrt{ab} e^{-\theta_2 x},
$$

for some constants $\theta_1, \theta_2 > 0$, by using crucially, as in [14], the fact that the Airy function is asymmetric and has fast decay at $+\infty$. The proof will be based on fixed point arguments and on the fact that the functions $g_1$ and $g_2$ are in some sense localized in space. This allows us to obtain exponential decay for $x \geq 0$.

Thus, by taking the limit as $n \to +\infty$ in (32), we obtain:

$$
\forall t \in \mathbb{R}, \forall x \geq 0, \quad |\eta(t, x)| \leq \theta_1 \sqrt{ab} e^{-\theta_2 x},
$$

(35)

By using the invariance of the KdV equation under the transformation $t \to -t, x \to -x$, we prove that in fact:

$$
\forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad |\eta(t, x)| \leq \theta_1 \sqrt{ab} e^{-\theta_2 |x|}.
$$
Finally by using (H2), we will obtain estimate (26).

More precisely, we will prove the following lemmas:

**Lemma 1** (Asymptotic behavior of \( \eta_{n,n} \)). \( \text{Let } t \in \mathbb{R}. \) There exists \( a_0' > 0 \) such that if \( 0 < a < a_0' \), then for some sequence \( t_n \to -\infty \), we have:

\[
\eta_{n,n}(t - t_n) \to 0 \quad \text{in } L^\infty(\mathbb{R}) \text{ as } n \to +\infty.
\]

**Remark.** In fact, by a classical argument, the result is true for any sequence \( t_n \to -\infty \).

**Lemma 2** (Exponential estimate for \( x > 0 \)). \( \text{There exists } a_0'' > 0, \theta_1 = \theta_1(\lambda_1, \lambda_2) > 0, \theta_2 = \theta_2(\lambda_1, \lambda_2) > 0 \) such that if \( 0 < a < a_0'' \), then

\[
\forall t \in \mathbb{R}, \forall x \geq 0, \quad |\eta_{n,n}(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}.
\]

Assuming Lemma 1 and Lemma 2 (which will be proved in Steps 3 and 4, respectively), we finish the proof of Proposition 1.

Fix \( t \in \mathbb{R} \) and \( x \geq 0 \). Recall that, \( \forall n \in \mathbb{N} \), we have:

\[ \eta(t) = \eta_{n,n}(t - t_n) + \eta_{n,n}(t - t_n). \]

Lemma 1 yields

\[ \eta_{n,n}(t - t_n, x) \to 0 \quad \text{as } n \to \infty. \]

Therefore,

\[ \eta_{n,n}(t - t_n, x) \to \eta(t, x) \quad \text{as } n \to \infty. \]

It follows from Lemma 2 that \( \forall n \in \mathbb{N} \),

\[ |\eta_{n,n}(t - t_n, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}. \]

Hence, we obtain, letting \( n \) go to \( \infty \),

\[ |\eta(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}. \]

Thus,

\[
\forall t \in \mathbb{R}, \forall x \geq 0, \quad |\eta(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}.
\]

Let us now prove the result for \( x \leq 0 \). We use the symmetry of equation (1) under the following transformation: if \( u(t, x) \) is solution of (1) then \( \tilde{u}(t, x) = u(-t, -x) \) is also a solution of (1).

Similarly, the function \( \tilde{\eta}(t, x) = \eta(-t, -x) \) satisfies

\[ \tilde{\eta}_t + \tilde{\eta}_{xxx} - \tilde{x}_t \tilde{\eta}_x = \tilde{g}_1 + \tilde{g}_2 x - (\tilde{\eta}^5)_x, \]

where \( \tilde{x}_t(t) = x_t(-t), \tilde{g}_1(t, x) = g_1(-t, -x), \) and \( \tilde{g}_2(t, x) = g_2(-t, -x) \). Note that \( \tilde{\eta} \) satisfies the same \( H^1 \) bound and the same \( L^2 \) compacteness assumption as \( \eta \), inherited from the assumptions on \( \varepsilon \) (see (56) and (57) below).

Therefore, we can apply the same argument to \( \tilde{\eta} \) (Lemma 1 and Lemma 2), and we obtain

\[
\forall t \in \mathbb{R}, \forall x \geq 0, \quad |\tilde{\eta}(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}.
\]
By (38) and (39), we obtain:

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad |\eta(t, x)| \leq \sqrt{aD1} e^{-\theta|t|x|}.$$ 

Since

$$\varepsilon(s, y) = \lambda^{-1/2}(s)\eta(s, \lambda(s)y),$$

by (H2), we have

$$|\varepsilon(s, y)| \leq \lambda^{-1/2} \sqrt{aD1} e^{-\theta\lambda^{-1}|y|},$$

which concludes the proof of Proposition 1.

Therefore, to complete the proof of Proposition 1, we only have to prove Lemmas 1 and 2.

Step 2. Preliminaries on classical KdV equations.

Here, we recall some notations and results about the linear Airy operator and the Cauchy problem for the generalized KdV equations that will be useful in Steps 3 and 4, and later.

We begin with some standard facts about the Airy operator. Recall that the solution of the so-called Airy equation:

$$\partial_t \zeta + \zeta_{xxx} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

$$\zeta(0, x) = \zeta_0, \quad x \in \mathbb{R},$$

for \(\zeta_0 \in L^2(\mathbb{R})\), is given by \(\zeta(t) = S(t)\zeta_0\), where \(S(t)\) represents the convolution with \((3t)^{-1/3} \text{Ai}(x(3t)^{-1/3})\). Here, \(\text{Ai}\) is the Airy function, i.e.

$$\text{Ai}(x) = \frac{1}{2\pi} \int \exp \left( \frac{i|\xi|^3}{3} + i\xi x \right) d\xi.$$ 

Recall that \(\text{Ai}\) is a \(C^\infty\) function, and that it satisfies the following estimates (see [8] and references therein):

(41) \((1 + |x|)^{1/2} |\text{Ai}(x)| + |\text{Ai}'(x)| \leq C(1 + |x|)^{1/4}, \quad \forall x \in \mathbb{R},\)

(42) \((1 + x)^{1/2} |\text{Ai}(x)| + |\text{Ai}'(x)| \leq C(1 + x)^{1/4} e^{-\frac{2}{3}|x|^{3/2}}, \quad \forall x \geq 0,\)

for some \(C > 0\).

We also recall that \(S(t)\) is an isometry in \(L^2(\mathbb{R})\) and in \(H^1(\mathbb{R})\), and that, for \(\zeta_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})\), we have

(43) \(\forall t \in \mathbb{R}, \quad |S(t)\zeta_0| \leq C t^{-1/3} |\zeta_0|_{L^1}.\)

We go on with some notation and results of Kenig, Ponce and Vega [11] (see also Bourgain [3] for another approach based on bilinear estimates).

For \(1 \leq p, q \leq +\infty\), and \(\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), we define

$$|\xi|_{L^p_T L^q_x} \equiv \left( \int_{-\infty}^{+\infty} \left( \int_{-T}^{T} |\xi(t, x)|^q \, dt \right)^{p/q} \, dx \right)^{1/p},$$

and

$$|\xi|_{L^p_T L^q_x} \equiv \left( \int_{-T}^{T} \left( \int_{-\infty}^{+\infty} |\xi(t, x)|^p \, dx \right)^{q/p} \, dt \right)^{1/q}.$$
with $T = t$ to indicate the case where $(-T, T) = \mathbb{R}$.

The following norm is introduced in [11] to deal with the global Cauchy problem for the critical KdV equation for small initial data in $L^2(\mathbb{R})$

\begin{equation}
\Omega(\xi) = \max\left( |\xi|_{L^\infty_t L^2_x}, |\xi|_{L^4_t L^6_x}, |\xi|_{L^5_t L^\infty_x} \right).
\end{equation}

In particular, as a consequence of estimates (3.6), (3.7), (3.8), (3.11) and (3.12) in [11], we have, for $\xi_0: \mathbb{R} \to \mathbb{R}$, and $\zeta: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

\begin{equation}
\Omega(S(t)\xi_0) \leq C|\xi_0|_{L^2},
\end{equation}

\begin{equation}
\Omega\left( \int_0^t S(t - t') (\zeta^5)_{x} (t') \, dt' \right) \leq C(\Omega(\zeta))^5,
\end{equation}

and

\begin{equation}
\Omega\left( \int_0^t S(t - t') (\zeta^5 - \zeta^5_2)_{x} (t') \, dt' \right) \leq C[\Omega(\zeta_1) + \Omega(\zeta_2)]^d \Omega(\zeta_1 - \zeta_2).
\end{equation}

These estimates are used to show the global existence of small $L^2$ solutions of the critical generalized KdV equation via the contraction principle. Indeed, Theorem 2.8 in [11] says that for $\xi_0 \in L^2(\mathbb{R})$, with $|\xi_0|_{L^2}$ small enough, there exists a unique global solution $\xi \in C(\mathbb{R}, L^2(\mathbb{R}))$ of

\begin{equation}
\zeta_t + \zeta_{xxx} + (\zeta^5)_x = 0,
\end{equation}

satisfying $\xi(0) = \xi_0$. Moreover, we have

\begin{equation}
\Omega(\xi) \leq C|\xi_0|_{L^2},
\end{equation}

where $C$ is independent of $|\xi_0|_{L^2}$.

By Corollary 2.9 in [11], when $\xi_0 \in H^1(\mathbb{R})$, if $|\xi_0|_{L^2}$ is small enough, then the solution $\xi$ of (48) is global in time in $H^1(\mathbb{R})$, and we have:

\begin{equation}
\Omega(\xi_t) \leq C|\xi_0|_{H^1}.
\end{equation}

Recall that these techniques are also applied to the problem of local existence in time in the case of an arbitrary $\xi_0 \in L^2(\mathbb{R})$.

Recall also a similar estimate ((3.7) in [11])

\begin{equation}
\sup_{t \in [-T,T]} \left| \frac{\partial}{\partial x} \int_0^t S(t - t') \xi(t') \, dt' \right|_{L^2} \leq C|\xi|_{L^2}.
\end{equation}

Another norm is used in [11] to deal with the local in time Cauchy problem in $H^1(\mathbb{R})$ for the KdV equation

\begin{equation}
\zeta_t + \zeta_{xxx} + (\zeta^2)_x = 0.
\end{equation}

For $T > 0$, we define

\begin{equation}
\Sigma^T(\zeta) = \max\left( \sup_{(0,T)} |\zeta|_{H^1}, |\zeta_x|_{L^\infty_t L^\infty_x}, |\zeta_{xxx}|_{L^\infty_t L^2_x}, (1 + T)^{-1}|\zeta|_{L^2(L^\infty_x)} \right).
\end{equation}
We have the following estimates in [11]:

\[ \sum^T \left( S(t) \xi_0 \right) \leq C |\xi_0|_{H^1}, \]

\[ \sum^T \left( \int_0^t S(t-t') \zeta \, dt' \right) \leq CT^{1/2} \left( |\zeta|_{L^2_x L^2_t} + |\zeta_x|_{L^2_x L^2_t} \right). \]

Note that a proof of (55) is given in Appendix A of [14] as a straightforward consequence of the estimates of [11].

**Step 3.** Decay properties of \( I, \eta \) as \( n \to +\infty \) for fixed \( t \).

In view of the compactness assumption (H3), we claim the following:

**Claim.** There exists a sequence \( t_n \to -\infty \), and \( \eta_{-\infty} \in H^1(\mathbb{R}) \) such that \( \eta(t_n) \to \eta_{-\infty} \) in \( L^2(\mathbb{R}) \) strong as \( n \to +\infty \).

Let us prove the claim, for which assumption (H3) is crucial.

First, note that \( |\eta(t)|_{L^2} = |\varepsilon(t)|_{L^2} \) and \( |\eta_x(t)|_{L^2} = \lambda^{-1}(t)|\varepsilon_x(t)|_{L^2} \), and so

\[ (56) \quad \forall t \in \mathbb{R}, \quad |\eta(t)|_{H^1} \leq (1 + \lambda^{-1}) a. \]

Let \( (t_n) \) be any sequence such that \( t_n \to -\infty \). By (H3) and (H2), \( \eta(t_n) \) satisfies

\[ (57) \quad \forall \varepsilon_0 > 0, \exists R_0(\varepsilon_0), \text{ such that } |\eta(t_n)|_{L^2(|y| > R_0)} \leq \varepsilon_0, \]

moreover, by (56) we have:

\[ |\eta(t_n)|_{H^1} \leq Ca. \]

Since \( H^1(\mathbb{R}) \to L^2_{loc}(\mathbb{R}) \), there exists a subsequence of \( (t_n) \), still denoted by \( (t_n) \) and a function \( \eta_{-\infty} \in L^2_{loc}(\mathbb{R}) \), such that \( \eta(t_n) \to \eta_{-\infty} \) in \( L^2_{loc}(\mathbb{R}) \) as \( n \to +\infty \). Since \( |\eta(t_n)|_{H^1} \leq Ca \), we have \( \eta_{-\infty} \in H^1(\mathbb{R}) \), and \( |\eta_{-\infty}|_{H^1} \leq Ca \).

Finally, by (57), we have

\[ \eta(t_n) \to \eta_{-\infty} \text{ in } L^2(\mathbb{R}) \text{ strong as } n \to \infty \]

and the claim is proved.

This property and scattering results for equation (33) allows us to prove Lemma 1.

**Proof of Lemma 1.** In order to use directly the arguments of [11] on the critical KdV equation, we need to remove the term \( -\chi_i(t + t_n)(\eta_{L,n})_x \) in equation (33). This is easily done by setting:

\[ \mathcal{P}_{L,n}(t, x) = \eta_{L,n}(t, x - x(t + t_n) + x(t_n)). \]

Then \( \mathcal{P}_{L,n} \) satisfies

\[ (58) \quad \begin{cases} \frac{d}{dt}(\mathcal{P}_{L,n}) + (\mathcal{P}_{L,n})_{xxx} + (\mathcal{P}_{L,n}^5)_x = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \mathcal{P}_{L,n}(0, x) = \eta(t_n, x), & x \in \mathbb{R}. \end{cases} \]
Since we have $\eta(t, x) \to \eta_{-\infty}$ in $L^2$ strong, it is natural to introduce the solution $\eta_{L}$ of:

$$
\begin{align*}
(\eta_{L})_{t} + (\eta_{L})_{xxx} + (\eta_{L})_{x} &= 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\eta_{L}(0, x) &= \eta_{-\infty}(x), & x \in \mathbb{R}.
\end{align*}
$$

Observe that we have $|\eta(t_n)|_{L^2} \leq b$, and so $|\eta_{-\infty}|_{L^2} \leq b$. Therefore, if $b$ is small enough, then $\eta_{L}$ is global in $L^2$. In fact, since $\eta_{-\infty} \in H^1(\mathbb{R})$, then $\eta_{L}$ is also global in $H^1$ (see Corollaries 2.9 and 2.12 in [11]). The same argument can be applied to $\eta_{L,n}$, and then, for $b$ small enough, we have (see Step 2 of the proof of Proposition 1)

$$
\forall s \in \mathbb{R}, \quad |\eta_{L}(s)|_{L^2} + |\eta_{L,n}(s)|_{L^2} \leq C b, \quad |\eta_{L}(s)|_{L^2} + |(\eta_{L,n})_{x}(s)|_{L^2} \leq C a.
$$

**Step 1** of the proof of Lemma 1. We claim that if $|\eta_{-\infty}|_{L^2}$ is small enough then

$$
\eta_{L}(t) \to 0 \quad \text{in } L^\infty(\mathbb{R}) \text{ as } t \to \infty.
$$

Indeed, first, we use the scattering result in $L^2$ of Kenig, Ponce and Vega [11], Theorem 2.14 (critical case). It says that if $|\eta_{0}|_{L^2}$ is small enough then

$$
\lim_{t \to +\infty} |\eta_{L}(t) - S(t) \eta_{+}|_{L^2} = 0,
$$

where

$$
\eta_{+} = \eta_{-\infty} - \int_{0}^{\infty} S(-s) ((\eta_{L})^{4}(\eta_{L})_{x})(s) \, ds \in L^2(\mathbb{R}).
$$

In fact, we can prove that if in addition $\eta_{-\infty} \in H^1(\mathbb{R})$, then

$$
\lim_{t \to +\infty} |\eta_{L}(t) - S(t) \eta_{+}|_{H^1} = 0.
$$

Indeed, for $|\eta_{-\infty}|_{L^2}$ small enough, it is shown in [11] that

$$
L^2 = \lim_{t \to +\infty} \int_{0}^{t} S(-s) ((\eta_{L})^{4}(\eta_{L})_{x})(s) \, ds
$$

exists by using the fact that $|\eta_{L,n}|_{L^\infty L^{10}} < \infty$ (see p. 595 of [11]).

Now, recall that when the Cauchy problem for the critical KdV equation (48) is solved in $H^1(\mathbb{R})$, i.e. $\zeta_0 \in H^1(\mathbb{R})$, if $|\zeta_0|_{L^2}$ is small enough, then $\zeta$ is global in $H^1(\mathbb{R})$ and we have (see (50)):

$$
|\zeta_{x}|_{L^\infty L^{10}} < \infty.
$$

Here, since $\eta_{-\infty} \in H(\mathbb{R})$, for $a$ small enough, we have $|\eta_{L,n}|_{L^\infty L^{10}} < \infty$. Therefore, it follows from (51) and Holder’s inequality (used twice) that, for $t, t' \in \mathbb{R}$,
\[
\frac{\partial}{\partial x} \int_S S(-s) (\overline{\eta}_1)^4 (\overline{\eta}_1)_x(s) \, ds \left|_{L^2} \right.
\]
\[
\leq C \int_{x \in \mathbb{R}} \left( \int_t^{t'} \left| \overline{\eta}_1(x(s)) \right|^2 \, ds \right)^{1/2} \, dx
\]
\[
\leq C \int_{x \in \mathbb{R}} \left( \int_t^{t'} \left| \overline{\eta}_1(s) \right|^{10} \, dx \right)^{2/5} \left( \int_t^{t'} \left| \overline{\eta}_1(x(s)) \right|^{10} \, ds \right)^{1/10} \, dx
\]
\[
\leq C \left( \int_{x \in \mathbb{R}} \left( \int_t^{t'} \left| \overline{\eta}_1(s) \right|^{10} \, dx \right)^{1/2} \left( \int_t^{t'} \left| \overline{\eta}_1(x(s)) \right|^{10} \, ds \right)^{1/2} \right)^{4/5}
\]
so that by (62)
\[
H^1 \left\{ \lim_{t \to \infty} \int_0^t S(-s) (\overline{\eta}_1)^4 (\overline{\eta}_1)_x(s) \, ds \right\}
\]
exists. Therefore, \( \eta_+ \in H^1(\mathbb{R}) \) and
\[
(63) \quad \left| \overline{\eta}_1(t) - S(t) \eta_+ \right|_{H^1} \to 0 \quad \text{as } n \to \infty.
\]
In particular,
\[
(64) \quad \left| \overline{\eta}_1(t) - S(t) \eta_+ \right|_{L^\infty} \to 0 \quad \text{as } n \to \infty.
\]
Second, we prove that for any \( \tilde{\zeta}_0 \in H^1(\mathbb{R}) \), we have
\[
(65) \quad \left| S(t) \tilde{\zeta}_0 \right|_{L^\infty} \to 0 \quad \text{as } t \to +\infty.
\]
Indeed, let \( \tilde{\zeta}_0 : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function of compact support in \( \mathbb{R} \). We use (43) and the fact that \( S(t) \) is an isometry in \( H^1(\mathbb{R}) \), then
\[
\left| S(t) \tilde{\zeta}_0 \right|_{L^\infty} \leq \left| S(t) \tilde{\zeta}_0 \right|_{L^\infty} + \left| S(t) (\tilde{\zeta}_0 - \tilde{\zeta}_0) \right|_{L^\infty}
\]
\[
\leq C t^{-1/3} \left| \tilde{\zeta}_0 \right|_{L^1} + \left| S(t) (\tilde{\zeta}_0 - \tilde{\zeta}_0) \right|_{H^1}
\]
\[
\leq C t^{-1/3} \left| \tilde{\zeta}_0 \right|_{L^1} + \left| \tilde{\zeta}_0 \right|_{H^1}.
\]
For \( \tilde{\rho} > 0 \), choose \( \tilde{\zeta}_0 \) such that \( \left| \tilde{\zeta}_0 - \tilde{\zeta}_0 \right|_{H^1} < \tilde{\rho} \). Next, \( \tilde{\zeta}_0 \) being chosen, there exists \( \tilde{t} \) such that \( C t^{-1/3} \left| \tilde{\zeta}_0 \right|_{L^1} < \tilde{\rho} \). Therefore, for \( t > \tilde{t} \), we have \( \left| S(t) \tilde{\zeta}_0 \right|_{L^\infty} \leq 2\tilde{\rho} \), which proves (65).

Hence, we have
\[
\left| \overline{\eta}_1(t) \right|_{L^\infty} \leq \left| \overline{\eta}_1(t) - S(t) \eta_+ \right|_{L^\infty} + \left| S(t) \eta_+ \right|_{L^\infty} \to 0, \quad \text{as } t \to \infty.
\]
Thus (61) is proved.

**Step 2** of the proof of Lemma 1. We show
\[
(66) \quad \sup_{-\infty < t < \infty} \left| \overline{\eta}_1(t) - \overline{\eta}_{1n}(t) \right|_{L^\infty} \to 0, \quad \text{as } n \to 0.
\]
Indeed, we have \( |\eta(t_n)|_{H^1} \leq Ca, |\eta_{-\infty}|_{H^1} \leq Ca \), and then, by (49), if \( a \) is small enough, it follows that:

\[
\Omega(\eta_{1,n}) + \Omega(\eta_1) \leq Ca.
\]

It follows from (47) that

\[
\Omega(\eta_1 - \eta_{1,n}) \leq C|\eta(t_n) - \eta_{-\infty}|_{L^2} + C\left[\Omega(\eta_1) + \Omega(\eta_{1,n})\right]^4 \Omega(\eta_1 - \eta_{1,n}).
\]

Therefore, if we take \( a \) small enough so that

\[
\frac{1}{2} \text{sup}_{t \in \mathbb{R}} |\eta_{1,n}(t) - \eta_{1,n}(t)|_{L^2} \leq \frac{1}{2} \Omega(\eta_1 - \eta_{1,n}) \leq C|\eta(t_n) - \eta_{-\infty}|_{L^2},
\]

which proves that

\[
\text{sup}_{t \in \mathbb{R}} |\eta_{1,n}(t) - \eta_{1,n}(t)|_{L^2} \to 0, \quad \text{as } n \to 0.
\]

Since there exists \( C > 0 \) independent of \( s \) and \( n \), such that \( |\eta_{1,n}(s)|_{H^1} + |\eta_{1,n}(s)|_{H^1} \leq C \), by the Gagliardo–Nirenberg inequality \( |u|^2_{L^2} \leq 2|u|_{L^2}|u_x|_{L^2} \), we obtain:

\[
\text{sup}_{-\infty < t < \infty} |\eta_1(t) - \eta_{1,n}(t)|_{L^\infty} \to 0, \quad \text{as } n \to 0.
\]

Finally, let \( t \in \mathbb{R} \), we have, by Steps 1 and 2,

\[
|\eta_{1,n}(t - t_n)|_{L^\infty} = |\eta_{1,n}(t - t_n)|_{L^\infty} \leq \text{sup}_{t \in \mathbb{R}} |\eta_{1,n}(t) - \eta_1(t)|_{L^\infty} + |\eta_1(t - t_n)|_{L^\infty} \to 0,
\]

as \( n \to \infty \).

Thus, the proof of Lemma 1 is complete.

**Step 4.** Uniform exponential estimate on the half plane for \( \eta_{1,n} \).

Let us first remark the following control on term \( x_1 \).

By the orthogonality assumption (H1), if \( b \) is small enough then assumption (H2) implies that:

\[
\forall s \in \mathbb{R}, \quad \frac{\lambda_1}{2} \leq x_1(s) \leq \frac{3\lambda_2}{2}.
\]

Indeed, let us recall the results of Lemma 4 and Lemma 12 of [14].

**Lemma 3 [14].** Let \( \varepsilon \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) be a solution of equation (18) satisfying (H1). Let \( b = \text{sup}_{s \in \mathbb{R}} |\varepsilon(s)|_{L^2} \). There exists \( \varepsilon_0 \) and \( C > 0 \) such that if \( b < \varepsilon_0 \), then:

(i) Relations between \( \lambda_3/\lambda \) and \( x_1/\lambda = 1 \).

\[
\frac{\lambda_3}{\lambda} \left( \frac{1}{4} \int Q^4 - \int y(Q^3)\varepsilon \right) - \left( \frac{x_1}{\lambda} - 1 \right) \int (Q^3)\varepsilon
\]

\[
= \int L((Q^3)\varepsilon) + 10 \int (Q^3)\varepsilon Q^1\varepsilon^2 - \int (Q^3)\varepsilon R(\varepsilon),
\]

and
\[
\frac{-\lambda_x}{\lambda} \int y Q_{yy}y + \left( \frac{x_x}{\lambda} - 1 \right) \left( \frac{1}{2} \int Q^2 - \int Q_{yy}y \right)
= 20 \left( \int Q^3 Q_{yy}y - 10 \int Q_{yy} Q^3 y^2 - \int Q_{yy} R(y) \right),
\]
where \(R(y) = 10 Q^3 y^3 + 5 Q y^4 + y^5\).

(ii) Estimates on \(\lambda_x/\lambda\) and \(x_x/\lambda - 1\).

\[
(70) \quad \left| \frac{\lambda_x}{\lambda} \right| + \left| \frac{x_x}{\lambda} - 1 \right| \leq C b.
\]

It follows that if \(C b < 1/2\) and \(b < b_0\), then \(1/2 \leq x_x/\lambda \leq 3/2\), and so by (H2), \(\lambda_1/2 \leq x_x \leq 3\lambda_2/2\), and thus (67) is proved.

By using strongly (41) and (42) (that is the decay properties of the Airy function at \(+\infty\)), we prove Lemma 2 which claims that \(\eta_{\Omega,n}\) is controlled by a sum of exponentially decaying functions.

**Proof of Lemma 2.** – Note that the function \(\eta_{\Omega,n}\) satisfies (34) and so since \(\eta_n = \eta_{\Omega,n} + \eta_{\Pi,n}\), we have

\[
(\eta_{\Omega,n} x) + (\eta_{\Pi,n} x) - x_x(t + t_n)(\eta_{\Pi,n} x)
= g_1(t + t_n) + g_2(t + t_n) - (\eta_{\Pi,n} + 5 \eta_{\Pi,n} \eta_{\Pi,n}) + 10 \eta_{\Pi,n} \eta_{\Pi,n} + 10 \eta_{\Pi,n} \eta_{\Pi,n} + 5 \eta_{\Pi,n} \eta_{\Pi,n}) x,
\]
where \(g_1\) and \(g_2\) are defined in (30) and (31). Moreover, we have

\[
\eta_{\Pi,n}(0) = 0.
\]

**Step 1** of the proof of Lemma 2. We will need the following two lemmas concerning the linear Airy equation, proved in Appendix A.

**Lemma 4 (Airy equation property).** – Let \(\zeta\) be the solution of:

\[
\begin{cases}
\zeta_t + \zeta_{xxxx} = g_4(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\zeta(0, x) = 0, & x \in \mathbb{R},
\end{cases}
\]

where \(g \in L^\infty_1 L^2_x\). Assume that, for some constants \(\delta_1, \delta_2 > 0\),

\[
\forall t \in (0, 1), \forall x \in \mathbb{R}, \quad |g(t, x)| \leq \delta_1 e^{-\delta_2 x}.
\]

Then, there exists \(C = C(\delta_2) > 0\) such that

\[
\forall t \in (0, 1), \forall x \in \mathbb{R}, \quad |\zeta(t, x)| \leq C \frac{t^{1/4}}{\delta_1} e^{-\delta_2 x}.
\]

**Lemma 5 (Shifted Airy equation property).** – Let \(\zeta\) be the solution of:

\[
\begin{cases}
\zeta_t + \zeta_{xxxx} - x_x \zeta = g_1(t, x) + g_2(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
\zeta(0, x) = 0, & x \in \mathbb{R},
\end{cases}
\]

where \(g_1, g_2 \in L^\infty_1 L^2_x\). Assume that, for some constants \(\delta_1, \delta_2, \text{ and } \delta_3 > 0\),

\[
\forall t \geq 0, \forall x \in \mathbb{R}, \quad |g_1(t, x)| + |g_2(t, x)| \leq \delta_1 e^{-\delta_3 x}.
\]
and

\[ \forall t \geq 0, \quad 0 < \delta_3 < x_t(t). \]

If \( \delta_2 < \sqrt{\delta_3} \), then there exists \( C = C(\delta_2, \delta_3) > 0 \) such that

\[ \forall t \geq 0, \forall x \in \mathbb{R}, \quad |\xi(t, x)| \leq C_1 e^{-\delta_2 x}. \]

**Step 2** of the proof of Lemma 2. Exponential decay in the half plane for small time.

We claim that for some \( t_0 > 0, \theta_1 > 0, \theta_2 > 0 \), we have

\[ \forall t \geq 0, t \geq t_0, \forall x \in \mathbb{R}, \quad |\eta_{n, t}(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_2 x}. \]

As in the proof of Lemma 8 of [14], we use a fixed point argument on an auxiliary function which satisfies a simple KdV type equation (in order to eliminate some interacting terms).

For \( n \in \mathbb{N} \), set

\[ Q_n(x) = \lambda^{-1/2}(t_n) Q(\lambda^{-1}(t_n) x). \]

We consider:

\[ \xi_n(t, x) = \eta_{n, t}(t, x - x(t + t_n) + x(t_n)) + \lambda^{-1/2}(t + t_n) Q(\lambda^{-1}(t + t_n) (x - x(t + t_n) + x(t_n))) - Q_n(x), \]

then \( \xi_n \) satisfies the following equation:

\[ \xi_{nt} + \xi_{nxxx} = -((\xi_n + \eta_{1, n} + Q_n)^5 + (Q_n)_{xx} - \overline{Q}^5_{1, n})_x. \]

Indeed, we have:

\[
\xi_n(t, x) = \eta(t + t_n, x - x(t + t_n) + x(t_n)) - \overline{Q}_{1, n}(t, x)
+ \lambda^{-1/2}(t + t_n) Q(\lambda^{-1}(t + t_n) (x - x(t + t_n) + x(t_n))) - Q_n(x)
= \lambda^{-1/2}(t + t_n) (\xi + \overline{Q}(t + t_n, \lambda^{-1}(t + t_n) (x - x(t + t_n) + x(t_n))))
- Q_n(x) - \overline{Q}_{1, n}(t, x)
= u(t + t_n, x + x(t_n)) - Q_n(x) - \overline{Q}_{1, n}(t, x).
\]

(Recall that \( u(t, x) = \lambda^{-1/2}(t)(\xi + \overline{Q})(t, \lambda^{-1}(t)(x - x(t))) \).) Since \( u \) and \( \overline{Q}_{1, n} \) satisfy

\[ u_t + u_{xxx} + (u^5)_x = 0, \]

\[ (\overline{Q}_{1, n})_t + (\overline{Q}_{1, n})_{xxx} + (\overline{Q}^5_{1, n})_x = 0, \]

we obtain

\[ \xi_{nt} + \xi_{nxxx} = -((u^5(\cdot + t_n, \cdot + x(t_n)))_x + (Q_n)_{xx} - \overline{Q}^5_{1, n})_x \]

\[ = -((\xi_n + \eta_{1, n} + Q_n)^5 + (Q_n)_{x} - \overline{Q}^5_{1, n})_x. \]

Moreover, \( \xi_n(0) = \eta_{1, n}(0) = 0 \).

We are going to use a fixed point argument on \( \xi_n \). For \( \theta_1, \theta_2 > 0 \), and \( 0 < t_0 < 1 \) to be chosen later, we define:
\[ \mathcal{M} = \{ \xi \in C((0, t_0), H^1(\mathbb{R})) \mid \text{such that } \Sigma^0(\xi) \leq \sqrt{ab}, \]
\[ \text{and } \langle \xi(t), x \rangle \leq \sqrt{ab} \theta(1 - e^{-\theta_0 t}) \text{ for all } t \in (0, t_0), \forall x \in \mathbb{R} \}, \]
equipped with the norm of \( L^\infty((0, t_0), H^1(\mathbb{R})) \).
In order to prove that the solution \( \xi_n \) of (79) belongs to \( \mathcal{M} \), for \( t_0 > 0 \) small enough, we only need to show that:
\[
\Phi : \xi \in \mathcal{M} \mapsto \Phi(\xi)(t) = - \int_0^t S(t - s) \left( (\xi(s) + \eta_{1,n} + Q_n)^5 + (Q_n)_{xx} - \eta_{1,n}^5 \right) \, ds
\]
(80)
maps \( \mathcal{M} \) into itself and that it is a contraction for the norm \( \Sigma^0 \).
(i) Estimates for the \( \Sigma^0 \) norm. First, we note that since
\[ \forall t \in \mathbb{R}, \quad \lambda_1 \leq \lambda(t) \leq \lambda_2, \]
and
\[ 0 \leq Q(x) \leq C \, e^{-|x|}, \]
we have
(81)
\[ 0 \leq Q_n(x) = \lambda^{-1/2}(t_n) Q_0(\lambda^{-1}(t) x) \leq \lambda^{-1/2} Q_0(\lambda^{-1} x) \leq C \, e^{-\lambda_1^{-1}|x|}. \]
Similarly, since \( (Q_n)_{xx} = \lambda^2(t_n) Q_n - Q_n^2 \), we obtain:
(82)
\[ \|(Q_n)_{xx}\| + \|(Q_n)_{xxx}\| \leq C \, e^{-\lambda_1^{-1}|x|}. \]
We have:
\[ |F(t, x)| + |F_x(t, x)| \]
\[ = \left| \left( \langle \xi_n + Q_n, \eta_{1,n} \rangle + (Q_n)_{xx} - \eta_{1,n}^5 \right) + \left( \langle (\xi_n + Q_n) + \eta_{1,n}, (Q_n)_{xx} - \eta_{1,n}^5 \rangle \right) \right| \]
\[ \leq C \, e^{-\lambda_1^{-1}|x|} + C \| \xi_n + Q_n + (\xi_n + Q_n)_x \| (|\xi_n + Q_n|^2 + \|\eta_{1,n}\|)^4 \]
\[ + \|\xi_n + Q_n\| |\eta_{1,n}| + |\xi_n + Q_n|^2 |\eta_{1,n}|^2 + |\xi_n + Q_n| |\eta_{1,n}|^3 + |\eta_{1,n}|^4 \]
\[ + \|\eta_{1,n}\| |\xi_n + Q_n| |\xi_n + Q_n|^3 + |\xi_n + Q_n|^2 |\eta_{1,n}| + |\xi_n + Q_n| |\eta_{1,n}|^2 + |\eta_{1,n}|^3 \].
By using \( |\eta_{1,n}|_{L^\infty} \leq |\eta_{1,n}|_{H^1} \leq C \), and \( |\xi_n + Q_n|_{L^\infty} \leq |\xi_n + Q_n|_{H^1} \leq \Sigma^0(\xi_n) + C \leq C' \), we obtain:
\[ |F(t, x)| + |F_x(t, x)| \leq C \, e^{-\lambda_1^{-1}|x|} + C \| \xi_n + Q_n \| + |(\xi_n + Q_n)_x| + |(\eta_{1,n})_x| \].
Therefore, since \( t_0 < 1 \),
\[ |F|_{L^2_0 L^2_1} + |F_x|_{L^2_0 L^2_1} \leq C. \]
By (55), we have:
\[ \Sigma^0(\Phi(\xi_n)) \leq C t_0^{1/2} \left( |F|_{L^2_0 L^2_1} + |F_x|_{L^2_0 L^2_1} \right) \leq C t_0^{1/2}. \]
Therefore, by taking \( t_0 \) small enough, depending on \( a \) and \( b \), we obtain \( \Sigma^0(\Phi(\xi)) \leq \sqrt{ab} \). By similar arguments, and by possibly choosing a smaller \( t_0 \), \( \Phi \) is a contraction for the norm \( \Sigma^0 \).

(ii) Decay property. Here, we will use Lemma 4. We need only show an exponential estimate on \( F(t,x) \) for \( 0 < t < t_0 \) and \( x \geq 0 \).

Set
\[
\theta_2 = \frac{1}{2} \min \left( \sqrt{\frac{\lambda_1}{2\lambda_2}}, \frac{1}{2\lambda_1} \right).
\]

Recall that
\[
F(s) = (\xi_n(s) + \eta_{1,n} + Q_n)^5 + (Q_n)_{xx} - \eta_{1,n}^5.
\]

By (81), (82), and since \( n > 2 \), we have, with \( \rho_{ab} < 1 \),
\[
|F| \leq C e^{-\lambda_1^{-1}x} + \frac{C}{|\xi_n + Q_n|} \left( \frac{\xi_n + Q_n}{4} + \frac{\xi_n + Q_n}{4} |\eta_{1,n}| \right)
+ \frac{C}{2|\xi_n + Q_n|^2} \left( |\eta_{1,n}|^2 + \frac{2|\xi_n + Q_n|}{4} |\eta_{1,n}| + |\eta_{1,n}|^4 \right)
\leq C \left[ e^{-\lambda_1^{-1}x} + \frac{\xi_n + Q_n}{4} \right]
+ C e^{-\lambda_1^{-1}x} + C\theta_1 e^{-\rho_{ab}x} \leq C(1 + \theta_1) e^{-\theta_2 x},
\]
where \( C \) is independent of \( \theta_1 \).

By using Lemma 4 and (80), we obtain:
\[
\forall t \in (0, t_0), \forall x \in \mathbb{R}, \quad |\Phi(\xi_n)(t)| \leq C t^{1/4} (1 + \theta_1) e^{-\theta_2 x}.
\]

If \( \theta_1 > 1 \) and if \( t_0 \) is such that \( C(1 + \theta_1) t_0^{1/4} \leq \sqrt{ab} \), then
\[
\forall t \in (0, t_0), \forall x \in \mathbb{R}, \quad |\Phi(\xi_n)(t)| \leq \sqrt{ab} e^{-\rho_{ab}x} \leq \theta_1 \sqrt{ab} e^{-\rho_{ab}x}
\]
and so \( \Phi \) maps \( M \) into itself. Therefore, using the uniqueness for the Cauchy problem in \( \Sigma^0 \), we have \( \xi_n \in M \).

Now, let us return to \( \eta_{1,n} \). By (70), we have
\[
|\lambda(t + t_n) - \lambda(t_n)| + |x(t + t_n) - x(t_n)| \leq C b t_0,
\]
which proves that, taking \( t_0 \) small enough, by the decay properties of \( Q \) and its derivatives, we have:
\[
\forall t \in \mathbb{R}, \quad |\lambda^{-1/2}(t + t_n) Q(\lambda^{-1}(t + t_n)(x - x(t + t_n) + x(t_n)) - Q_n(x))| \leq C b e^{-x/2\lambda_1},
\]
for \( t \) small enough.

Therefore, (78) is proved with \( \theta_2 \) defined by (83).

Step 3 of the proof of Lemma 2. Exponential decay in the half plane for large time.

Here, \( \theta_2 \) is still fixed by (83). We define:
\[
t_1 = \sup \left\{ t' > 0, \ (78) \ is \ satisfied \ \forall t \in (0, t') \right\}.
\]

Assuming \( t_1 < +\infty \), we find a contradiction for \( \theta_1 \) well-chosen, which will complete the proof of Lemma 2. The contradiction follows from the equation of \( \eta_{1,n} \) and Lemma 5.
Recall that the equation satisfied by $\eta_{\mu,n}$ is

$$
\begin{cases}
(\eta_{\mu,n})_t + (\eta_{\mu,n})_{xxx} = x_t (t + t_0) (\eta_{\mu,n})_x \\
\eta_{\mu,n}(0, x) = 0, \quad x \in \mathbb{R},
\end{cases}
$$

(84)

where $g_1$ and $g_2$ are defined in (30), (31). Moreover, $\eta_{\mu,n}$ satisfies the initial condition

$$
\eta_{\mu,n}(0, x) = 0.
$$

For all $0 < t < t_1$, we have

$$
\forall x \in \mathbb{R}, \quad |\eta_{\mu,n}(t, x)| \leq \sqrt{ab} \theta_1 e^{-\theta_1 x}.
$$

(85)

On the other hand, we have the following estimates on $g_1$ and $g_2$, $\forall t \in (0, t_1), \forall x \in \mathbb{R}$,

$$
|g_1(t, x)| \leq C b e^{-\frac{1}{2} |x|} \leq C b e^{-\theta_2 |x|},
$$

(86)

$$
|g_2(t, x)| \leq C \sqrt{ab} e^{-\frac{1}{2} |x|} \leq C \sqrt{ab} e^{-\theta_2 |x|}.
$$

(87)

(Recall that in (83), we have chosen $\theta_2 = \frac{1}{\sqrt{a_2}}$.) Indeed, by the decay properties of $Q$:

$$
\forall x \in \mathbb{R}, \quad |Q(x)| + |Q_x(x)| \leq C e^{-|x|},
$$

(H2), and Lemma 3(ii), we obtain:

$$
|g_1(t, x)| \leq C b e^{-\frac{1}{2} |x|} (1 + |x|) \leq C b e^{-\frac{1}{2} |x|}.
$$

Note that, by (56), we have

$$
\forall t \in \mathbb{R}, \quad |\eta(t)|_{L^\infty} \leq 2 |\eta(t)|_{L^2}^{1/2} |\eta(t)|_{H^1}^{1/2} \leq C \sqrt{ab}.
$$

(88)

Thus, using again the decay properties of $Q$ and its derivatives, (H2), Lemma 3(ii), and (88), we have:

$$
|g_2(t, x)| \leq C b e^{-\frac{1}{2} |x|} + C e^{-\frac{1}{2} |x|} (|\eta(t)|_{L^\infty} + |\eta(t)|_{H^1}^{4}) \leq C \sqrt{ab} e^{-\frac{1}{2} |x|}.
$$

Now, set

$$
G(t, x) = \eta_5^5 - (\eta_{\mu,n})^5 = (\eta_{\mu,n} + \eta_{\mu,n})^5 - (\eta_{\mu,n})^5
$$

$$
= - (\eta_{\mu,n}^5 + 5 \eta_{\mu,n}^4 \eta_{\mu,n} + 10 \eta_{\mu,n}^3 \eta_{\mu,n}^2 + 10 \eta_{\mu,n}^2 \eta_{\mu,n}^3 + 5 \eta_{\mu,n}^4).
$$

We have, $\forall t \in (0, t_1)$,

$$
|G| \leq |\eta_{\mu,n}| (|\eta_{\mu,n}|^4 + 5 |\eta_{\mu,n}|^3 |\eta_{\mu,n}| + 10 |\eta_{\mu,n}|^2 |\eta_{\mu,n}|^2 + 10 |\eta_{\mu,n}| |\eta_{\mu,n}|^3 + 5 |\eta_{\mu,n}|^4).
$$

Now, note that $|\eta_{\mu,n}|_{L^\infty} \leq |\eta_{\mu,n}|_{L^\infty} + |\eta_{\mu,n}|_{L^\infty}$. By (60), we have

$$
|\eta_{\mu,n}|_{L^\infty} + |\eta_{\mu,n}|_{L^\infty} \leq 2 |\eta_{\mu,n}|_{L^2}^{1/2} (|\eta_{\mu,n}|_{L^2}^{1/2} + 2 |\eta_{\mu,n}|_{L^2}^{1/2} |\eta_{\mu,n}|_{L^2}^{1/2}) \leq C \sqrt{ab},
$$

where

$$
(89)
$$
and so

\[ |\eta_{II,n}|_{L^\infty} + |\eta_n|_{L^\infty} \leq C \sqrt{ab} \leq Ca. \]

It follows that, \( \forall t \in (0, t_1) \), \( \forall x \in \mathbb{R} \),

\[ |G(t, x)| \leq Ca^4|\eta_{II,n}(t, x)|. \]

By (85), we obtain

\[ \forall t \in (0, t_1), \forall x \in \mathbb{R}, \quad |G(t, x)| \leq C\theta_1a^4\sqrt{ab}e^{-\theta_2x}. \]

Finally, we obtain the following estimate:

\[ \forall t \in (0, t_1), \forall x \in \mathbb{R}, \quad |\eta_{II,n}(t, x)| \leq C(1 + a^4\theta_1)\sqrt{ab}e^{-\theta_2x}. \]

Recall that \( \eta_{II,n} \) satisfies:

\[ (\eta_{II,n})_t + (\eta_{II,n})_{xxx} - x_i(t + t_n)(\eta_{II,n})_x = \text{g}_1(t + t_n) + \text{g}_2(t + t_n) + G_x. \]

Observe that by (67), we have \( x_i = \lambda^{-3}x > \lambda^{-3}\frac{3\lambda}{2} \). Therefore, since \( \theta_2 < \sqrt{\frac{\lambda}{2\lambda - 2}} \) (see (83)), we can apply Lemma 5. It follows that

\[ \forall t \in [0, t_1], \forall x \in \mathbb{R}, \quad |\eta_{II,n}(t, x)| \leq C(1 + a^4\theta_1)\sqrt{ab}e^{-\theta_2x}, \]

where \( C \) is independent of \( \theta_1 \).

Now, we note that a similar estimate holds for \( t \in [0, t_1 + \delta^*] \), for some \( \delta^* > 0 \). Indeed, the estimate for \( t \in [t_1, t_1 + \delta^*] \) is based on the same technique. For \( t_0 > 0 \), set \( \tilde{\eta}_{II,n}(t, x) = \eta_{II,n}(t_1 - \frac{t_0}{2} + t, x), \tilde{\eta}_{II,n}(0, x) = \eta_{II,n}(t_1 - \frac{t_0}{2}, x) \). Using a fixed point argument in an sufficiently small interval of time \([0, t_0]\), as in Step 1, and using an analogue of Lemmas 4 and 5 for the homogeneous Airy equation (see also Lemma 7 in [14]), we have:

\[ \forall t \in (0, t_0), \forall x \in \mathbb{R}, \quad |\tilde{\eta}_{II,n}(t, x)| \leq C^{-1/4}e^{-\theta_2x} \]

and so

\[ \forall t \in \left(t_1, t_1 + \frac{t_0}{2}\right), \forall x \in \mathbb{R}, \quad |\eta_{II,n}(t, x)| \leq Ce^{-\theta_2x}. \]

Of course, for the moment the constant \( C \) above is not small.

To conclude, we insert the above estimate in the equation of \( \eta_{II,n} \) and we find, by the same technique as above, for \( 0 < \delta^* < t_0/2 \),

\[ \forall t \in [0, t_0 + \delta^*], \forall x \in \mathbb{R}, \quad |\eta_{II,n}(t, x)| \leq \left(C(1 + a^4\theta_1)\sqrt{ab} + C(\delta^*)^{1/4}\right)e^{-\theta_2x}. \]

Thus, by choosing \( \delta^* > 0 \) small enough, we have:

\[ \forall t \in [0, t_1 + \delta^*], \forall x \in \mathbb{R}, \quad |\eta_{II,n}(t, x)| \leq C'(1 + a^4\theta_1)\sqrt{ab}e^{-\theta_2x}, \]

for a constant \( C' > 0 \) independent of \( \theta_1 \). (See a similar argument in [14], proof of Proposition 3.)
Therefore, we obtain a contradiction with the definition of $t_1$ if:

$$C'(1 + a^4 \theta_1) < \theta_1.$$ 

This relation is easily satisfied with any $\theta_1 > 2C'$ and taking $a$ such that $a^4 < \frac{1}{C'}$. The proof of Lemma 2 is now complete.

3. Comparison between different norms and reduction to a linear equation

First, we show the following proposition, comparing the $L^2$ and the $H^1$ norm of a global bounded solution of (18) satisfying the assumption of Proposition 1.

**Proposition 2** (Comparison between $L^2$ and $H^1$ norms). – Under the assumptions of Proposition 1, there exist $a_1 > 0$ and $C > 0$ such that if $a < a_1$ then

$$b \leq a \leq Cb,$$

where

$$a = \sup_{s \in \mathbb{R}} |\varepsilon(s)|_{H^1}, \quad b = \sup_{s \in \mathbb{R}} |\varepsilon(s)|_{L^2}.$$

**Remark.** – In the proof of Proposition 2, we will crucially use that in Proposition 1, we have obtained $|\varepsilon(s, y)| \leq \theta_1 \sqrt{a^b e^{-\theta_2|y|}}$, and not $|\varepsilon(s, y)| \leq \theta_1 a e^{-\theta_2|y|}$.

**Proof.** – The proof is based on the Viriel identity for equation (18) (see also [14]), the exponential decay on $\varepsilon$ obtained in Proposition 1, and a suitable continuity property of $\varepsilon(s)$ with respect to the initial data.

First, we claim two lemmas. Let

$$I(s) = \frac{1}{2} \int y \varepsilon^2(s);$$

note that by Proposition 1, $I(s)$ is defined for all $s \in \mathbb{R}$.

**Lemma 6** (Viriel type identity). – There exists a constant $C > 0$ such that

$$\frac{d}{ds} (\lambda I)(s) \leq CB^2 - \frac{3}{2} \lambda_1 |\varepsilon(s)|^2_{H^1}.$$ 

**Lemma 7** (Lower bound on the $H^1$-norm of $\varepsilon$). – There exist $a'_1 > 0$, $c > 0$, and $\sigma > 0$ such that if $0 < a < a'_1$ and

$$|\varepsilon(s_0)|_{H^1} \geq \frac{a}{2},$$

for some $s_0 \in \mathbb{R}$, then

$$\forall s \in (s_0, s_0 + \sigma), \quad |\varepsilon(s)|_{H^1} \geq ca.$$ 

First, we prove Proposition 2 assuming Lemmas 6 and 7. Then, we prove Lemma 6. The proof of Lemma 7 is given in Appendix B1.

Since $\varepsilon$ satisfies the assumptions of Proposition 1, if $a < a_0$, then

$$\forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |\varepsilon(s, y)| \leq \theta_1 \sqrt{a^b e^{\theta_2|y|}},$$

where $\theta_1$ and $\theta_2$ are independent of $a$ and $b$. 
It follows that we have

$$\forall s \in \mathbb{R}, \quad |I(s)| \leq C a b \int |y| \exp \left(-\theta_2 |y|\right) dy \leq C a b.$$  

By Lemma 6, we have:

$$\forall s \in \mathbb{R}, \quad (\lambda I)\varepsilon(s) \leq C b^2 - \frac{3}{2} \lambda_1 |\varepsilon(s)|^2_{H^1}.$$  

By the definition of $a$, there exists $s_0 \in \mathbb{R}$ such that

$$\frac{a}{2} \leq |\varepsilon(s_0)|_{H^1} \leq a.$$  

By Lemma 7, if $a < a'_1$, we have:

$$\forall s \in (s_0, s_0 + \sigma), \quad |\varepsilon(s)|_{H^1} \geq c a.$$  

By identities (92), (93) and (90) we will deduce that there exists $C > 0$ such that $C a < b$.

Indeed, by (93) and (94), we have, for $C, C' > 0$ independent of $a$ and $b$,

$$\forall s \in (s_0, s_0 + \sigma), \quad (\lambda I)\varepsilon(s) \leq C b^2 - C'a^2.$$  

Integrating the above formula between $s_0$ and $s_0 + \sigma$, we obtain:

$$\lambda(s_0 + \sigma) I(s_0 + \sigma) - \lambda(s_0) I(s_0) \leq C b^2 - \frac{3}{2} \lambda_1 a^2.$$  

But by (92), we have

$$|\lambda(s_0 + \sigma) I(s_0 + \sigma) - \lambda(s_0) I(s_0)| \leq C a b.$$  

Therefore, we deduce

$$C b^2 - C'a^2 \geq -C'' a b,$$  

and so, for some $C > 0$, $C(b^2 + ab) \geq a^2$,

which proves that there exists $C > 0$ such that $C a \leq b$.

Proof of Lemma 6. – By [14], Lemma 5, the function $s \mapsto I(s)$ is $C^1$ and satisfies:

$$I_s + \frac{\lambda}{\lambda} I = \frac{\lambda}{\lambda} \int y \left(\frac{Q}{2} + y Q_y\right) \varepsilon + \left(\frac{\lambda}{\lambda} - 1\right) \left(\int y Q_y \varepsilon - \frac{1}{2} \int \varepsilon^2\right)$$

$$- \frac{3}{2} (L \varepsilon, \varepsilon) + \int \varepsilon^2 - 10 \int Q^3 \left(\frac{Q}{2} + y Q_y\right) \varepsilon^2$$

$$+ 10 \int \left(\frac{2 Q^3}{3} - y Q^2 Q_y\right) \varepsilon^3 + 5 \int \left(\frac{3 Q^2}{2} - y Q Q_y\right) \varepsilon^4$$

$$+ \int (4Q - y Q_y) \varepsilon^5 + \frac{5}{6} \int \varepsilon^6.$$  

Let us prove (89). First, note that if $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function with fast decay at $\pm \infty$, then

$$\forall s \in \mathbb{R}, \quad \left| \int \chi \varepsilon(s) \right| \leq |\chi|_{L^2} \left| \varepsilon(s) \right|_{L^2} \leq C b,$$
and
\[ \forall s \in \mathbb{R}, \quad \left| \int \chi e^2(s) \right| \leq |\chi|_{L^\infty} |\varepsilon(s)|_{L^2}^2 \leq Cb^2. \]

Moreover, take \( a \leq 1 \), so that, \( \forall s \in \mathbb{R}, \ |\varepsilon(s)|_{L^\infty} \leq |\varepsilon(s)|_{H^1} \leq a \leq 1 \). Therefore, for \( i \geq 2 \), we have:
\[ \forall s \in \mathbb{R}, \quad \int |\varepsilon|^i \leq |\varepsilon(s)|_{L^\infty}^{i-2} \int e^2(s) \leq Cb^2. \]

Second, recall that by Lemma 3, we have, for \( b \ll a \) small enough,
\[ \left| \frac{\lambda_s}{\lambda} \right| + \left| \frac{s_x}{\lambda} - 1 \right| \leq Cb; \]

therefore,
\[ \left| \frac{\lambda_s}{\lambda} \int y \left( \frac{Q}{2} + yQ_y \right) \varepsilon + \left( \frac{s_x}{\lambda} - 1 \right) \left( \int y Q_y \varepsilon - \frac{1}{2} \int e^2 \right) + \int \varepsilon^2 \right. \]
\[ \left. - 10 \int Q^3 \left( \frac{Q}{2} + yQ_y \right) \varepsilon^2 + 10 \int \left( \frac{2Q^3}{3} - yQ^2 Q_y \right) \varepsilon^3 \right. \]
\[ + 5 \int \left( \frac{3Q^2}{2} - yQ Q_y \right) \varepsilon^4 + \int (4Q - yQ_y) \varepsilon^5 + \frac{5}{6} \int \varepsilon^6 \right| \leq Cb^2. \]

By using the above identity and (95), we obtain:
\[ \forall s \in \mathbb{R}, \quad (\lambda I)_s(s) \leq \lambda(s)Cb^2 - \frac{3}{2} \lambda(s) (L \varepsilon(s), \varepsilon(s)). \]

Note that
\[ (L \varepsilon, \varepsilon) = \int e^2 + \int e^2 - 5 \int Q^4 e^2 \geq |\varepsilon|_{H^1}^2 - C|\varepsilon|_{L^2}^2 \geq |\varepsilon|_{H^1}^2 - Cb^2, \]

and so
\[ \forall s \in \mathbb{R}, \quad (\lambda I)_s(s) \leq \lambda(s)C' b^2 - \frac{3}{2} \lambda(s)|\varepsilon(s)|_{H^1}^2. \]

Finally, since \( \lambda_1 \leq \lambda(s) \leq \lambda_2 \), we obtain
\[ \forall s \in \mathbb{R}, \quad (\lambda I)_s(s) \leq C\lambda_2 b^2 - \frac{3}{2} \lambda_1 |\varepsilon(s)|_{H^1}^2. \]

The proof of Lemma 6 is complete. \( \Box \)

In the rest of this section, using very strongly Proposition 1 and Proposition 2, we show that if, for fixed \( \lambda_1 \) and \( \lambda_2 \), there exists a sequence \( \varepsilon_n \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \), of solutions of
\[ \varepsilon_{ns} = (L \varepsilon_n)_y + \frac{\lambda_{ns}}{\lambda_n} \left( \frac{Q}{2} + yQ_y \right) + \left( \frac{s_{ns}}{\lambda_n} - 1 \right) Q_y + \frac{\lambda_{ns}}{\lambda_n} \left( \frac{\varepsilon_n}{2} + y\varepsilon_{ny} \right) \]
\[ + \left( \frac{s_{ns}}{\lambda_n} - 1 \right) \varepsilon_{ny} - 10 \left( Q^3 e_n^2 + 10 Q^2 e_n^3 + 5 Q e_n^4 + e_n^5 \right), \quad (s, y) \in \mathbb{R}^2, \tag{96} \]
satisfying (H1), (H2), (H3) (without any uniformity in \( n \) for (H3)), and such that:

\[
(b_n = \sup_{s \in \mathbb{R}} |\varepsilon_n(s)|_{L^2} \to 0 \quad \text{as} \quad n \to +\infty,
\]

then there exists a nontrivial, bounded, function \( w \) which is entire solution of a linear equation, and which has exponential decay at infinity in space. The existence of such a solution \( w \) will lead to a contradiction in Part B.

The function \( w \) is constructed as the limit of a renormalization of the \( \varepsilon_n \). The precise convergence result is the following:

**Proposition 3** (Convergence to a linear problem). – Let \( \lambda_1, \lambda_2 > 0 \). Consider a sequence \( \varepsilon_n \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) of solutions of (96) satisfying (H1), (H2) and (H3) (without any uniformity in \( n \) for (H3)). Assume that

\[
(b_n = \sup_{s \in \mathbb{R}} |\varepsilon_n(s)|_{L^2} \to 0 \quad \text{as} \quad n \to +\infty,
\]

then,

(i) There exist a sequence \( (s_n) \in \mathbb{R} \) and a subsequence \( (\varepsilon_{n'} \) such that:

\[
\frac{\varepsilon_{n'}(s_{n'} + s)}{b_{n'}} \to w(s) \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}))
\]

where \( w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) satisfies:

\[
w \neq 0,
\]

\[
w_s - (Lw)_y = \alpha(s) \left( \frac{Q}{2} + y Q_y \right) + \beta(s) Q_y, \quad (s, y) \in \mathbb{R}^2,
\]

for some continuous functions \( \alpha \) and \( \beta \).

(ii) Moreover, there exist \( C > 0 \) and \( \theta_2 > 0 \) such that \( w \) satisfies:

\[
(\text{H1}') \quad \forall s \in \mathbb{R}, \quad (w(s), Q^3) = 0, \quad (w(s), Q_y) = 0,
\]

\[
(\text{H2}') \quad \forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |w(s, y)| \leq C e^{-\theta_2 |s|}.
\]

**Proof.** –

**Step 1.** Renormalization and formal asymptotic.

By the definition of \( b_n \), for any \( n \in \mathbb{N} \), there exists \( s_n \in \mathbb{R} \), such that

\[
|\varepsilon_n(s_n)|_{L^2} \geq \frac{b_n}{2},
\]

Set

\[
w_n(s, y) = \frac{\varepsilon_n(s + s_n, y)}{b_n}.
\]

We claim the following lemma on \( w_n \):

**Lemma 8** (Properties of the sequence \( (w_n) \)). – The sequence \( (w_n) \) satisfies the following properties:
(i) **Nonvanishing:**

\[ |w_n(0)|_{L^2} \geq \frac{1}{2}, \]  

(ii) **\( H^1 \)-bound:**

\[ \forall s \in \mathbb{R}, \quad |w_n(s)|_{H^1} \leq C, \]  

(iii) **Uniform exponential decay:**

\[ \forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |w_n(s, y)| \leq C e^{-\theta|y|}. \]  

Moreover, for any \( n \in \mathbb{N} \), the function \( w_n \) satisfies:

\[ w_{ns} = (Lw_n)_y + \alpha_n \left( \frac{Q}{2} + y Q_y \right) + \beta_n Q_y \]

\[ + b_n \tilde{\alpha}_n \left( \frac{w_n}{2} + y w_{ny} \right) + b_n \tilde{\beta}_n w_{ny} + b_n (F_n + G_n), \]

where

\[ \alpha_n(s) = \frac{\int L(Q^3)w_n(s)}{\int Q^4}, \quad \beta_n(s) = \frac{20 \int Q^2 w_n(s)}{\int Q^4}, \]

\[ F_n = \left( \frac{\tilde{\alpha}_n - \alpha_n}{b_n} \right) \left( \frac{Q}{2} + y Q_y \right) + \left( \frac{\tilde{\beta}_n - \beta_n}{b_n} \right) Q_y, \]

\[ G_n = -(10Q^3w_n^2 + 10b_n Q^2w_n^3 + 5b_n^2 Q w_{ny} + b_n^2 w_{ny}^2), \]

and \( \tilde{\alpha}_n, \tilde{\beta}_n \) satisfy

\[ |\tilde{\alpha}_n(s) - \alpha_n(s)| \leq C b_n, \quad |\tilde{\beta}_n(s) - \beta_n(s)| \leq C b_n. \]

**Proof of Lemma 8.** – By (98), we have

\[ |w_n(0)|_{L^2} \geq \frac{1}{2}. \]

By Proposition 2, we have

\[ \forall s \in \mathbb{R}, \quad |e_n(s)|_{H^1} \leq C b_n, \]

therefore,

\[ \forall s \in \mathbb{R}, \quad |w_n(s)|_{H^1} \leq C. \]

By Proposition 1 and Proposition 2, we have

\[ \forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |e_n(s, y)| \leq C b_n e^{-\theta|y|}, \]

thus,

\[ \forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |w_n(s, y)| \leq C e^{-\theta|y|}. \]

Moreover, replacing \( e_n(s + s_n) \) by \( b_n w_n(s) \) in (96), and dividing by \( b_n \), we find:
\( w_{ns} = (L w_n)_s + \theta_n \left( \frac{Q}{2} + y Q_y \right) + \beta_n Q_y + b_n \theta_n \left( \frac{w_n}{2} + y w_{ny} \right) \)

(106)

\[ + b_n \beta_n y w_{ny} - b_n (10 Q^3 w_n^2 + b_n 10 Q^2 w_n^3 + 5 b_n Q w_n^4 + b_n^3 w_n^5)_n \, , \]

where

\[
\theta_n(s) = \frac{1}{b_n} \frac{\lambda_{ns} (s + s_n)}{\lambda_n (s + s_n)} - 1, \quad \beta_n(s) = \frac{1}{b_n} \frac{x_{ns} (s + s_n)}{\lambda_n (s + s_n)} - 1.
\]

Recall that, by Lemma 3, we have

(107)

\[
\frac{\lambda_{ns}}{\lambda_n} \left( \frac{1}{2} \int Q^4 - \int y (Q^3)_y \xi_n \right) \left( \frac{x_{ns}}{\lambda_n} - 1 \right) \int (Q^3)_y \xi_n
\]

\[ = \int L((Q^3)_y) \xi_n - 10 \int (Q^3)_y Q^3 \xi_n^2 - \int (Q^3)_y (10 Q^2 \xi_n^3 + 5 Q \xi_n^4 + \xi_n^5), \]

and

(108)

\[
\frac{\lambda_{ns}}{\lambda} \int y Q^2 \xi_n + \left( \frac{x_{ns}}{\lambda_n} - 1 \right) \left( \frac{1}{2} \int Q^2 - \int Q y \xi_n \right)
\]

\[ = 20 \int Q^3 Q^2 \xi_n^2 - 10 \int Q Q^2 \xi_n^2 - \int Q (10 Q^2 \xi_n^3 + 5 Q \xi_n^4 + \xi_n^5). \]

Replacing \( \xi_n (s + s_n) \) by \( b_n w_{ns} (s) \) and dividing by \( b_n \), it follows that:

(109)

\[
\theta_n \left( \frac{1}{2} \int Q^4 - b_n \int y (Q^3)_y w_n \right) - b_n \beta_n \int (Q^3)_y w_n
\]

\[ = \int L((Q^3)_y) w_n
\]

\[ - 10 b_n \int (Q^3)_y Q^3 w_n^2 - b_n^2 \int (Q^3)_y (10 Q^2 w_n^3 + 5 b_n Q w_n^4 + b_n^2 w_n^5), \]

and

(110)

\[
\theta_n \int y Q^2 w_n + \beta_n \left( \frac{1}{2} \int Q^2 - b_n \int Q y w_n \right)
\]

\[ = 20 \int Q^3 Q^2 w_n^2 - 10 b_n \int Q Q^3 w_n^2 - b_n^2 \int Q (10 Q^2 w_n^3 + 5 b_n Q w_n^4 + b_n^2 w_n^5). \]

Let

\[
\alpha_n(s) = \frac{\int L((Q^3)_y) w_n(s)}{\int Q^4}, \quad \beta_n(s) = \frac{20 \int Q^3 Q^2 w_n(s)}{\int Q^2},
\]

by (109) and (110), and the uniform bound on the \( H^1 \) norm of \( w_n \), we have, for \( n \) large enough

(111)

\[
|\theta_n(s) - \alpha_n(s)| \leq C b_n, \quad |\beta_n(s) - \beta_n(s)| \leq C b_n.
\]

This concludes the proof of Lemma 8. \( \square \)

Formally, when we pass to the limit as \( n \to \infty \), we expect to obtain that \( \alpha_n \to \alpha, \beta_n \to \beta \) and \( w_n \to w \), where \( w \) is solution of a limit equation:

(112)

\[
w_s - (L w)_y = \alpha(s) \left( \frac{Q}{2} + y Q_y \right) + \beta(s) Q_y, \quad (s, y) \in \mathbb{R} \times \mathbb{R},
\]

which satisfies (H1') and (H2'). In the two following steps, we justify such a convergence result.
**Step 2.** Some properties of the limit equation.

Consider first the following linear equation:

\[ w_{1s} - (Lw_1)_y = 0, \quad (s, y) \in \mathbb{R} \times \mathbb{R}. \]

We show that this equation is globally well-posed in \( H^1 \). This result makes use of the norm \( \Sigma^T \) introduced in [11], and defined in Step 2 of the proof of Proposition 1.

**Lemma 9 (Well-posedness of (113) in \( H^1 \)).** – Let \( w_0 \in H^1(\mathbb{R}) \). Then there exists a unique global solution \( w_1 \in C(\mathbb{R}, H^1(\mathbb{R})) \) of (113) satisfying \( w_1(0) = w_0 \).

**Remark.** – One can also show that equation (113) is globally well-posed in \( L^2 \).

**Proof of Lemma 9.** – First, let \( \tilde{w}_1(s, y) = w_1(s, y - s) \) and \( \tilde{Q}(s, y) = Q(y - s) \), so that solve the equation

\[ w_{1s} + w_{1yy} - w_{1y} + 5(\tilde{Q}^4w_1)_y = 0, \quad w_1(0) = w_0, \]

is equivalent to solve

\[ \tilde{w}_{1s} + \tilde{w}_{1yyy} + 5(\tilde{Q}^4\tilde{w}_1)_y = 0, \]

with initial condition \( \tilde{w}_1(0) = w_0 \in H^1 \).

Recall that we have defined:

\[ \Sigma^S(\zeta) = \max \left( \sup_{0 < s < s_0} |\zeta|_{H^1}, |\xi|_{L^2_s}, |\zeta|_{L^2_{\infty}L^\infty_s}, (1 + S)^{-1}|\zeta|_{L^2_{\infty}L^\infty_s} \right). \]

For \( 0 < s_0 < 1 \) and \( K > 0 \) to be chosen later, define:

\[ M = \{ \tilde{w}_1 \in C((-s_0, s_0), H^1(\mathbb{R})): \Sigma^{s_0}(\tilde{w}_1) \leq (K + 1) |w_0|_{H^1} \}. \]

Set, for \( \tilde{w}_1 \in M \) and \( s \in (-s_0, s_0) \),

\[ \Phi(\tilde{w}_1(s)) = S(s)w_0 - \int_0^s S(s - s')(S\tilde{Q}^4\tilde{w}_1)_y \, ds'. \]

By (54) and (55), we have

\[ \Sigma^{s_0}(\Phi(\tilde{w}_1)) \leq C|w_0|_{H^1} + Cx_0^{1/2} \left[ \left| (\tilde{Q}^4\tilde{w}_1)_y \right|_{L^2_{\infty}L^2_s} + \left| (\tilde{Q}^4\tilde{w}_1)_{yy} \right|_{L^2_{\infty}L^2_s} \right]. \]

By the decay properties of \( \tilde{Q} \), we have

\[ \forall s \in (-1, 1), \forall y \in \mathbb{R}, \quad |\tilde{Q}(s, y)| + |\tilde{Q}_y(s, y)| + |\tilde{Q}_{yy}(s, y)| \leq Ce^{-|y|}; \]

therefore \( \forall s \in (-s_0, s_0), \forall y \in \mathbb{R}, \)

\[ \left| (\tilde{Q}^4\tilde{w}_1)_y(s, y) \right|^2 + \left| (\tilde{Q}^4\tilde{w}_1)_{yy}(s, y) \right|^2 \leq C e^{-8|y|} \left[ |\tilde{w}_1(s, y)|^2 + |\tilde{w}_{1y}(s, y)|^2 + |\tilde{w}_{1yy}(s, y)|^2 \right]. \]

It follows that
\[
\left| \left( \frac{\partial^2}{\partial t^2} \tilde{w}_1 \right)_y \right|^2_{L^2_y L^2_y} + \left| \left( \frac{\partial^2}{\partial t^2} \tilde{v}_1 \right)_{yy} \right|^2_{L^2_y L^2_y} \leq C s_0 \sup_{t \in (-s_0, s_0)} \left| \tilde{w}_1(s) \right|_{H^1_y} + \left| \tilde{v}_1(y) \right|_{L^2_y} e^{-8y^2} \left| L_y^\top \right| \leq C \left( \Sigma^{s_0}(\tilde{w}_1) \right)^2,
\]

where \( C \) is an universal constant.

Thus, eventually,
\[
\Sigma^{s_0}(\Phi(\tilde{w}_1)) \leq C|w_0|_{H^1} + C s_0^{1/2} \Sigma^{s_0}(\tilde{w}_1) \leq C|w_0|_{H^1} + C(K + 1)s_0^{1/2}|w_0|_{H^1}.
\]

It follows that for \( K = C \) and \( s_0 = \left( \frac{1}{\int (1 + y)^2} \right)^2 \), which is independent of \( |w_0|_{H^1} \), \( \Phi \) maps \( \mathcal{M} \) into itself. By a similar argument, one can show that \( \Phi \) is a contraction on \( \mathcal{M} \), for \( s_0 \) small enough, independent of \( |w_0|_{H^1} \).

It follows that for any \( w_0 \in H^1(\mathbb{R}) \), there exists a unique local solution of (113) in \( H^1(\mathbb{R}) \), on an certain interval of time \([-s_0, s_0]\), independent of \( |w_0|_{H^1} \). Therefore, by a standard iteration argument, there exists a global solution \( w_1 \) of (113). The proof of Lemma 9 is thus complete.

Let us exhibit some explicit solutions of (112) which will be crucial in what follows.

**Lemma 10 (Explicit solutions of (112) and (113)).**

(i) **Explicit solutions of (113):** for any \( \gamma, \delta \in \mathbb{R} \), the function
\[
w_1 = \gamma \left( \frac{Q}{2} + y Q_y \right) + (-2\gamma s + \delta) Q_y
\]

is a solution of (113).

(ii) **Explicit solutions of (112):** for any continuous functions \( \alpha \) and \( \beta \), the function
\[
w = \gamma(s) \left( \frac{Q}{2} + y Q_y \right) + \delta(s) Q_y,
\]

where \( \gamma(s) = \int_0^s \alpha(s') \, ds' \), and \( \delta(s) = \int_0^s (\beta(s') - 2\gamma(s')) \, ds' \), is a solution of (112).

The interest of the explicit solutions given in Lemma 10(ii) is the following: whenever we have a solution \( w \) of (112), we can set \( w_1 = w - (\gamma(s) \left( \frac{Q}{2} + y Q_y \right) + \delta(s) Q_y) \), so that \( w_1 \) is a solution of (113). By Lemma 9, this will allow us to treat equation (112).

**Proof of Lemma 10.** – First, we recall some useful calculations.

(a) \( L(Q_y) = 0 \). Indeed, we have
\[
L(Q_y) = -Q_{yy} + 5Q^4 Q_y = -Q + Q^5 + Q_y = 5Q^4 Q_y = 0,
\]

where we have used \( Q_{yy} = Q - Q^5 \).

(b) \( L\left( \frac{Q}{2} + y Q_y \right) \). Indeed,
\[
L\left( \frac{Q}{2} + y Q_y \right) = -\left( \frac{Q}{2} + y Q_y \right)_{yy} + \frac{Q}{2} + y Q_y - 5Q^4 \left( \frac{Q}{2} + y Q_y \right)
\]
\[
= -\frac{5}{2} Q_{yy} - y Q_{yy} + \frac{Q}{2} + y Q_y - 5Q^4 - 5y Q^4 Q_y
\]
\[
= -\frac{5}{2} Q + \frac{5}{2} Q^5 - y Q_y - y (Q^5) + \frac{Q}{2} + y Q_y - 5Q^5 - 5y Q^4 Q_y
\]
\[
= -2Q.
\]
(i) Using (a) and (b), we have:
\[ w_{1s} = -2\gamma_0 Q_y, \]
and
\[ -(Lw_{1y}) = -\gamma_0 \left( L \left( \frac{Q}{2} + y Q_y \right) \right)_y + (-2\gamma_0 + \delta_0) L(Q_y) = 2\gamma_0 Q_y, \]
therefore,
\[ w_{1s} - (Lw_{1y}) = 0. \]

(ii) By using (a) and (b), and the expressions of \( \gamma \) and \( \delta \), we have
\[ \alpha(s) \left( \frac{Q}{2} + y Q_y \right) + (\beta(s) - 2\gamma(s)) Q_y, \]
and
\[ -(Lw)_y = -\gamma(s) \left( L \left( \frac{Q}{2} + y Q_y \right) \right)_y + \delta(s) L(Q_y) = 2g(s) Q_y, \]
therefore
\[ w_s - (Lw)_y = \alpha(s) \left( \frac{Q}{2} + y Q_y \right) + \beta(s) Q_y. \]

Thus, Lemma 10 is proved. \( \Box \)

**Step 3.** Convergence to the limit problem.

By Lemma 8, the sequence \( (w_n) \) satisfies:
\[ \left| w_n(0) \right|_{L^2} \geq \frac{1}{2}, \quad \left| w_n(0) \right|_{H^1} \leq C, \quad \forall y \in \mathbb{R}, \quad \left| w_n(0, y) \right| \leq C e^{-\delta_0 |y|}. \]

Therefore, there exists a subsequence, still denoted by \( (w_n) \), and \( w_0 \in H^1(\mathbb{R}) \), such that
\[ w_n(0) \rightarrow w_0 \quad \text{in } L^2 \text{ strong}. \]

In particular, \( |w_0|_{L^2} \geq 1/2 \), and so \( w_0 \neq 0 \).

We claim the following lemma, which completes the proof of Proposition 3.

**Lemma 11 (Convergence of the sequence \( (w_n) \)).** – We have
\[ w_n \rightarrow w \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R})), \]
where \( w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) satisfies:
\[ w \neq 0, \quad w(0) = w_0, \quad w_s - (Lw)_y = \alpha(s) \left( \frac{Q}{2} + y Q_y \right) + \beta(s) Q_y, \quad (s, y) \in \mathbb{R}^2, \]
for some continuous functions \( \alpha \) and \( \beta \).
Moreover, there exist $C > 0$ and $\theta_2 > 0$ such that $w$ satisfies

\begin{align*}
(H1') & \quad \forall s \in \mathbb{R}, \quad (w(s), Q^3) = 0, \quad (w(s), Q_y) = 0, \\
(H2') & \quad \forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |w(s, y)| \leq C e^{-\theta_2 |y|}.
\end{align*}

The proof of Lemma 11 is given in Appendix B2.

**Part B: Linear estimates**

The objective of this section is to show that the solution $w$ obtained in Part A does not exist, which will conclude the proof of Theorem 1.

Throughout Part B, let $w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of:

\begin{equation}
(117) \quad w_y - (Lw)_y = \alpha(s) \left( \frac{Q}{2} + yQ_y \right) + \beta(s) Q_y, \quad (s, y) \in \mathbb{R} \times \mathbb{R},
\end{equation}

where $\alpha$ and $\beta$ are continuous functions of $s$. Let $w_1 \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of:

\begin{equation}
(118) \quad w_{1y} - (Lw_1)_y = 0, \quad (s, y) \in \mathbb{R} \times \mathbb{R}.
\end{equation}

Consider the following assumptions:

\begin{itemize}
  \item [(H1')] Orthogonality conditions: $\forall s \in \mathbb{R}, \quad (w(s), Q^3) = 0, \quad (w(s), Q_y) = 0$.
  \item [(H2')] Exponential decay condition: $\forall (s, y) \in \mathbb{R} \times \mathbb{R}, \quad |w(s, y)| \leq C e^{-c|y|}$, for some constants $C > 0$ and $c > 0$.
\end{itemize}

We claim the following result.

**Theorem 3** (A linear Liouville theorem for equation (117)). – Let $w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (117) satisfying (H1’) and (H2’); then

\[ w \equiv 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}. \]

**Remark.** – The same result is true with different orthogonality conditions. Indeed, (H1’) can be replaced by $(w, \chi_1) = 0$, where $\chi_1$ and $\chi_2$ satisfy $(\chi_1, \chi_2 + yQ_y) \neq 0$ and $(\chi_2, Q_y) \neq 0$. See Section 5.

As an easy consequence of the proof of Theorem 3, and of explicit solutions of equation (117), we obtain the following result:

**Corollary 1** (Characterization of stationary solutions of (118)). – Let $w_1 \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (118) satisfying (H2’), then

\[ w_1 \equiv \delta_0 Q_y \]

for some constant $\delta_0 \in \mathbb{R}$.

**Remark.** – In fact, we obtain by Corollary 1 a characterization of the stationary solutions of (118).

From Theorem 3, we obtain a contradiction in the proof of Theorem 1. Therefore, now Theorem 3 implies Theorem 1.
The operator $L$ is a very special operator (see Titchmarsh [22]), whose linear structure is explicit through a change of variable. However, the proof of Theorem 3 is in fact completely nonlinear, in the sense that it is based on the study of the time variation of some quantities of energy type and Viriel type.

4. Identities for equations (117) and (118)

We begin this section with a lemma which gives four identities for equation (118). Recall that equation (118) is well-posed in $H^1$ (see Lemma 9).

Let us recall that equation (18) has four important quantities:

\begin{align*}
M_0 &= \int Q\varepsilon(s) + \frac{1}{2} \int \varepsilon^2(s), \\
E(Q + \varepsilon(s)) &= \lambda^2(s)E_0, \\
J_0(s) &= \int \varepsilon(s) \left( \int_0^s \frac{Q}{2} + zQ_z \right), \\
I_0(s) &= \frac{1}{2} \int y\varepsilon^2(s).
\end{align*}

They are closely related to four important quantities for the linear problem (118), which can be seen as a limit problem of (18):

\begin{align*}
(w_1, Q), \\
(Lw_1, w_1),
\end{align*}

\begin{align*}
J_1(s) &= \int w_1(s) \left( \int_0^s \frac{Q}{2} + zQ_z \right), \\
I_1(s) &= \frac{1}{2} \int yw_1^2(s).
\end{align*}

Indeed, for equation (118), we prove the following identities.

**Lemma 12 (Identities for equation (118)).** Let $w_1 \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (118) such that $yw_1^2(0) \in L^1(\mathbb{R})$ and $J_1(0)$ is defined, then, $\forall s \in \mathbb{R}$,

\begin{itemize}
  \item[(i)] $(w_1(s), Q) = (w_1(0), Q)$,
  \item[(ii)] $(Lw_1(s), w_1(s)) = (Lw_1(0), w_1(0))$,
  \item[(iii)] $\frac{d}{ds}J_1(s) = 2(w_1(0), Q)$,
  \item[(iv)] $\frac{d}{ds}I_1(s) = -H(w_1(s), w_1(s))$, where, for $w \in H^1(\mathbb{R})$,
\end{itemize}

\begin{align*}
H(w, w) &= -((Lw)_y, yw) = \frac{3}{2}(Lw, w) - \int w^2 + 10 \int Q^3 \frac{Q}{2} + yQ_y w^2 \\
&= -(L_1 w, w),
\end{align*}

where

\begin{align*}
L_1 w &= -\frac{3}{2}w_{yy} + \frac{1}{2}w - \frac{5}{2}Q^4 w + 10yQ^3 Q_y w.
\end{align*}

**Proof.** To prove this lemma, we make formal calculations that can be justified rigorously by the regularization arguments used in [14], Lemmas 5 and 6. (The proof would be exactly the same, we have to consider a sequence $w_{1n}(0) \in H^1(\mathbb{R})$, such that $w_{1n}(0) \to w_1(0)$ in $H^1$.)

The lemma is a consequence of some algebraic properties of the operator $L$. Let us first recall (see Part A, proof of Lemma 10):

\begin{itemize}
  \item[(a)] $L(Q_y) = 0$.
  \item[(b)] $L(\frac{Q}{2} + yQ_y) = -2Q$.
\end{itemize}
First, by taking the scalar product of equation (118) with $Q$, and integrating by parts, we obtain
\[
\frac{d}{ds}(w_1, Q) = (Lw_1)_y, Q) = -(Lw_1, Q) = -(w_1, LQ_y) = 0,
\]
since $LQ_y = 0$. It follows that $\forall s \in \mathbb{R}$, $(w_1(s), Q) = (w_1(0), Q)$.

Next, by taking the scalar product of (118) with $Lw_1$, we have, formally,
\[
0 = \frac{1}{2}(w_1, Lw_1) + (w_1, (Lw_1)_y)) = \frac{1}{2} \frac{d}{ds}(Lw_1, w_1),
\]
and so $\forall s \in \mathbb{R}$, $(Lw_1(s), w_1(s)) = (Lw_1(0), w_1(0))$.

By multiplying the equation by $\int_0^s \frac{Q}{2} + zQ_z$, we have
\[
\frac{d}{ds} j(s) = \int (Lw_1)_y, \int_0^s \frac{Q}{2} + zQ_z).
\]
Since
\[
L \left( \int_0^s \frac{Q}{2} + zQ_z, \right) = L \left( \frac{Q}{2} + yQ_y \right) = -2Q,
\]
we obtain
\[
\frac{d}{ds} j(s) = 2 \int Qw_1(s) = 2 \int Qw_1(0).
\]

We turn to (iv). By taking the scalar product of equation (118) with $yw_1$, we have
\[
\frac{1}{2} \frac{d}{ds} \int yw_1^2 = ((Lw_1)_y, yw_1) = -(Lw_1, w_1) - (Lw_1, yw_1y).
\]

Using the expression of $Lw_1 = -w_{1yy} + w_1 - 5Q^4w_1$, and then integrating by parts, we obtain:
\[
(Lw_1, yw_1y) = \int (-w_{1yy} + w_1 - 5Q^4w_1)yw_1y
\]
\[
= \frac{1}{2} \int w_{1y}^2 - \frac{1}{2} \int w_1^2 + 10 \int Q^3Q_zw_{1y}^2 + \frac{5}{2} \int Q^4w_1^2
\]
\[
= \frac{1}{2} (Lw_1, w_1) - \int w_1^2 + 10 \int Q^3Q_z \left( \frac{Q}{2} + yQ_y \right) w_1^2,
\]
and thus
\[
\frac{1}{2} \frac{d}{ds} \int yw_1^2 = -H(w_1, w_1),
\]
where $H(w_1, w_1)$ is given by
\[
H(w_1, w_1) = \frac{3}{2} (Lw_1, w_1) - \int w_1^2 + 10 \int Q^3Q_z \left( \frac{Q}{2} + yQ_y \right) w_1^2.
\]
Another expression of $H$ is given by $H(w_1, w_1) = (L_1w_1, w_1)$, where
\[
L_1w_1 = -\frac{3}{2} w_{1yy} + \frac{1}{2}w_1 - \frac{5}{2}Q^4w_1 + 10yQ^3Q_yw_1.
The proof of Lemma 12 is now complete. □

Remark. – In particular, if we consider a solution of (118) which satisfies (H2'), then J(s) is uniformly bounded in time, and thus we necessarily have (w(s), Q) = 0, for all s ∈ R.

Now, we turn to equation (117). Let w be a solution of (117) and define the functions J and I by

\[ J(s) = \int w(s, y) \left( \int_0^y \frac{Q}{2} + zQ_z \right) \, dy, \]

\[ I(s) = \frac{1}{2} \int yw^2(s, y) \, dy. \]

We show the following lemma giving relations for equation (117) inherited from equation (118) and its special solutions.

**Lemma 13 (Identities for equation (117)).** – Let \( w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) be a solution of (117) such that \( yw^2(0) \in L^1(\mathbb{R}) \), and J(0) is defined, then, \( \forall s \in \mathbb{R} \),

(i) \( (w(s), Q) = (w(0), Q) \),

(ii) \( (Lw(s), w(s)) = (Lw(0), w(0)) - 4(\int_0^s \alpha(s') \, ds')(w(0), Q) \),

(iii) \( \frac{d}{ds} J(s) = 2(w(0), Q) \),

(iv) \( \frac{d}{ds} I(s) = -H(w, w) + \alpha(s) \int yQ_y + yQ_y w(s) + \beta(s) \int yQ_y w(s) \), where \( H(w, w) \) is defined in Lemma 12.

If in addition, the solution \( w \) satisfies (H2'), then, \( \forall s \in \mathbb{R} \),

(v) \( (w(s), Q) = 0 \),

(vi) \( (Lw(s), w(s)) = (Lw(0), w(0)) \),

(vii) \( J(s) = J(0) \).

**Proof.** – The proof is very similar to the one of Lemma 12. One needs only care in the calculation about the contribution of the additional term \( \alpha(s)(\frac{Q}{2} + yQ_y) + \beta(s)Q_y \) in the right-hand side of equation (117).

First, by taking the scalar product with Q, we obtain

\[ \frac{d}{ds}(w, Q) = ((Lw)_y, Q) + \alpha(s)(Q \cdot \frac{Q}{2} + yQ_y) + \beta(s)(Q, Q_y). \]

But

\[ \int Q \left( \frac{Q}{2} + yQ_y \right) = \frac{1}{2} \int Q^2 - \frac{1}{2} \int Q^2 = 0, \]

by integration by parts and

\[ \int QQ_y = 0, \]

by parity. On the other hand, as in the proof of Lemma 12, we have \( ((Lw)_y, Q) = -(w, LQ_y) = 0 \).

Therefore, we obtain \( \frac{d}{ds}(w(s), Q) = 0 \), and so

\[ \forall s \in \mathbb{R}, \quad (w(s), Q) = (w(0), Q). \]
Next, when we take the scalar product of equation (117) by $Lw$, instead of $\frac{d}{ds}(Lw, w) = 0$, we obtain the following expression:

$$\frac{1}{2} \frac{d}{ds}(Lw, w) = \alpha(s) \left( Lw, \frac{Q}{2} + y Q_y \right) + \beta(s)(Lw, Q_y).$$

We have

$$\left( Lw, \frac{Q}{2} + y Q_y \right) = \left( w, L \left( \frac{Q}{2} + y Q_y \right) \right) = -2(w, Q)$$

and

$$(Lw, Q_y) = (w, LQ_y) = 0.$$ 

Therefore, (121)

$$\frac{1}{2} \frac{d}{ds}(Lw, w) = -2\alpha(s)(w(0), Q).$$

By taking the scalar product of equation (117) by $\int_0^y (\frac{Q}{2} + z Q_z)$, we obtain:

$$\frac{d}{ds} J(s) = 2(w(0), Q) + \alpha(s) \int \left( \frac{Q}{2} + y Q_y \right) \left( \int_0^y \frac{Q}{2} + z Q_z \right) + \beta(s) \int Q_y \left( \int_0^y \frac{Q}{2} + z Q_z \right).$$

Note that by parity, we have

$$\int \left( \frac{Q}{2} + y Q_y \right) \left( \int_0^y \frac{Q}{2} + z Q_z \right) = 0;$$

on the other hand

$$\int Q_y \left( \int_0^y \frac{Q}{2} + z Q_z \right) = -\int Q \left( \frac{Q}{2} + y Q_y \right) = 0.$$ 

We conclude

$$\frac{d}{ds} J(s) = 2(w(0), Q).$$

For the Viriel type identity, we argue as in the proof of Lemma 12, and we obtain

$$\frac{1}{2} \frac{d}{ds} \int y w^2(s) = -H(w(s), w(s)) + \alpha(s) \int y \left( \frac{Q}{2} + y Q_y \right) w(s) + \beta(s) \int y Q_y w(s).$$

Therefore, we have already proved (i), (ii), (iii), and (iv).

Now, let us assume that the solution $w$ satisfies (H2$^\prime$). Assumption (H2$^\prime$) implies that

$$\forall s \in \mathbb{R}, \quad |J(s)| \leq C \int |w(s, y)| \, dy \leq C \int e^{-c|y|} \, dy \leq C,$$

for some constant $C > 0$. This completes the proof.
which means that $J(s)$ is uniformly bounded in $s \in \mathbb{R}$ (note that an $H^1$ bound on $w$ would not be sufficient). Since $\frac{d}{ds}J(s) = 2(w(0), Q)$, we necessarily have

$$\forall s \in \mathbb{R}, \quad (w(s), Q) = (w(0), Q) = 0$$

and so $J(s) = J(0)$.

By (121), we conclude that

$$\forall s \in \mathbb{R}, \quad (Lw(s), w(s)) = (Lw(0), w(0)).$$

Therefore (v), (vi), and (vii) are proved. □

5. Choice of orthogonality conditions and positiveness of the quadratic form $H$

By Section 3, we know that if $w$ is a global bounded $H^1$ solution of equation (117) satisfying (H2'), then $w$ is orthogonal to $Q$ for any time and $(Lw, w)$ is a conserved quantity. Moreover, we have the following Viriel type identity for $w$:

$$\frac{d}{ds}I(s) = -H(w(s), w(s)) + \alpha(s) \int y \left( \frac{Q}{2} + yQ_y \right) w(s) + \beta(s) \int yQ_y w(s).$$

The objective is this section is to control the right-hand side of this identity under suitable orthogonality conditions on $w$ and to give it a sign. This will be done in two steps.

In fact, in view of identity (122), it is clearly interesting to use a solution $w$ such that

$$\left( w(s), yQ_y \right) = \left( w(s), y \left( \frac{Q}{2} + yQ_y \right) \right) = 0.$$ (123)

Indeed, under these orthogonality conditions, the Viriel identity will reduce to:

$$\frac{d}{ds}I(s) = -H(w(s), w(s)).$$

Step 1. Adapted orthogonality conditions for the Viriel identity.

The next lemma shows that, using the explicit solutions of (118), it is possible to modify the function $w$ so that it satisfies the orthogonality conditions (123).

**Lemma 14.** – Let $w \in C(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (117). Then there exist two continuous functions $s \mapsto \gamma(s)$ and $s \mapsto \delta(s)$ such that:

$$\overline{w} = w + \gamma(s) \left( \frac{Q}{2} + yQ_y \right) + \delta(s)Q_y$$ (124)

satisfies

$$\forall s \in \mathbb{R}, \quad (\overline{w}, yQ_y) = \left( \overline{w}, y \left( \frac{Q}{2} + yQ_y \right) \right) = 0.$$ (H2')

Moreover, $\overline{w}$ is solution of

$$\overline{w}_s - (L\overline{w})_y = \overline{w}(s) \left( \frac{Q}{2} + yQ_y \right) + \mathcal{F}(s)Q_y,$$ (125)
Proof. – The proof follows from the explicit solutions of (118) and a transversality condition. First, we claim that
\[ Z_y Q_y \] is strictly positive. Indeed, since
\[ R Q (Q^2 - C y Q_y)^2 > 0, \]
we have
\[ \frac{1}{\mu_0} \int y Q_y \left( \frac{Q}{2} + y Q_y \right) \] is strictly positive. Indeed, since \( \int Q (Q^2 + y Q_y) = 0 \), we have
\[ \mu_0 = \int \left( \frac{Q}{2} + y Q_y \right)^2 = \int \left( \frac{Q}{2} + y Q_y \right) \left( \frac{Q}{2} + y Q_y \right) = \int y Q_y \left( \frac{Q}{2} + y Q_y \right) > 0. \]
Next, set
\[ w(s) = \frac{1}{\mu_0} \int y Q_y w(s), \quad \delta(s) = -\frac{1}{\mu_0} \int y \left( \frac{Q}{2} + y Q_y \right) w(s), \]
and consider \( \overline{w}(s) \) as given in (124). Then, \( \forall s \in \mathbb{R} \),
\[ \langle \overline{w}(s), y Q_y \rangle = (w(s), y Q_y) + y(s) \mu_0 + \delta(s) \left( Q_y, y Q_y \right) = 0, \]
by \( (Q_y, y Q_y) = 0 \) and the expression of \( y(s) \). Similarly, we have
\[ \forall s \in \mathbb{R}, \quad \left( \overline{w}(s), y \left( \frac{Q}{2} + y Q_y \right) \right) = 0. \]
Observe that \( y \) and \( \delta \) are \( C^1 \) functions of \( s \) by the arguments of [14], proof of Lemma 4.
Now, by direct calculations, \( \overline{w} \) satisfies (125), with \( \overline{w}(s) = \alpha(s) + y'(s) \) and \( \overline{\beta}(s) = \beta(s) - 2y(s) + \delta'(s) \).
Since \( \overline{w}(s) \) is orthogonal to \( y Q_y \) for all \( s \in \mathbb{R} \), we find an expression of \( \overline{w}(s) \) by taking the scalar product of equation (125) with \( y Q_y \). We obtain
\[ y \left( \frac{Q}{2} + y Q_y \right) \] and so
\[ \overline{w}(s) = \frac{1}{\mu_0} \int \overline{w}(s) L \left( \left( y Q_y \right)_y \right) = \frac{1}{\mu_0} \int \overline{w}(s) L \left( y Q_y \right)_y. \]
Similarly, we have
\[ \overline{\beta}(s) = \frac{1}{\mu_0} \int \overline{w}(s) L \left( y \left( \frac{Q}{2} + y Q_y \right) \right)_y. \] Since
we obtain
\[ \bar{\beta}(s) = \frac{1}{\mu_0} \int \overline{w}(s)(Q - 3Q^5 - L(y^2 Q_{yy})). \]

Therefore, starting from a global, bounded solution \( w \) of (117), we can obtain, through a simple transformation, a global bounded solution \( \overline{w} \) of (125) (which is in fact the same equation, with different coefficients), satisfying (H2').

Assume that \( w \) satisfies (H2'), then \( \overline{w} \) also satisfies (H2') from the fact that \( |y(s)| + |\delta(s)| \leq C \). Therefore, we can apply Lemma 13 to \( \overline{w} \). We obtain:
\[ \forall s \in \mathbb{R}, \quad (\overline{w}(s), Q) = 0, \]
which is a third orthogonality property on \( \overline{w} \). This will be crucial in the next step.

**Step 2.** Positivity property of \( H \).
In general, \( H(w, w) \) has no sign. In fact, we will show that there is a space of dimension 2 where \( H \) is definite positive. Indeed, we now claim a lower positive bound on \( H(w, w) \), in the case where \( w \) is orthogonal to \( Q \) and \( y(Q^2 + yQ_y) \).

**Proposition 4** (Positiveness of \( H \) under the orthogonality condition). – If \( w \in H^1(\mathbb{R}) \) satisfies
\[ (w, Q) = \left( w, y\left( \frac{Q}{2} + yQ_y \right) \right) = 0 \]
then
\[ H(w, w) \geq \frac{1}{10} (Lw, w). \]

**Remark.** – We point out that the assumption \( (w, yQ_y) = (w, y(\frac{Q}{2} + yQ_y)) = 0 \) would not be sufficient to have \( H(w, w) \geq 0 \). The property \( (w, Q) = 0 \) is fundamental in Proposition 4. Indeed, there are functions \( w \) such that \( H(w, w) < 0 \) and \( (w, yQ_y) = 0 \).

**Remark.** – It is also possible to show under the assumptions of Proposition 4 that \( H(w, w) \geq \rho(w, w) \), for some \( \rho > 0 \).

**Remark.** – Recall that from Weinstein [24], Proposition 2.7, we have \( (Lw, w) \geq 0 \) for any \( w \) such that \( (w, Q) = 0 \).

**Proof of Proposition 4.** – See Appendix C. The proof is based on a pointwise minoration by a quadratic form of classical type. The index is proved to be 2. Direct calculations then allow us to reduce the proof to check that two scalar products satisfy given conditions which would be checked numerically.

**6. Conclusion of the proof of Theorem 3**

We are now able to conclude the proof of Theorem 3.

Let \( w \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) be a solution of (117) satisfying (H1') and (H2') as in the statement of Theorem 3.
Consider \( \bar{w} \) as given in Lemma 14. The function \( \bar{w} \) satisfies the same kind of equation as \( w \) (with different coefficients in the right hand side, see (125)), and in addition, we have

\[
\forall s \in \mathbb{R}, \quad (\bar{w}(s), y Q) = \left( \left( \bar{w}(s), y \left( \frac{Q}{2} + y Q \right) \right) = 0. \right)
\]

Since \( w \) satisfies (H2'), \( \bar{w} \) also satisfies (H2'), and by Lemma 13(v), (vi) and (iv), we have

\[
(\bar{w}(s), Q) = 0,
\]
\[
(L \bar{w}(s), \bar{w}(s)) = (L \bar{w}(0), \bar{w}(0)),
\]
\[
\frac{d}{ds} \bar{T}(s) = -\frac{d}{ds} \int y \bar{w}^2(s) = -H(\bar{w}, \bar{w}).
\]

We apply Proposition 4 to \( \bar{w}(s) \), for all \( s \in \mathbb{R} \), and we find

\[
\frac{d}{ds} \bar{T}(s) \leq -\frac{1}{10} (L \bar{w}(s), \bar{w}(s)) = -\frac{1}{10} (L \bar{w}(0), \bar{w}(0)).
\]

Recall that (Proposition 2.7 in [24]),

\[
\inf_{(w, Q) = 0} (Lw, w) = 0.
\]

It follows that \( (L \bar{w}(0), \bar{w}(0)) \geq 0 \). But, since \( \bar{w} \) satisfies (H2'), the function \( \bar{T}(s) \) is uniformly bounded in \( s \in \mathbb{R} \). Therefore, we have:

\[
\forall s \in \mathbb{R}, \quad (L \bar{w}(s), \bar{w}(s)) = (L \bar{w}(0), \bar{w}(0)) = 0.
\]

We need the following lemma:

**Lemma 15.** – Assume that \( w \in H^1(\mathbb{R}) \) satisfies \( (w, Q) = 0 \) and \( (Lw, w) = 0 \), then

\[
w = \gamma_0 \left( \frac{Q}{2} + y Q \right) + \delta_0 Q,
\]

for some \( \gamma_0, \delta_0 \in \mathbb{R} \).

**Proof of Lemma 15.** – Note first that \( (Q^3, \frac{Q}{2} + y Q) \neq 0 \). Indeed, by integration by parts, we have

\[
\mu_1 = \int Q^3 \left( \frac{Q}{2} + y Q \right) = \frac{1}{4} \int Q^4 > 0.
\]

Let

\[
\gamma_0 = \frac{1}{\mu_1} \int Q^3 w,
\]

and

\[
\delta_0 = \frac{1}{\int Q^3} \int Q y w,
\]
then
\[ \tilde{w} = w - \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) - \delta_0 Q_y \]
satisfies \((\tilde{w}, Q^3) = (\tilde{w}, Q_y) = 0\). By (20), we have
\[ (L \tilde{w}, \tilde{w}) \geq (\tilde{w}, \tilde{w}). \]

But
\[
(L \tilde{w}, \tilde{w}) = (L w, w) + 2 \left( w, L \left( \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) + \delta_0 Q_y \right) \right) \\
+ \left( L \left( \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) + \delta_0 Q_y \right), \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) + \delta_0 Q_y \right) \\
= (L w, w) - 2 \gamma_0 (w, Q) - 2 \gamma_0 \left( Q, \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) + \delta_0 Q_y \right) \\
= (L w, w) = 0,
\]
since \((w, Q) = (L w, w) = 0\).

It follows that \(\tilde{w} = 0\), and thus
\[ w = \gamma_0 \left( \frac{O}{2} + \gamma Q_y \right) + \delta_0 Q_y. \]

Thus, Lemma 15 is proved. \(\square\)

Observe that \(w = \tilde{w} - \gamma(s) \left( \frac{O}{2} + \gamma Q_y \right) - \delta(s) Q_y \), and so as in the proof of Lemma 15, we have \((L w, w) = (L \tilde{w}, \tilde{w}) = 0\). On the other hand, we have \((w, Q) = 0\), so that we can apply Lemma 15 to \(w(s)\), for all \(s \in \mathbb{R}\). Thus,
\[ w(s) = a(s) \left( \frac{O}{2} + \gamma Q_y \right) + b(s) Q_y. \]
Since for all \(s \in \mathbb{R}\), we have \((w(s), Q^3) = (w(s), Q_y) = 0\), we see that
\[ a(s) \int \left( \frac{O}{2} + \gamma Q_y \right) Q^3 = \frac{a(s)}{4} \int Q^4 = 0, \]
\[ b(s) \int Q_y^2 = 0, \]
which gives that \(w \equiv 0\) on \(\mathbb{R} \times \mathbb{R}\).

The proof Theorem 3 is thus complete.

The proof of Corollary 1 follows easily.

Proof of Corollary 1. – We have seen in the proof of Theorem 3 that the only solutions of (117) satisfying (H2') are of the form
\[ w(s, y) = \gamma(s) \left( \frac{O}{2} + \gamma Q_y \right) + \delta(s) Q_y. \]
Thus, we have

\[ w_s = (Lw)_y = \gamma' \left( \frac{Q}{2} + yQ_y \right) + (\delta' - 2\gamma')Q_y, \]

and so, if \( w \) is a solution of (118), then we have \( \gamma' = 0 \) and \( \delta' - 2\gamma' = 0 \). Therefore, \( \gamma = \gamma_0 \) and \( \delta = 2\gamma_0 \). Since \( w \) is supposed to be uniformly bounded in \( y \) and \( s \), we have \( \gamma_0 = 0 \), and then \( \delta = \delta_0 \in \mathbb{R} \). Finally, \( w(s, y) = \delta_0 Q_y(y) \), where \( \delta_0 \in \mathbb{R} \) is a constant. This concludes the proof of Corollary 1.

By Part A and Part B, we obtain the following Liouville theorem for \( \varepsilon \) which is equivalent to Theorem 1.

**Proposition 5 (Liouville theorem for \( \varepsilon \)).** Let \( \varepsilon \in C(\mathbb{R}, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R})) \) be a solution of (18) on \( \mathbb{R} \times \mathbb{R} \) satisfying:

(H1) Orthogonality conditions:

\[ \forall s \in \mathbb{R}, \quad (\varepsilon(s), Q^3) = (\varepsilon(s), Q_y) = 0. \]

(H2) \( H^1 \) bounds: There exists \( \lambda_1, \lambda_2 > 0 \) such that

\[ \forall s \in \mathbb{R}, \quad \lambda_1 \leq \lambda(s) \leq \lambda_2. \]

(H3) \( L^2 \) compactness: \( \forall \delta_0 > 0, \exists \delta_0 > 0, \) such that

\[ \forall s \in \mathbb{R}, \quad |\varepsilon(s)|_{L^2(|y|> \delta_0)} \leq \delta_0. \]

There exists \( a_1 > 0 \) such that if

\[ |\varepsilon(0)|_{H^1} \leq a_1, \]

then

\[ \varepsilon \equiv 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}. \]

**Remark.** If \( E_0 < 0 \), it suffices to assume in Proposition 5 that \( |\varepsilon_0|_{L^2} \) is small (and not \( |\varepsilon(0)|_{H^1} \) small). See at the end of the proof of (H4) in Section 2.

**Part C: Proof of Theorem 2**

We use Proposition 5 to prove the following result, which is equivalent to Theorem 2 stated in the introduction.

Let \( \varepsilon \in C(\mathbb{R}_+, H^1(\mathbb{R})) \cap L^\infty(\mathbb{R}_+, H^1(\mathbb{R})) \) be defined as in the introduction by:

\[ \varepsilon(t, y) = \lambda^{1/2}(t)u(t, \lambda(t)(x + x(t))) - Q(y), \]

and recall the relation between the variables of time \( t \) and \( s \):

\[ s = \int_0^t \frac{dt'}{\lambda^3(t')}, \quad \text{or equivalently,} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}. \]
PROPOSITION 6. – Assume that for some \( \lambda_1, \lambda_2 > 0 \), we have

\[
\forall s \geq 0, \quad \lambda_1 \leq \lambda(s) \leq \lambda_2.
\]

There exists \( a_2 > 0 \) such that if

\[
|\varepsilon(0)|_{H^1} \leq a_2,
\]

then

\[
\varepsilon(x) \to 0 \quad \text{in } H^1(\mathbb{R}),
\]

as \( s \to +\infty \).

Before proving Proposition 6, we state and prove a general monotonicity result concerning small solutions of the generalized KdV equation. This monotonicity property says that in some sense the ‘mass’ of a small solution can only travel to the left. This kind of result will be crucial in the proof of Proposition 6.

For \( K > 0 \), to be chosen later, we define:

\[
\forall x \in \mathbb{R}, \quad \phi(x) = \phi_K(x) = c Q \left( \frac{x}{K} \right),
\]

\[
\psi(x) = \psi_K(x) = \int_{-\infty}^{x} \phi(y) \, dy,
\]

where

\[
c = \frac{K}{\int_{-\infty}^{+\infty} Q}.
\]

so that

\[
(126) \quad \forall x \in \mathbb{R}, \quad 0 \leq \psi(x) \leq 1, \quad \lim_{x \to -\infty} \psi(x) = 0, \quad \lim_{x \to +\infty} \psi(x) = 1.
\]

Let \( z \) be a solution of:

\[
z_t + z_{xxx} + (z^5)_x = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},
\]

and define, for \( \sigma > 0 \),

\[
\forall t \geq 0, \quad I(t) = I_\sigma(t) = \int z^2(t, x)\psi(x - \sigma t) \, dx.
\]

We claim the following lemma:

**Lemma 16** (Monotonicity of \( I \) for small solutions of (127)). – Let \( z \in C([0, +\infty), H^1(\mathbb{R}) \cap L^\infty([0, +\infty), H^1(\mathbb{R}) \) be a solution of (127). For any \( \sigma > 0 \), if \( K \geq \sqrt{2/\sigma} \), and

\[
(128) \quad \sup_{t \geq 0} |z(t)|_{L^\infty} \leq d_0 = \left( \frac{3\sigma}{20} \right)^{1/4},
\]

then the function \( I \) is nonincreasing in \( t \).
Note that for a given $z(0)$ is small in $H^1$, then (128) is implied by the energy conservation.

**Proof of Lemma 16.** — We recall that if $x \mapsto \varphi(x)$ is a $C^3$ function such that:

$$|\varphi(x)| + |\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \leq C,$$

for some constant $C > 0$, then $t \mapsto \int z^2(t, x)\varphi(x)\,dx$ is $C^1$ and

$$\frac{d}{dt} \int z^2(t) \varphi = -3 \int z^2(t) \varphi' + \int z^2(t) \varphi^{(3)} + \frac{5}{3} \int z^6(t) \varphi'.$$

(Formally, one can obtain (129) by taking the scalar product of (127) by $z\varphi$ and integrating by parts. See for example the proof of Lemma 5 in [14].)

Therefore, we have:

$$\mathcal{I}'(t) = -3 \int z^2(t) \varphi'(x - \sigma t) - \sigma \int z^2(t) \varphi'(x - \sigma t) + \int z^2(t) \varphi''(x - \sigma t) + \frac{5}{3} \int z^6(t) \varphi'(x - \sigma t).$$

(130)

Note that $\psi''(x) = \varphi'(x) = \frac{c}{K^2} Q_x\left(\frac{x}{K}\right)$, and $\psi^{(3)}(x) = \frac{c}{K^3} Q_{xx}\left(\frac{x}{K}\right)$. Since $Q_{xx} = Q - Q^5 \leq Q$, we have:

$$\forall x \in \mathbb{R}, \quad \varphi''(x) \leq \frac{c}{K^2} Q\left(\frac{x}{K}\right) = \frac{1}{K^2} \varphi(x).$$

Thus, we have:

$$\mathcal{I}'(t) \leq -3 \int z^2(t) \varphi(x - \sigma t) - \sigma \int z^2(t) \varphi(x - \sigma t) + \int z^2(t) \varphi''(x - \sigma t) + \frac{5}{3} \int z^6(t) \varphi(x - \sigma t).$$

Since we choose $K \geq \sqrt{2/\sigma}$, we find

$$\mathcal{I}'(t) \leq -3 \int z^2(t) \varphi(x - \sigma t) - \frac{\sigma}{2} \int z^2(t) \varphi(x - \sigma t) + \frac{5}{3} \int z^6(t) \varphi(x - \sigma t).$$

Next, since $d^3 \leq d^3_0 = \frac{3\sigma}{2\varphi}$, we obtain:

$$\frac{5}{3} \int z^6(t) \varphi(x - \sigma t) \leq \frac{5}{3} \left|z(t)\right|^4_{L^\infty} \int z^2(t) \varphi(x - \sigma t) \leq \frac{5d^4}{3} \int z^2(t) \varphi(x - \sigma t) \leq \frac{\sigma}{4} \int z^2(t) \varphi(x - \sigma t),$$

and so,

$$\forall t \geq 0, \quad \mathcal{I}'(t) \leq -3 \int z^2(t) \varphi(x - \sigma t) - \frac{\sigma}{4} \int z^2(t) \varphi(x - \sigma t) \leq 0.$$

Thus Lemma 16 is proved. \qed
Now, we prove Proposition 6.

**Proof of Proposition 6.** – First, by arguing as in the proof of (H4) at the beginning of Part A, we have

\[ a = \sup_{s \geq 0} |\varepsilon(s)|_{H^1} \leq C \sqrt{a_2}. \]

Assume for the sake of contradiction that, for some sequence \( s_n \to +\infty \), we have

\[ \varepsilon(s_n) \neq 0, \quad \text{in } H^1. \]

Since \( |\varepsilon(s_n)|_{H^1} \leq C \) and \( \lambda_1 \leq \lambda(\varepsilon(s_n)) \leq \lambda_2 \), there exists a subsequence of \( (s_n) \), which we still denote by \( (s_n) \), \( \tilde{\varepsilon}_0 \in H^1(\mathbb{R}) \) and \( \tilde{\lambda}_0 > 0 \) such that

\[ \tilde{\varepsilon}_0 \neq 0, \quad \varepsilon(s_n) \to \tilde{\varepsilon}_0, \quad \text{in } H^1, \quad \text{and} \quad \lambda(s_n) \to \tilde{\lambda}_0. \]

Note that \( |\tilde{\varepsilon}_0|_{H^1} \leq a \).

Denote by \( \tilde{\varepsilon} \) the solution of (18) with \( \tilde{\varepsilon}(0) = \tilde{\varepsilon}_0 \) and \( \tilde{\lambda}, \tilde{\varepsilon} \) such that \( \tilde{\varepsilon} \) satisfies \( (\tilde{\varepsilon}, \mathcal{Q}^3) = (\tilde{\varepsilon}, Q^3) = 0 \). Set \( v(t, y) = Q(y) + \varepsilon(t, y) = \lambda^{1/2}(t)u(t, \lambda(t)y + x(t)), \) and \( \tilde{\varepsilon} = Q + \tilde{\varepsilon} \).

Let us start with crucial properties of \( L^2 \) interaction between the regions \( y \) large and \( y \) small, which will allow us to establish an \( L^2 \) compactness property on \( \tilde{v} \). Then, we will reach a contradiction by showing that \( \tilde{\varepsilon} \equiv 0 \), using the Liouville theorem for \( \varepsilon \), i.e. Proposition 5.

**LEMMA 17** (Stability of weak convergence with respect to time). – We have:

\begin{align*}
(134) \quad & \forall s \in \mathbb{R}, \quad \varepsilon(s_n + s) \to \tilde{\varepsilon}(s) \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to +\infty, \\
(135) \quad & \forall s \in \mathbb{R}, \quad v(s_n + s) \to \tilde{v}(s) \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to +\infty.
\end{align*}

**Proof.** – See Appendix D.

**LEMMA 18** (\( L^2 \) compactness of \( v \) on the right). – There exists \( a_4 > 0 \) such that if \( 0 < a < a_4 \), then we have the following property: \( \forall \delta_0 > 0, \exists R_2 = R_2(\delta_0) > 0 \) such that

\[ \forall t \geq 0, \quad \int_{y > R_2} v^2(t, y) \, dy \leq \delta_0. \]

**LEMMA 19** (Irreversibility of the loss of mass on the left). – There exists \( a_3 > 0, \quad a_3 = a_3(\lambda_1, \lambda_2), \) such that if \( 0 < a < a_3 \) then, for all \( \delta_0 > 0, \varepsilon_0 \in (0, 1), \) there exists \( R_1 = R_1(\delta_0, \varepsilon_0, \lambda_1, \lambda_2) > 0 \) such that for all \( \gamma_0 > R_1 \) and \( t_0 \geq 0, \)

\[ \forall t \geq t_0, \quad \int_{y < -\frac{\gamma_0}{1 - \varepsilon_0}} v^2(t, y) \, dy \geq (1 - \varepsilon_0) \int_{y < -\gamma_0} v^2(t_0, y) \, dy - \delta_0. \]

Assuming these three lemmas, let us conclude the proof of Proposition 6.

To show \( L^2 \) compactness of \( \tilde{\varepsilon} \) is equivalent to show \( L^2 \) compactness of \( \tilde{v} \). We claim that \( \tilde{v} \) is \( L^2 \) compact:

\[ \forall \delta_0 > 0, \exists R_0 = R_0(\delta_0) > 0, \quad \text{such that } \forall s \in \mathbb{R}, \quad \int_{|y| > R_0} \tilde{v}^2(s) < \delta_0. \]
Assume for the sake of contradiction that for some $\delta_0 > 0$, we have

\begin{equation}
\forall y_0 > 0, \exists s_1(y_0) \in \mathbb{R}, \text{ such that } \int_{|y| > y_0} \widetilde{v}^2(s_1(y_0)) \geq \delta_0. \tag{138}
\end{equation}

Set

$$m_0 = \int \widetilde{v}^2(0) = \int \widetilde{v}^2(s), \quad \forall s \in \mathbb{R}, \quad M_0 = \int v^2(0) = \int v^2(s), \quad \forall s \geq 0.$$  

Note that by (135), we have $M_0 \geq m_0$. We are going to use Lemmas 17, 18 and 19 to obtain a contradiction from (138).

First, $\forall y_0 > 0$, $s_1(y_0)$ being defined in (138) by:

$$\int_{|y| < y_0} \widetilde{v}^2(s_1(y_0)) = \int \widetilde{v}^2(s_1(y_0)) \leq m_0 - \delta_0.$$  

For fixed $y_0$, we have, by (135), since $v(s_n) \to \widetilde{v}(0)$ in $L^2_{\text{loc}}$,

$$v(s_1(y_0) + s_n) \to \widetilde{v}(s_1(y_0)), \quad \text{in } L^2_{\text{loc}} \text{ as } n \to +\infty,$$

and so

$$\forall y_0 > 0, \exists N(y_0) \in \mathbb{N}, \text{ such that } s_1(y_0) + s_{N(y_0)} \geq 0,$$

$$\int_{|y| < y_0} \left| v(s_1(y_0) + s_{N(y_0)}) - \widetilde{v}(s_1(y_0)) \right|^2 \leq \frac{\delta_0}{2}.$$  

Let $s_2(y_0) = s_1(y_0) + s_{N(y_0)} \geq 0$, we obtain

$$\int_{|y| < y_0} v^2(s_2(y_0)) \leq \int_{|y| < y_0} \left| v(s_2(y_0)) - \widetilde{v}(s_1(y_0)) \right|^2 + \int_{|y| < y_0} \widetilde{v}^2(s_1(y_0))$$

$$\leq \frac{\delta_0}{2} + m_0 - \delta_0 \leq m_0 - \frac{\delta_0}{2}.$$  

By Lemma 18, there exists $R_2 = R_2(\delta_0/4) > 0$, such that

$$\forall s \geq 0, \int_{y > R_2} v^2(s) \leq \frac{\delta_0}{4}.$$  

Let us fix $\gamma_0 = \max(R_2(\delta_0/8, \varepsilon_0), R_2(\delta_0/4))$. Now, apply Lemma 19 with $\varepsilon_0 = \frac{\delta_0}{16M_0}$ and $\delta_0 = \frac{\delta_0}{16}$. For all $y_0 \geq \gamma_0$, we have, $\forall y > s_2(y_0)$,

\begin{equation}
\int_{y \leq -\frac{\delta_0}{16M_0}} v^2(s) \geq \left(1 - \frac{\delta_0}{16M_0}\right) \int_{y < -\gamma_0} v^2(s_2(y_0)) - \frac{\delta_0}{16} \int_{y < -\gamma_0} v^2(s_2(y_0)) = \frac{\delta_0}{8}. \tag{139}
\end{equation}

Therefore, for $y_0 \geq \gamma_0$, we have, $\forall s \geq s_2(y_0)$.
by using the conservation of the $L^2$ norm for $v(t)$. Then, cutting $\int v^2(s)$ into three pieces, and using (139), we have:

$$\int_{|y|<\frac{\delta_0}{2\lambda}} v^2(s) \leq \int_{y<-\lambda y_0} v^2(s) + \int_{|y|<\frac{\lambda y_0}{2}} v^2(s) + \int_{y>\lambda y_0} v^2(s)$$

$$\leq \frac{\delta_0}{8} + m_0 - \frac{\delta_0}{2} + \frac{\delta_0}{4} \leq m_0 - \frac{\delta_0}{8}$$

In particular, $\forall y_0 \geq y_0C$, there exists $n(y_0)$ such that

$$\forall n \geq n(y_0), \quad \int_{|s|<\frac{\lambda y_0}{2}} v^2(s_n) \leq m_0 - \frac{\delta_0}{8}.$$

Since $v(s_n) \to \tilde{v}(0)$ in $L^2_{loc}$, as $n \to \infty$, we obtain

$$\forall y_0 \geq y_0C, \quad \int_{|y|<\frac{\lambda y_0}{2}} \tilde{v}^2(0) \leq m_0 - \frac{\delta_0}{8} \quad \text{and} \quad \int \tilde{v}^2(0) \leq m_0 - \frac{\delta_0}{8}$$

which is a contradiction with the definition of $m_0$ ($m_0 > 0$).

Therefore $\tilde{v}$ satisfies the $L^2$ compactness property (137). Thus $\tilde{e}$ also satisfies the same property. Note that the smallness condition $\forall s \in \mathbb{R}, |\tilde{e}(s)|_{H^1} \leq C\sqrt{\alpha_2}$ is implied by (132) and (133). Thus, for $\alpha_2$ small enough, Theorem 1 or Proposition 5, implies that

$$\tilde{e} \equiv 0 \quad \text{on} \ \mathbb{R} \times \mathbb{R},$$

and in particular $\tilde{\delta}_0 \equiv 0$, which is a contradiction.

Thus the proof of Proposition 6 is complete.

**Proof of Lemma 18.** – The proof is similar to the one of Proposition 1 in Part A, using a similar decomposition of the function $\eta$. The localized part of the solution will be controlled as before and the nonlinear part will be controlled using the monotonicity Lemma 16.

**Step 1.** Decomposition of $\varepsilon$.

We use the function $\eta$ introduced in the proof of Proposition 1 (see Part A)

$$\eta(t,x) = \lambda^{-1/2}(t)e\{t, \lambda^{-1}(t)x\}.$$

Recall from the proof of Proposition 1 that we can split $\eta$ into two parts:

$$\eta(t,x) = \eta_l(t,x) + \eta_r(t,x),$$

where $\eta_l$ satisfies

$$(\eta_l)_t + (\eta_l)_{xxx} - x_t(t)(\eta_l)_x = -\left(\eta_l^4\right)_x, \quad (t,y) \in \mathbb{R}^+ \times \mathbb{R},$$
and $\eta_{II}$ satisfies
\[
(\eta_{II})_t + (\eta_{II})_{xxx} - x_t (\eta_{II})_x = g_1(t) + g_2(t) - (\eta^5 - (\eta_J)^5)_x, \quad (t, y) \in \mathbb{R}^+ \times \mathbb{R},
\]
\[
\eta_{II}(0) = 0,
\]
($g_1$ and $g_2$ are defined in the proof of Proposition 1).

Note that unlike in the proof of Proposition 1, we need not have a parameter $n$ in the splitting.

Lemma 2 still applies to $\eta_{II}$, and so we have
\[
\forall t \geq 0, \forall x \geq 0, \quad |\eta_{II}(t, x)| \leq \sqrt{ab} \eta_1 e^{-d_2 x}.
\]

It follows that $\eta_{II}$ satisfies a compactness property such as (136) and thus we need only deal with $\eta_1$.

**Step 2. Compacteness of $\eta_1$ on the right.**

It follows from similar technique as in Lemma 16. Set
\[
\overline{\eta}_1(t, x) = \eta_1(t, x - x(t) + x(0)),
\]
so that $\overline{\eta}_1$ satisfies:
\[
(\overline{\eta}_1)_t + (\overline{\eta}_1)_{xxx} + (\overline{\eta}_1^5)_x = 0,
\]
\[
\overline{\eta}_1(0) = \eta(0).
\]

Note that $|\eta(0)|_{H^1} \leq Ca$, and so, if $a$ is small enough, $\overline{\eta}_1$ is global in $H^1$ and we have
\[
\forall t \geq 0, \quad |\overline{\eta}_1(t)|_{L^2} \leq |\eta_1(t)|_{H^1} \leq Ca
\]
(we refer to Step 2 of the proof of Lemma 19).

We claim that if $a < a_0$, with $Ca_0 \leq d_0$ where $C$ is the constant in (141) and $d_0$ is defined in Lemma 16, then
\[
\forall \delta_0 > 0, \exists R_2(\delta_0) > R_0(\delta_0), \text{ such that } \forall t \geq 0, \quad \int_{y > R_2} \eta_1^2(t, y) \, dy \leq 2 \delta_0.
\]

The idea is the same as in Step 2 of the proof of Lemma 19.

Fix
\[
\sigma = \frac{1}{2 \lambda_2^2}, \quad K = \sqrt{\frac{2}{\sigma}}.
\]

By applying Lemma 16 to the function $\overline{\eta}_1(t, x + x_0)$, we have
\[
\forall t \geq 0, \quad \int \overline{\eta}_1(t, x) \psi(x - \sigma t - x_0) \, dx \leq \int \overline{\eta}_1^2(0, x) \psi(x - x_0) \, dx.
\]

Note that from the dominated convergence theorem and the definition of $\psi$, we have
Thus, \( \forall \delta_0 > 0, \exists R_2 \) such that for \( x_0 > R_2 \),

\[
\int \eta_2^2(0, x) \psi(x - x_0) \, dx \to 0, \quad \text{as } x_0 \to +\infty.
\]

Recall that \( x(t) - x(0) \geq \sigma t \), thus for \( x > x_0 + x(t) - x(0) \), we have

\[
x - \sigma t - x_0 \geq x(t) - x(0) - \sigma t + 2R_2 - x_0 \geq \frac{R_2}{2}.
\]

In conclusion,

\[
\int_{x > 2R_2} \eta_2^2(t, x) \, dx \leq \int_{x > 2R_2 + x(t) - x(0)} \eta_2^2(t, x) \, dx
\]

\[
\leq \frac{1}{\psi(R_2/2)} \int_{x > 2R_2 + x(t) - x(0)} \eta_2^2(t, x) \psi(x - \sigma t - x_0) \, dx
\]

\[
\leq \frac{1}{\psi(R_2/2)} \int \eta_2^2(t, x) \psi(x - \sigma t - x_0) \, dx \leq \frac{\delta_0}{\psi(R_2/2)}.
\]

Thus we have proved the claim and \( \eta_1 \) satisfies the \( L^2 \) compactness property on the right (136).

Since \( v(t, y) = Q(y) + \varepsilon(t, y) = Q(y) + \lambda^{1/2}(t) \eta(t, \lambda(t)y) \), and since \( \lambda_1 \leq \lambda(t) \leq \lambda_2 \), it is then clear that \( v \) also satisfies (136).

Thus, Lemma 18 is proved. \( \square \)

**Proof of Lemma 19.** – We may assume that \( t_0 = 0 \) by invariance by translation in time. It is a consequence of the fact that Lemma 16 still holds in a weak form for large solutions of the generalized KdV equation.

**Step 1.** Almost monotonicity property for \( u \).

Let

\[
\mathcal{I}_{x_0}(t) = \int u^2(t, x) \psi(x - x(0) - \sigma t - x_0) \, dx.
\]

**Lemma 20 (Almost monotonicity for \( u \)).** – Let \( \sigma > 0, K > 0 \) such that

\[
\sigma \leq \frac{1}{4\lambda^2}, \quad K \geq \frac{1}{\sigma}.
\]

There exists \( a_3 = a_3(\lambda_1, \sigma) \) such that if

\[
\sup_{t \geq 0} \| \varepsilon(t) \|_{H^1} = a \leq a_3,
\]

then for \( C = C(\lambda_1, \lambda_2, \sigma, K) \),

\[
\forall x_0 \leq 0, \forall t \geq 0, \quad \mathcal{I}_{x_0}(t) = \mathcal{I}_{x_0}(0) \leq C e^{a_0/K}.
\]
Proof of Lemma 20. – As in the proof of Lemma 16, since \( K > \frac{\sqrt{2}}{\sigma} \), we have:

\[
I_{x_0}'(t) \leq -3 \int u_2^2(t) \phi(x - x(0) - \sigma t - x_0) - \frac{\sigma}{2} \int u_1^2(t) \phi(x - x(0) - \sigma t - x_0)
+ \frac{5}{3} \int u^6(t) \phi(x - x(0) - \sigma t - x_0).
\]

Here, since \( u \) is not uniformly small, the way to treat the last term in the right hand side of the above formula is different from the one of Lemma 16. We will treat the regions where \( |u(t, x)| \) is large and where it is small in a different way. We will see that the contribution of the region that makes \( I_{x_0} \) increase is controlled by a term which is integrable in time, which allows us to conclude.

Recall that

\[
u(t, x) = \lambda^{-1/2} Q(\lambda^{-1} (x - x(t))) + \lambda^{-1/2} e(t, \lambda^{-1} (x - x(t))),
\]

\( \forall t \geq 0, \quad |\varepsilon(t)|_{L^\infty} \leq |\varepsilon(t)|_{H^1} \leq a \leq a_3, \quad \forall x \in \mathbb{R}, \ 0 \leq Q(x) \leq C e^{-|x|}.
\]

Let \( d_0 = \left( \frac{d_0}{2} \right)^{1/4} \) and let \( a_3 > 0 \) be such that

\[
a_3 \leq \min \left( \frac{d_0}{2}, 1, 1 \right), \quad \frac{|x_t|}{\lambda} - 1 \leq \frac{1}{2}
\]

(see Lemma 3).

There exists \( C_0 = C_0(\sigma, \lambda_1, \lambda_2) > 0 \) such that \( \forall t \geq 0, \forall x \in \mathbb{R} \), such that \( |x - x(t)| \geq C_0 \), we have

\[
|u(t, x)| \leq d_0.
\]

Indeed, for all \( x \in \mathbb{R} \) such that \( |x - x(t)| \geq C_0 \),

\[
|u(t, x)| \leq \lambda^{-1/2} Q(\lambda^{-1} (x - x(t))) + \lambda^{-1/2} |\varepsilon(t)|_{L^\infty} \leq C \lambda^{-1/2} e^{-\lambda^{-1} C_0} + \lambda^{-1/2} a
\]

\[
\leq C \lambda_1^{-1/2} e^{-C_0/\lambda_2} + \lambda_1^{-1/2} a_3 \leq C \lambda_1^{-1/2} e^{-C_0/\lambda_2} + \frac{d_0}{2} \leq d_0,
\]

for \( C_0 \) large enough.

As in the proof of Lemma 16, it follows that

\[
\frac{5}{3} \int_{|x-x(t)| \geq C_0} u^6 \phi(x - x(0) - \sigma t - x_0) \leq \frac{5}{3} d_0^6 \int_{|x-x(t)| \geq C_0} u^2 \phi(x - x(0) - \sigma t - x_0)
\]

\[
\leq \frac{\sigma}{4} \int u^2 \phi(x - x(0) - \sigma t - x_0)
\]

and so,

\[
I_{x_0}'(t) \leq -3 \int u_2^2(t) \phi(x - x(0) - \sigma t - x_0) - \frac{\sigma}{4} \int u_1^2(t) \phi(x - x(0) - \sigma t - x_0)
+ \frac{5}{3} \int u^6(t) \phi(x - x(0) - \sigma t - x_0)
\]

\[
\leq \frac{5}{3} \int u^2 \phi(x - x(0) - \sigma t - x_0).
\]
Now, since $\forall t \geq 0$, $|u(t)|_{L^\infty} \leq C$, we have

$$I_{x_0}^t(t) \leq C \int_{|y| \leq C_0} \phi(y - \sigma t + x(t) - x(0) - x_0) \, dy \leq C \int_{|y| \leq C_0} e^{-\frac{1}{4\lambda_2} |y - \sigma t + x(t) - x(0) - x_0|} \, dy.$$  

Since $x_0 = \lambda^{-3} x_0$, it follows from (142) and (H2), that

$$\forall t \geq 0, \quad \frac{1}{2\lambda_2^2} \leq x_0(t) \leq \frac{3}{2\lambda_1^2}, \quad x(t) \geq x(0) + \frac{t}{2\lambda_2^2}.$$  

Since $x_0 \leq 0$, $x(t) - x(0) \geq \frac{t}{2\lambda_2^2}$, and $\sigma \leq \frac{1}{4\lambda_2^2}$, we have $\forall y \in \mathbb{R}$ such that $|y| < C_0$:

$$|y - \sigma t + x(t) - x(0)| \geq -|y| + |\sigma t + x(t) - x(0) - x_0| \geq -C_0 + \frac{t}{4\lambda_2^2} - x_0.$$  

Thus, $\forall t \geq 0$,

$$I_{x_0}^t(t) \leq C \int_{|y| \leq C_0} e^{-\frac{1}{4\lambda_2^2 |y|} |y - \sigma t + x(t) - x(0)|} \, dy \leq C e^{-\frac{1}{4\lambda_2^2} |y_0|} e^{\frac{t}{2\lambda_2^2}}.$$  

By integration between $0$ and $t$, using crucially the exponential decay in time, it follows that

$$\forall t \geq 0, \quad I_{x_0}^t(t) - I_{x_0}(0) \leq C e^{\frac{t}{K}},$$

where $C$ depends on $\lambda_1, \lambda_2, \sigma$ and $K$; thus Lemma 20 is proved.

**Step 2.** Conclusion of the proof.

The proof is based on the variation of a quantity of the form $\int u^2 \psi$ and the relations between $\int u^2 \psi$ and $\int_{x < x_0} u^2$ and the relation between $\sigma$ and $x_0$.

(i) We interpret the condition on $v$ at $t = 0$ in terms of $u$.

Fix $\sigma = \frac{1}{4\lambda_2^2}, \quad K = \sqrt{\frac{2}{\sigma}} = 2\lambda_2 \sqrt{2}.$

We claim that for $R_1$ large enough (depending only on $\varepsilon_0, \lambda_1$ and $\lambda_2$), we have:

$$\forall y \geq R_1, \quad \int u^2(0, x) \left(1 - \psi \left( x - x(0) + \frac{3\lambda_1}{4} y_0 \right) \right) \, dx \geq (1 - \varepsilon_0) \int_{y < -y_0} v^2(0, y).$$

Recall that

$$v(t, y) = \lambda^{-1/2}(t) u(t, \lambda(t) y + x(t));$$

we obtain

$$\int u^2(0, x) \left(1 - \psi \left( x - x(0) + \frac{3\lambda_1}{4} y_0 \right) \right) \, dx = \int v^2(0, y) \left(1 - \psi \left( \lambda(0) y + \frac{3\lambda_1}{4} y_0 \right) \right) \, dy \geq \int_{y < -y_0} v^2(0, y) \left(1 - \psi \left( \lambda(0) y + \frac{3\lambda_1}{4} y_0 \right) \right) \, dy.$$
We have
\[ \forall y \leq -y_0 \leq -R_1, \quad \lambda(0)y + \frac{3\lambda_1}{4}y_0 \leq -\frac{\lambda_1 y_0}{4} \leq -\frac{\lambda_1 R_1}{4}. \]

If
\[ (146) \]
\[ \psi\left(-\frac{\lambda_1 R_1}{4}\right) \leq \varepsilon_0, \]
from the fact that \( \psi \) is a nondecreasing function, we obtain
\[ \int u^2(0, x) \left(1 - \psi\left(x - x(0) - \sigma t + \frac{3\lambda_1}{4}y_0\right)\right) \, dx \geq (1 - \varepsilon_0) \int v^2(0, y) \, dy. \]
and thus claim (145) is proved.

(ii) Now, we use Lemma 20 on \( u \).

The values of \( \sigma \) and \( K \) being fixed as above, we can apply Lemma 20 to \( u \), with \( x_0 = -\frac{3\lambda_1}{4}y_0 \) as before.

For \( a \leq a_3(\lambda_1, \lambda_2) \), since \( \int u^2(t) = \int u^2(0) \), we obtain:
\[ \forall t \geq 0, \quad \int u^2(t, x) \left(1 - \psi\left(x - x(0) - \sigma t + \frac{3\lambda_1}{4}y_0\right)\right) \, dx \geq (1 - \varepsilon_0) \int v^2(0, y) \, dy - Ce^{-\frac{3\lambda_1}{4}y_0}. \]

If \( R_1 \) satisfies:
\[ (147) \]
\[ Ce^{-\frac{3\lambda_1}{4}y_0} \leq C e^{-\frac{3\lambda_1 R_1}{4k}} \leq \frac{\delta_0}{2}, \]
then
\[ \forall t \geq 0, \quad \int u^2(t, x) \left(1 - \psi\left(x - x(0) - \sigma t + \frac{3\lambda_1}{4}y_0\right)\right) \, dx \geq (1 - \varepsilon_0) \int v^2(0, y) \, dy - \frac{\delta_0}{2}. \]

(iii) We go back to \( v(t) \).

We have,
\[ \int v^2(t, y) \, dy = \int \int u^2(t, x) \, dx \geq \int \int u^2(t, x) \, dx. \]

Since \( x(t) - x(0) \geq \frac{t}{2\lambda_2} = 2\sigma t \geq \sigma t \) (see (144), in the proof of Lemma 20), we obtain:
\[ \int v^2(t, y) \, dy \geq \int \int u^2(t, x) \, dx \]
\[ \geq \int \int u^2(t, x) \left(1 - \psi\left(x - x(0) - \sigma t + \frac{3\lambda_1}{4}y_0\right)\right) \, dx \]
\[ = \int \int u^2(t, x) \left(1 - \psi\left(x - x(0) - \sigma t + \frac{3\lambda_1}{4}y_0\right)\right) \, dx. \]
Now, observe that if \( x > \frac{1}{2} y_0 + \sigma t + x(0) \), then \( x - x(0) - \sigma t + \frac{3\lambda_1}{4} y_0 \geq \frac{\lambda_1}{4} R_1 \).

We have \( \lim_{x \to +\infty} \psi(x) = 1 \), therefore, for \( R_1 \) large,

\[
1 - \psi \left( \frac{\lambda_1}{4} R_1 \right) \leq \frac{\delta_0}{2} \int u_0^2 \ dx.
\]

Since \( \psi \) is nondecreasing, we obtain

\[
\int_{x \geq \frac{1}{2} y_0 + \sigma t + x(0)} u^2(t, x) \left( 1 - \psi \left( x - x(0) - \sigma t + \frac{3\lambda_1}{4} y_0 \right) \right) \ dx \leq \frac{\delta_0}{2}.
\]

Finally, for \( y_0 \geq R_1 \), where \( R_1 \) satisfies (146), (147) and (148), we obtain

\[
\int_{y \leq -\frac{y_0}{\delta_2} \sqrt{2}} \int_{y \leq -y_0} v^2(t, y) \ dy \geq (1 - \varepsilon_0) \int_{y \leq -y_0} v^2(t_0, y) \ dy - \delta_0;
\]

thus, the proof of Lemma 19 is complete. \( \square \)

Appendix A

In this appendix, we prove the Lemmas 4 and 5.

Let us recall the following estimates of the Airy function:

\[
(1 + |x|)^{1/2} |\text{Ai}(x)| + |\text{Ai}'(x)| \leq C (1 + |x|)^{1/4}, \quad \forall x \in \mathbb{R},
\]

\[
(1 + x)^{1/2} |\text{Ai}(x)| + |\text{Ai}'(x)| \leq C (1 + x)^{1/4} e^{-\frac{2}{3} x^{3/2}}, \quad \forall x \geq 0,
\]

for some \( C > 0 \).

We give a useful technical tool.

CLAIM. – Let \( \delta_2, \delta_3 > 0 \), satisfying \( \delta_2 < \sqrt{\delta_3} \). Then, there exist \( r_1 = r_1(\delta_2, \delta_3) > 0 \), and \( r_2 = r_2(\delta_2, \delta_3) > 0 \) such that,

\[
\forall a, b > 0, \quad \frac{2}{3} a^{3/2} b^{-1/2} - \delta_2 \left( a - \frac{\delta_3 b}{3} \right) \geq r_1 a + r_2 b.
\]

Proof of claim (151). – First note that by dividing (151) by \( b \) and setting \( x = a/b \), (151) is equivalent to

\[
\frac{2}{3} x^{3/2} - \delta_2 x + \frac{\delta_2 \delta_3}{3} \geq r_1 x + r_2.
\]
For $r > 0$, to be chosen later, let

$$f(x) = \frac{2}{3}x^{3/2} - \delta_2 x + \frac{\delta_2 \delta_3}{3} - rx.$$ 

We have

$$f'(x) = x^{1/2} - \delta_2 - r,$$

so that the minimum of $f$ on $[0, +\infty)$ is reached for $x_0 = (\delta_2 + r)^2$. The value of the minimum is given by

$$f(x_0) = \delta_2 \left[ \left( \frac{2}{3} \right) (\delta_2 + r)^2 - (\delta_2 + r)^2 + \frac{\delta_3}{3} \right]$$

$$= \delta_2 \left[ -\frac{1}{3}(\delta_2 + r)^2 + \frac{\delta_3}{3} \right].$$

Therefore, if

$$\delta_2 < \sqrt{\delta_3}$$

and if we choose $r = r_1 > 0$ such that

$$r_1 + \delta_2 < \sqrt{\delta_3},$$

then $f(x_0) = r_2 > 0$, and so $\forall x \geq 0$, we have $f(x) \geq f(x_0) = r_2 > 0$, which means that (152) is satisfied.

Thus the claim is proved.  

Proof of Lemma 4. – We have:

$$\zeta(t, x) = \int_0^t S(t - s) g_x(s) \, ds = \int_0^t (3(t - s))^{-2/3} \int_0^3 \text{Ai}'(y(3(t - s))^{-1/3}) g(s, x - y) \, dy \, ds,$$

since $S(t - s)$ represents the convolution with $(3(t - s))^{-1/3} \text{Ai}(x(3(t - s))^{-1/3})$.

Therefore, by (72), we have

$$|\zeta(t, x)| \leq \delta_1 \left[ (I) + (II) \right],$$

where

$$(I) = \int_0^t (3(t - s))^{-2/3} \int_{y < 0} |\text{Ai}'(y(3(t - s))^{-1/3})| e^{-\delta_2 (x - y)} \, dy \, ds,$$

and

$$(II) = \int_0^t (3(t - s))^{-2/3} \int_{y > 0} |\text{Ai}'(y(3(t - s))^{-1/3})| e^{-\delta_2 (x - y)} \, dy \, ds.$$

We use (149),
\[ (I) \leq \int_0^t (3(t-s))^{-2/3} \int_{y<0} \left( 1 + |y|^{1/4} (3(t-s))^{-1/12} \right) e^{-\delta_2 (s-y)} dy ds \]
\[ \leq e^{-\delta_2 x} \left( \int_0^t (3(t-s))^{-2/3} ds \right) \left( \int_{y<0} e^{\delta_2 y} dy \right) \]
\[ + e^{-\delta_2 x} \left( \int_0^t (3(t-s))^{-3/4} ds \right) \left( \int_{y<0} |y|^{1/4} e^{\delta_2 y} dy \right) \]
\[ \leq C (t^{1/3} + t^{1/4}) e^{-\delta_2 x} \leq C t^{1/4} e^{-\delta_2 x}, \]
for \( t \in (0, 1). \)

On the other hand, by (150), we have
\[ (II) \leq \int_0^t (3(t-s))^{-2/3} \int_{y>0} \left( 1 + |y|^{1/4} (3(t-s))^{-1/12} \right) \]
\[ \times \exp \left( -\frac{2}{3} |y|^{3/2} (3(t-s))^{-1/2} \right) e^{-\delta_2 (s-y)} dy ds. \]

Let \( \delta_1 > 0 \) be such that \( \sqrt{\delta_3} > \delta_2 \). We use claim (151), with a = y and \( b = 3(t-s) \), for \( y > 0 \), and \( t > s \). We obtain
\[ \frac{2}{3} |y|^{3/2} (3(t-s))^{-1/2} - \delta_2 (y - \delta_3 (t-s)) \geq r_1 y + 3r_2 (t-s) \geq r_1 y. \]

Therefore, we have:
\[ (II) \leq e^{-\delta_2 x} \left( \int_0^t (3(t-s))^{-2/3} e^{\delta_2 \delta_3 (t-s)} ds \right) \left( \int_{y>0} e^{-r_1 y} dy \right) \]
\[ + e^{-\delta_2 x} \left( \int_0^t (3(t-s))^{-3/4} e^{\delta_2 \delta_3 (t-s)} ds \right) \left( \int_{y>0} y^{1/4} e^{-r_1 y} dy \right). \]

Finally, if we take \( 0 < t < 1 \), we obtain
\[ (II) \leq C t^{1/4} e^{-\delta_2 x}. \]
This completes the proof of Lemma 4. \( \square \)

**Proof of Lemma 5.** – Note that, by (76), we have
\[ \forall t, s \in \mathbb{R}, \ t > s, \ \delta_3 (t-s) \leq x(t) - x(s). \]

Next, we have \( \xi(t, x) = \xi_1(t, x) + \xi_2(t, x), \) where
\[ \xi_1(t, x) = \int_0^t (3(t-s))^{-1/3} \int Ai(y(3(t-s))^{-1/3}) g_1(s, x - y + x(t) - x(s)) dy ds, \]
and

\[
\xi_2(t, x) = \int_0^t \left(3(t-s)\right)^{-2/3} \int_0^3 \left(y(3(t-s))^{-1/3}\right) g_2(s, x - y + x(t) - x(s)) dy ds.
\]

First, we treat \(\xi_1\). By (75), we have

\[
|\xi_1(t, x)| \leq \delta_1 \int_0^t \left(3(t-s)\right)^{-1/3} \int_0^3 |\text{Ai}(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} dy ds
\]

\[
\leq \delta_1 \left[ (I) + (II) \right],
\]

where

\[
(I) = \int_0^t \left(3(t-s)\right)^{-1/3} \int_{y<0} |\text{Ai}(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} dy ds,
\]

and

\[
(II) = \int_0^t \left(3(t-s)\right)^{-1/3} \int_{y>0} |\text{Ai}(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} dy ds.
\]

We use the estimates of the Airy function for \(y<0\) (149), and (154),

\[
(I) \leq \int_0^t \left(3(t-s)\right)^{-1/4} \int_{y<0} |y|^{-1/4} e^{-\delta_2(x-y+x(t)-x(s))} dy ds
\]

\[
\leq e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-1/4} e^{-\delta_2 \delta_3(t-s)} ds \right) \left( \int_{y<0} |y|^{-1/4} e^{\delta_2 y} dy \right)
\]

\[
\leq C \min \left( t^{3/4}, 1 \right) e^{-\delta_2 x}.
\]

On the other hand, by (150), we have:

\[
(II) \leq e^{-\delta_2 x} \int_0^t \left(3(t-s)\right)^{-1/3} \int_{y>0} \exp \left( -\frac{3}{2} \right) \left(3(t-s)\right)^{-1/2} e^{\delta_2(y-\delta_3(t-s))} dy ds.
\]

Again, we use claim (151), with \(a = y\) and \(b = 3(t-s)\), for \(y > 0\), and \(t > s\). We obtain

(155) \[ \frac{2}{3} \left(3(t-s)\right)^{-1/2} - \delta_2 (y - \delta_3 (t-s)) \geq r_1 y + 3r_2 (t-s). \]

We obtain finally that

\[
(II) \leq e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-1/3} e^{-3r_2(t-s)} ds \right) \left( \int_{y>0} e^{-r_1 y} dy \right)
\]

\[
\leq C \min \left( t^{2/3}, 1 \right) e^{-\delta_2 x}.
\]
Now, we deal with $\zeta_2$. We have, by (75),

\[
|\zeta_2(t, x)| \leq \delta_1 \int_0^t \left(3(t-s)\right)^{-2/3} \int |\text{Ai}'(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} \, dy \, ds
\]

\[
\leq \delta_1 [(\text{III}) + (\text{IV})],
\]

where

\[
(\text{III}) = \int_0^t \left(3(t-s)\right)^{-2/3} \int |\text{Ai}'(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} \, dy \, ds,
\]

and

\[
(\text{IV}) = \int_0^t \left(3(t-s)\right)^{-2/3} \int |\text{Ai}'(y(3(t-s))^{-1/3})| e^{-\delta_2(x-y+x(t)-x(s))} \, dy \, ds.
\]

We use the estimates of the derivative of the Airy function (149),

\[
(\text{III}) \leq \int_0^t \left(3(t-s)\right)^{-2/3} \int (1 + |y|^{1/4}(3(t-s))^{-1/12}) e^{-\delta_2(x-y+x(t)-x(s))} \, dy \, ds
\]

\[
\leq e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-2/3} e^{-\delta_2(y(t-s))} \, ds \right) \left( \int_{y>0} e^{\delta_2 y} \, dy \right)
\]

\[
+ e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-3/4} e^{-\delta_2(y(t-s))} \, ds \right) \left( \int_{y>0} |y|^{1/4} e^{\delta_2 y} \, dy \right)
\]

\[
\leq C \min \{ t^{1/4}, 1 \} e^{-\delta_2 x}.
\]

On the other hand, we have, by (150) and (155),

\[
(\text{IV}) \leq e^{-\delta_2 x} \int_0^t \left(3(t-s)\right)^{-2/3} \int (1 + |y|^{1/4}(3(t-s))^{-1/12}) \exp \left(-\frac{2}{3} y^{3/2}(3(t-s))^{-1/2} \right) e^{\delta_2(y-y(t-s))} \, dy \, ds
\]

\[
\leq e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-2/3} e^{-3r_2(t-s)} \, ds \right) \left( \int_{y>0} e^{-r_1 y} \, dy \right)
\]

\[
+ e^{-\delta_2 x} \left( \int_0^t \left(3(t-s)\right)^{-3/4} e^{-r_2(t-s)} \, ds \right) \left( \int_{y>0} y^{1/4} e^{-r_1 y} \, dy \right)
\]

\[
\leq C \min \{ t^{1/4}, 1 \} e^{-\delta_2 x}.
\]

Thus, we have proved (77).
Appendix B

B.1. Proof of Lemma 7

The proof follows from some estimates introduced in [11] to treat the KdV equation
\( \zeta_t + \zeta_{xxx} + (\zeta^2)_x = 0 \).

First, note that by invariance by translation in time of equation (18), we may assume that \( s_0 = 0 \).

Recall that we have defined \( \eta(t, x) = \lambda^{-1/2}(t) \varepsilon(t, \lambda^{-1}(t)x) \),
and that \( \eta \) satisfies
\( \eta_t + \eta_{xxx} - x_t \eta_x = g_1 + g_2 = \left( \eta^5 \right)_x \),
where \( g_1(t, x) = \lambda^{-7/2}(t) \left( \frac{\lambda_x(t)}{\lambda(t)} \right)^2 \frac{1}{2} Q(\lambda^{-1}(t)x) + (\lambda^{-1}(t)x) Q_x(\lambda^{-1}(t)x) \),
and
\( g_2(t, x) = \lambda^{-5/2} \left( \lambda_t - 1 \right) Q(\lambda^{-1}x) - 5\lambda^{-2} Q^2(\lambda^{-1}x) \eta 
- 10\lambda^{-3/2} Q^3(\lambda^{-1}x) \eta^2 - 10\lambda^{-1} Q^2(\lambda^{-1}x) \eta^3 - 5\lambda^{-1/2} Q(\lambda^{-1}x) \eta^4 \).

In order to eliminate the term \( x_t \eta_x \) in (156), we set \( \overline{\eta}(t, x) = \eta(t, x - x(t) + x(0)) \),
so that \( \overline{\eta} \) satisfies
\( \overline{\eta}_t + \overline{\eta}_{xxx} = \overline{\eta}_1 + \overline{\eta}_2 = \left( \overline{\eta}^5 \right)_x \),
where, for \( i = 1, 2 \), \( \overline{\eta}_i(t, x) = g_i(t, x - x(t) + x(0)) \).

Moreover, note that \( \overline{\eta}_0 = \lambda^{-1/2}(0) \varepsilon(s, \lambda^{-1}(0)x) \).

We use the norm \( \Sigma^T \) introduced in Step 2 of the proof of Proposition 1.

Control of \( \Sigma^T(\overline{\eta}) \), for \( T \) small enough.

By (55), we have, for \( T \in (0, 1) \),
\( \Sigma^T(w) \leq C \left\| \overline{\eta}(0) \right\|_{H^1} + C T^{1/2} \left\{ \left| g_1 \right|_{L^2_T L^2_x} + \left| g_1 \right|_{L^2_T L^2_x} + \left| g_2 \right|_{L^2_T L^2_x} + \left| g_{2x} \right|_{L^2_T L^2_x} + \left| g_{2xx} \right|_{L^2_T L^2_x} \right\} \).

We control the term in the right hand side of the above equation. First, by Lemma 3(ii), (H2), and \( |Q(x)| + |Q_x(x)| \leq C e^{-|x|} \), we have:
\( \left( \overline{\eta}^5 \right)_x \leq C b^2 \leq C a^2 \).

Next, by Lemma 3(ii), (H2), and \( \left\| \overline{\eta} \right\|_{L^\infty} \leq C a \leq C \), we have:
Therefore, for \( t \in (0, T) \), we have, from the space integrability of \( e^{-2\lambda_t^{-1}t} \),
\[
|\mathcal{F}_{2x}(t,x)|^2 + |\mathcal{F}_{2xx}(t,x)|^2 
\leq C e^{-2\lambda_t^{-1}t} \left[ a^2 + |\mathcal{P}(t,x)|^2 + |\mathcal{P}_x(t,x)|^2 + |\mathcal{P}_{xx}(t,x)|^2 \right].
\]

Therefore, for \( t \in (0, T) \), we have, from the space integrability of \( e^{-2\lambda_t^{-1}t} \),
\[
|\mathcal{F}_{2x}|^2_{L^2_xL^2_t} + |\mathcal{F}_{2xx}|^2_{L^2_xL^2_t} \leq C a^2 + C \sup_{0<T} \left[ \mathcal{P}(t) \right]_{H^1}^2 + \left[ \mathcal{P}_x \right]_{L^4_xL^\infty_t}^4 \left[ e^{-\lambda_1^{-1}t} \right]_{L^2_x}^2 
+ \left[ \mathcal{P}_{xx} \right]_{L^2_xL^2_t} \left[ e^{-\lambda_1^{-1}t} \right]_{L^2_x}^2 
\leq C a^2 + C \left[ (\Sigma^T(\mathcal{P}))^2 + (\Sigma^T(\mathcal{P}))^2 \right].
\]

For the term \( (\mathcal{P})_x \), we argue as in [11], proof of Theorem 2.1, pp. 580–583. We have
\[
|{(\mathcal{P})}_x|^2_{L^2_xL^2_t} + |{(\mathcal{P})}_{xx}|^2_{L^2_xL^2_t} \leq C |{\mathcal{P}}|^3_{L^2_xL^\infty_t} \left[ |{(\mathcal{P})}_x|_{L^2_xL^2_t} + |{(\mathcal{P})}_{xx}|_{L^2_xL^2_t} \right] 
\leq C (1 + T) (\Sigma^T(\mathcal{P}))^2 \leq 2C (\Sigma^T(\mathcal{P}))^2,
\]
by claim (4.10) in [11].

Therefore, we have
\[
\Sigma^T(\mathcal{P}) \leq C |{\mathcal{P}}(0)|_{H^1} + C T^{1/2} \left[ a + \Sigma^T(\mathcal{P}) + (\Sigma^T(\mathcal{P}))^2 \right].
\]

Note that by (56), we have \( |{\mathcal{P}}(0)|_{H^1} \leq Ca \), therefore, by taking \( C T^{1/2} \leq 1/2 \), we have
\[
\Sigma^T(\mathcal{P}) \leq 2(C + 1)a + (\Sigma^T(\mathcal{P}))^2,
\]
which implies
\[
\Sigma^T(\mathcal{P}) \leq C' a,
\]
for \( C' = 4(C + 1) \), by choosing \( a \) small enough.

Thus, we have proved that there exists \( T > 0, C > 0 \), and \( a'' > 0 \) such that for \( a < a'' \),
\[
(159) \quad \Sigma^T(\mathcal{P}) \leq C a.
\]

It is important to note that \( T \) and \( C \) do not depend on \( a \).

**Control of the variation of \( |{\mathcal{P}}(t)|_{H^1} \).**

Note that, for \( t \in (0, T) \),
\[
\mathcal{P}(t) - S(t)\mathcal{P}(0) = - \int_0^t S(t-s) (\mathcal{F}_1 + \mathcal{F}_{2x} - (\mathcal{P})_x) \, ds,
\]
so that, as before, for \( t \in (0, T) \),
\[
\Sigma^T(\mathcal{P}) - S(t)\mathcal{P}(0) \leq C_T^{1/2} \left[ |{g_1}|_{L^2_xL^2_t} + |{g_1x}|_{L^2_xL^2_t} + |{g_2x}|_{L^2_xL^2_t} + |{g_2xx}|_{L^2_xL^2_t} \right] 
+ \left[ |{(\mathcal{P})}_x|_{L^2_xL^2_t} + |{(\mathcal{P})}_{xx}|_{L^2_xL^2_t} \right] .
\]

By arguing exactly in the same way as before to estimate the right-hand side of the previous equation, and using (159), we find, for \( t \in (0, T) \),
\[ \Sigma'(\bar{\eta}(\cdot) - S(\cdot)\bar{\eta}(0)) \leq \text{Cat}^{1/2}. \]

On the other hand, we have \( |\varepsilon(h)|_{H^1} \geq a/2 \), and so
\[
|\bar{\eta}(0)|_{H^1}^2 = |\bar{\eta}_x(0)|_{L^2}^2 + |\bar{\eta}(0)|_{L^2}^2 = \lambda^{-2}(0) |\varepsilon_x(0)|_{L^2}^2 + |\varepsilon(0)|_{L^2}^2 \geq c_1^2 |\varepsilon(0)|_{H^1}^2,
\]
where \( c_1 = \min(1, \lambda_1^{-1}) \). It follows that
\[
|\bar{\eta}(0)|_{H^1} \geq c_1^2 a.
\]

Therefore, if \( Ct^{1/2} \leq c_1/4 \), then, using the fact that \( S(t) \) is an isometry in \( H^1 \), and the expression of \( \Sigma' \), we have
\[
|\bar{\eta}(t)|_{H^1} \geq |S(t)\bar{\eta}(0)|_{H^1} - |\bar{\eta}(t) - S(t)\bar{\eta}(0)|_{H^1} \\
\geq |\bar{\eta}(0)|_{H^1} - |\Sigma'(\bar{\eta}(\cdot) - S(\cdot)\bar{\eta}(0))|_{H^1} \\
\geq \left( \frac{c_1^2}{2} - Ct^{1/2} \right) a \geq \frac{c_1^2}{4} a.
\]

Finally, it follows that for \( T \) small enough, independent of \( a \),
\[
|\bar{\eta}(t)|_{H^1} \geq \frac{c_1^2}{4} a.
\]

Therefore, there exists \( S > 0 \) and \( c_2 > 0 \) such that
\[
|\bar{\eta}(s)|_{H^1} \geq c_2 a.
\]

**B.2. Proof of Lemma 11**

First, we remove linear terms related to the invariances of the equation, then we prove an \( L^2 \) convergence (using \( L^2 \) theory for equation (1)).

By Lemma 8, for any \( n \in \mathbb{N} \), the function \( w_n \) satisfies
\[
\begin{align*}
\dot{w}_n - (Lw_n)_y &= \alpha_n \left( \frac{Q}{2} + yQ_y \right) + \beta_n Q_y + b_n \bar{\alpha}_n \left( \frac{w_n}{2} + yw_{ny} \right) \\
&\quad + b_n \bar{\beta}_n w_{ny} + b_n (F_n + G_n),
\end{align*}
\]
(160)
where
\[
\begin{align*}
F_n &= \left( \alpha_n - \alpha_n \right) \left( \frac{Q}{2} + yQ_y \right) + \left( \beta_n - \beta_n \right) Q_y, \\
G_n &= -(10Q^3w_n^2 + 10bnQ^2w_n^3 + 5bn^2Qw_n^4 + b_n^3w_n^5). \\
\end{align*}
\]

**Step 1.** Reduction of the problem by scaling, translation and explicit solutions.

(i) First, note that by substracting to \( w_n \) an explicit solution of the linear problem \( w_{ni} = (Lw_n)_y + \alpha_n \left( \frac{Q}{2} + yQ_y \right) + \beta_n Q_y \), as constructed in Lemma 10, we can remove the term \( \alpha_n \left( \frac{Q}{2} + yQ_y \right) + \beta_n Q_y \) in the equation of \( w_n \).

Let
\[
w_{1,n} = w_n - \left[ y_n \left( \frac{Q}{2} + yQ_y \right) + \delta_n Q_y \right],
\]
where...
where

\[ y_n(s) = \int_0^s \alpha_n(s') \, ds' \] \quad \text{and} \quad \delta_n(s) = \int_0^s [\beta_n(s') - 2y_n(s')] \, ds', \]

then \( w_{1,n} \) satisfies

\[ (w_{1,n})_s - (L w_{1,n})_y = b_n \tilde{\alpha}_n \left( \frac{w_n}{2} + y w_n \right) + b_n \tilde{\beta}_n w_{ny} + b_n (F_n + G_n), \]

or, equivalently,

\[ (w_{1,n})_s + (w_{1,n})_{yy} - (w_{1,n})_y = -5 (Q^4 w_{1,n})_y + b_n \tilde{\alpha}_n \left( \frac{w_{1,n}}{2} + y (w_{1,n})_y \right) + b_n \tilde{\beta}_n (w_{1,n})_y + b_n (F_n + G_n + H_n), \]

where

\[ H_n = \tilde{\alpha}_n \left( \frac{w_n - w_{1,n}}{2} + y (w_n - w_{1,n})_y \right) + \tilde{\beta}_n (w_n - w_{1,n})_y \]

\[ = \tilde{\alpha}_n y_n \left( \frac{Q}{2} + y Q_x \right) + y \left( \frac{Q}{2} + y Q_y \right) \] \quad \text{and} \quad \delta_n \left( \frac{Q}{2} + y Q_y \right) \]

\[ + \tilde{\beta}_n \delta_n \left( \frac{Q}{2} + y Q_y \right) + \tilde{\beta}_n \delta_n Q_{yy}. \]

(ii) Set, \( \forall n \in \mathbb{N}, \forall s \in \mathbb{R}, \forall y' \in \mathbb{R}, \)

\[ w_{2,n}(s, y') = \mu_n^{-1/2}(s) w_{1,n}(s, \mu_n^{-1}(s)(y' - \sigma_n(s))), \]

where

\[ \mu_n(s) = e^{b_n \int_0^s \tilde{\alpha}_n(s') \, ds'} > 0, \]

\[ \sigma_n(s) = \int_0^s \mu_n(s')(b_n \tilde{\beta}_n(s') + 1) \, ds'. \]

Observe that we have

\[ \frac{d}{ds} \mu_n(s) = b_n \tilde{\alpha}_n(s) e^{b_n \int_0^s \tilde{\alpha}_n(s') \, ds'} = b_n \tilde{\alpha}_n(s) \mu_n(s), \]

\[ \frac{d}{ds} \sigma_n(s) = \mu_n(s)(b_n \tilde{\beta}_n + 1); \]

therefore,

\[ (w_{2,n})_s + \mu_n^{1/2}(w_{2,n})_{yy} \]

\[ = \mu_n^{-1/2}((w_{1,n})_s + (w_{1,n})_{yy})(s, \mu_n^{-1}(y' - \sigma_n)) \]

\[ - \mu_n^{-1/2}(b_n \tilde{\alpha}_n \left( \frac{w_{1,n}}{2} + y (w_{1,n})_y \right) + (b_n \tilde{\beta}_n + 1)(w_{1,n})_y (s, \mu_n^{-1}(y' - \sigma_n)) \]

\[ = -5 \mu_n^{-1/2} (Q^4 w_{1,n})_y (s, \mu_n^{-1}(y' - \sigma_n)) + b_n \mu_n^{-1/2}(F_n + G_n + H_n)(s, \mu_n^{-1}(y' - \sigma_n)). \]
Set
\[ Q_n(s, y') = \mu_n^{-1/2} Q(\mu_n^{-1}(y' - \sigma_n)), \]
\[ F_{2,n}(s, y') = \mu_n^{-7/2} F_n(s, \mu_n^{-1}(y' - \sigma_n)), \]
\[ G_{2,n}(s, y') = \mu_n^{-5/2} G_n(s, \mu_n^{-1}(y' - \sigma_n)), \]
\[ H_{2,n}(s, y') = \mu_n^{-7/2} H_n(s, \mu_n^{-1}(y' - \sigma_n)), \]
so that \( w_{2,n} \) satisfies
\[ \frac{1}{\mu_n} (w_{2,n})_s + (w_{2,n})_{y'y'} = -5 \left( Q_n^4 w_{2,n} \right)_{y'} + b_n \left[ F_{2,n} + (G_{2,n})_{y'} + H_{2,n} \right], \]
with initial condition
\[ w_{2,n}(0) = w_n(0); \]
(note that \( \mu_n(0) = 1, \sigma_n(0) = 0, \gamma_n(0) = \delta_n(0) = 0 \).

Finally, using the following change of variable
\[ s' = \int_0^s \mu_n^3(s'') \, ds'', \quad \text{or, equivalently,} \quad ds' = \mu_n^3(s) \, ds, \]
we obtain the following equation for \( w_{2,n}(s', y') \):
\[ (w_{2,n})_{s'} + (w_{2,n})_{y'y'} = -5 \left( Q_n^4 w_{2,n} \right)_{y'} + b_n \left[ F_{2,n} + (G_{2,n})_{y'} + H_{2,n} \right]. \]

Let us introduce some preliminary tools.

**Step 2.** Preliminaries on the local Cauchy problem in \( L^2 \) for the critical generalized KdV equation.

For \( \xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), and \( T > 0 \), we define:
\[ \eta^T_1(\xi) = \sup_{t \in (-T, T)} |\xi(t)|_{L^2}, \quad \eta^T_2(\xi) = |\xi|_{L^2_{-T}L^2_T}, \quad \eta^T_3(\xi) = |\xi|_{L^\infty_{-T}L^2_T}. \]
\[ \Omega^T(\xi) = \max_{j=1,2,3} \eta^T_j(\xi). \]

We claim the following estimates, which are direct consequences of the results of [11].

**Lemma 21.** (i) For \( \xi_0 \in L^2(\mathbb{R}) \),
\[ \Omega^T(S(t)\xi_0) \leq C|\xi_0|_{L^2}. \]
(ii) For \( g \in L^\infty((-T, T), L^2(\mathbb{R})) \),
\[ \Omega^T \left( \int_0^t S(t-s)g(s) \, ds \right) \leq CT \sup_{(-T,T)} |g|_{L^2}. \]
(iii) For \( i, j \in \mathbb{N} \setminus \{0\} \), such that \( i + j = 5 \), and for \( \xi_1, \xi_2 \) such that \( \eta^T_2(\xi_1) + \eta^T_3(\xi_1) + \eta^T_3(\xi_2) < \infty \), we have
\[
\Omega^T \left( \frac{\partial}{\partial x} \int_0^t S(t-s) \xi_1^i(s) \xi_2^j(s) \, ds \right)
\]

(163) \( \leq C \left( \eta_1^T(\xi_1) \right)^{i-1} \left( \eta_2^T(\xi_2) \right)^{j-1} \left( \eta_3^T(\xi_1) \eta_1^T(\xi_2) + \eta_3^T(\xi_1) \eta_2^T(\xi_2) + \eta_3^T(\xi_1) \eta_3^T(\xi_2) \right). \)

**Proof.** Estimate (161) follows from estimate (3.6) and (3.11) in [11], and (162) is a direct consequence of (161).

Next, take \( i, j \in \mathbb{N} \setminus \{0\} \), such that \( i + j = 5 \). By estimate (3.7) and (3.8) of [11], and applying Hölder inequality twice, we have:

\[
\eta_1^T \left( \frac{\partial}{\partial x} \int_0^t S(t-s) \xi_1^i(s) \xi_2^j(s) \, ds \right) + \eta_2^T \left( \frac{\partial}{\partial x} \int_0^t S(t-s) \xi_1^i(s) \xi_2^j(s) \, ds \right)
\]

\[
\leq C \left| \xi_1^i \xi_2^j \right|_{L^1_t L^2_x} \leq C \int \left( \int_T^1 \xi_1^i \xi_2^j \right)^{1/2}
\]

\[
\leq C \int_{x \in \mathbb{R}} \left| \xi_1^i \right|_{L^6_t L^3_x} \left| \xi_2^j \right|_{L^6_t L^3_x} \leq C \left( \eta_1^T(\xi_1) \right)^{i} \left( \eta_2^T(\xi_2) \right)^{j}.
\]

Next, by (3.12) of [11],

\[
\eta_3^T \left( \int_0^t S(t-s) \frac{\partial}{\partial x} (\xi_1^i \xi_2^j)(s) \, ds \right) \leq C \left| \frac{\partial}{\partial x} (\xi_1^i \xi_2^j) \right|_{L^5_t L^{10/9}_x}.
\]

Now, note that, by using Hölder inequality twice, \((i-1) + j = 4\),

\[
\left| \xi_1^i \xi_2^j \right|_{L^1_t L^2_x}^2 \leq \left| \xi_1^i \right|_{L^{6/5}_t L^{5/3}_x} \left| \xi_2^j \right|_{L^{6/5}_t L^{5/3}_x}^4 \leq \left| \xi_1^i \right|_{L^{1/3}_t L^{9}_x} \left| \xi_2^j \right|_{L^{1/3}_t L^{9}_x} \left| \xi_1^i \right|_{L^{1/3}_t L^{9}_x} \left| \xi_2^j \right|_{L^{1/3}_t L^{9}_x} \leq C \left( \eta_3^T(\xi_1) \right)^{i} \left( \eta_3^T(\xi_2) \right)^{j}.
\]

Therefore, (163) is proved, and the proof of Lemma 21 is complete. \( \square \)

Note also that if \( \chi : \mathbb{R} \to \mathbb{R} \), is such that \( |\chi(x)| + |\chi_x(x)| \leq C e^{-c|x|} \), for some constants \( C, c > 0 \), then we have

(164) \( \eta_2^T(\chi) \leq C T^{1/10}, \quad \eta_3^T(\chi) \leq C T^{1/2}. \)

As an application of these results, we prove that equation (113) is well posed in \( L^2 \).

**Corollary 2** (Well-posedness of (113) in \( L^2 \)). – Let \( w_0 \in L^2(\mathbb{R}) \). Then there exists a unique global solution \( w_1 \in C(\mathbb{R}, L^2(\mathbb{R})) \) of (113) satisfying \( w_1(0) = w_0 \).

**Proof.** – Recall that setting \( \tilde{w}_1(s, y) = w_1(s, y-s) \), it is equivalent to solve

\[
\tilde{w}_{1s} + \tilde{w}_{1yy} + 5 (\tilde{Q}^4 \tilde{w}_1)_y = 0, \quad \tilde{w}_1(0) = w_0,
\]

where \( \tilde{Q}(s, y) = Q(y-s) \) (see Lemma 9).

Let, for \( K > 0 \) and \( T \in (0, 1) \) to be chosen later,

\[
\mathcal{M} = \{ \tilde{w}_1 \in C([-T, T], L^2(\mathbb{R})); \Omega^T(\tilde{w}_1) \leq (K + 1)|w_0|_{L^2} \}.
\]
and
\[ \Phi(\tilde{w}(s)) = S(s)w_0 - \int_0^s S(s-s')\left(5\tilde{\Omega}^4\tilde{w}(s')\right)ds'. \]

Then, we have, by (161) and (163), and next (164):
\[ \Omega^T(\Phi(\tilde{w})) \leq C|w_0|_{L^2} + C\Omega^T(\tilde{w})\left(\eta_2^T(\tilde{Q})\right)^3 \left(\eta_2^T(\tilde{Q}) + \eta_2^T(\tilde{Q})\right) \]
\[ \leq C|w_0|_{L^2} + C T^{2/5} \Omega^T(\tilde{w}). \]

It follows that for \( K = C + 1 \) and \((C + 1)CT^{2/5} = 1\), \( \Phi \) maps \( \mathcal{M} \) into itself. Observe that the choice of \( T \) is independent of \( \|w_0\|_{L^2} \). By the same argument, we can show that by possibly choosing a smaller \( T \), \( \tilde{Q} \) is a contraction for the norm \( T \). Therefore, we have proved the existence of a unique solution \( \tilde{w} \) of (113) on \( (0, T) \), where \( T > 0 \) is independent of \( |w_0|_{L^2} \). By a standard iteration argument, it follows that equation (113) is globally well posed in \( L^2 \), and so Corollary 2 is proved. \( \square \)

**Step 3.** Convergence of the sequence \( (w_{2,n},) \).

Let \( w \in C(R, H^1(R)) \) be the solution of (113) with initial value \( w_0 \in H^1 \), as given in Lemma 9. We prove the following lemma:

**Lemma 22.** We have
\[ w_{2,n} \to w_1(s, y-s) \]
\[ \text{in } L^\infty_{loc}(R, L^2(R)). \]

**Proof.** We denote the variables of \( w_{2,n} \) by \( (s, y) \) instead of \((s', y')\) for simplicity.

Set
\[ z_n(s, y) = w_{2,n}(s, y) - w_1(s, y-s). \]

We verify easily that
\[ (z_n)_s + (z_n)_{yy} = -5(\tilde{\Omega}^4 z_n)_y - 5[(Q^4_n - \tilde{\Omega}^4)w_{2,n}]_y + b_n[F_{2,n} + (G_{2,n})_y + H_{2,n}]. \]

(Recall that \( \tilde{Q}(s, y) = Q(y-s) \).)

Thus,
\[ z_n(s) = S(s)\left(w_n(0) - w_0\right) \]
\[ + \int_0^s S(s-s')\left[-5(\tilde{\Omega}^4 z_n)_y - 5[(Q^4_n - \tilde{\Omega}^4)w_{2,n}]_y + b_n[F_{2,n} + (G_{2,n})_y + H_{2,n}]\right]ds'. \]

Fix \( T_0 > 0 \). Let \( T \in (0, 1) \), \( T < T_0 \) to be chosen later.

We claim that
\[ z_n \to 0 \]
\[ \text{in } L^\infty((-T, T), L^2(R)). \]

By repeating the argument on \( (T, 2T) \), \((-2T, -T)\),... and then iterating until \((-nT, nT)\) covers \((-T_0, T_0)\), we obtain
\[ w_{2,n} \to w_1(s, y-s) \]
\[ \text{in } L^\infty((-T_0, T_0), L^2(R)). \]
To prove (166), we estimate $\Omega^T(z_n)$. First, by (161), we have

$$\Omega^T (S(s)(w_n(0) - w_0)) \leq C \left| w_n(0) - w_0 \right|_{L^2}.$$  

Now, we treat separately the nonhomogeneous terms.

(i) For the term $(\tilde{Q}^4 z_n)_y$, we recall that

$$\forall s \in (-T_0, T_0), \forall y \in \mathbb{R}, \quad \left| \tilde{Q}(y) \right| + \left| \tilde{Q}_y(y) \right| \leq C e^{-|y|},$$

(the constant $C$ depends on $T_0$) so that, by (164),

$$\eta^T_2(\tilde{Q}) + \eta^T_3(\tilde{Q}) \leq CT^{1/10},$$

where the constant $C$ depends on $T_0$ but not on $T$.

Next, by using (163), we have

$$\Omega^T \left( \frac{\partial}{\partial x} \int_0^s S(s - s') \tilde{Q}^4(s') z_n(s') \, ds' \right) \leq C \left( \eta^T_2(\tilde{Q}) \right)^3 \left( \eta^T_2(\tilde{Q}) + \eta^T_3(\tilde{Q}) \right) \Omega^T(z_n)$$

$$\leq CT^{2/5} \Omega^T(z_n).$$

(ii) We consider the term $((\tilde{Q}_n^4 - \tilde{Q}^4)w_{2,n})_y$.

By Lemma 8(iii),

$$\forall s \in \mathbb{R}, \forall y \in \mathbb{R}, \quad \left| w_n(s, y) \right| \leq C e^{-\beta_2|y|},$$

and so

$$\forall s \in \mathbb{R}, \quad \left| \alpha_n(s) \right| + \left| \beta_n(s) \right| \leq D.$$

By Lemma 8, we also have

$$\left| \tilde{\alpha}_n - \alpha_n \right| + \left| \tilde{\beta}_n - \beta_n \right| \leq C b_n,$$

in particular,

$$\forall s \in \mathbb{R}, \quad \left| \tilde{\alpha}_n \right| + \left| \tilde{\beta}_n \right| \leq C,$$

and so

$$\forall s \in (-T_0, T_0), \quad \left| \mu_n(s) - 1 \right| + \left| \sigma_n(s) - s \right| \leq C b_n.$$  

Therefore, by the decay properties of $Q$ and $Q_y$, we have

$$\forall s \in (-T_0, T_0), \forall y \in \mathbb{R}, \quad \left| Q_n^4 - \tilde{Q}^4 \right| \leq C b_n e^{-c|y|}.$$  

Therefore, we obtain, by (163), and next (164),

$$\Omega^T \left( \int_0^s S(s - s') \left( (\tilde{Q}_n^4 - \tilde{Q}^4)w_{2,n}(s') \right) \, ds' \right)$$

$$\leq C b_n \left( \eta^T_2 \left( \left( \frac{Q_n^4 - \tilde{Q}^4}{b_n} \right)^{1/4} \right) \right) \left[ \eta^T_2 \left( \left( \frac{Q_n^4 - \tilde{Q}^4}{b_n} \right)^{1/4} \right) \right] \left( \eta^T_2 \left( \left( \frac{Q_n^4 - \tilde{Q}^4}{b_n} \right)^{1/4} \right) \right).$$

By the decay properties of \( Q, Q_y, Q_{yy} \), and the bounds on \( \alpha_n, \beta_n, |\alpha_n - \alpha_n|, |\beta_n - \beta_n|, \gamma_n \) and \( \delta_n \), we have

\[ \forall s \in (-T_0, T_0), \forall y \in \mathbb{R}, \quad |F_{2,n}(s, y)| + |H_{2,n}(s, y)| \leq C e^{-c|y|}, \]

so that by (162), we have

\[ \Omega^T \left( \int_0^s S(s - s')(F_{2,n} + H_{2,n})(s') \, ds' \right) \leq CT. \]

(iv) Finally, we treat the term \( (G_{2,n})_y \).

Note that if we set

\[ \tilde{\omega}_n(s, y) = \mu_n^{1/2}(s) w_n(s, \mu_n^{-1}(s)(y - \sigma_n(s))) \]

\[ = w_{2,n}(s, y) + \mu_n^{1/2}(s) \left[ y_n \left( \frac{Q}{2} + y Q_y \right) + \delta_n Q_y \right] (s, \mu_n^{-1}(s)(y - \sigma_n(s))), \]

then

\[ G_{2,n} = -(10Q_n^3 \tilde{\omega}_n^2 + 10b_n Q_n^2 \tilde{\omega}_n^3 + 5b_n^2 Q_n \tilde{\omega}_n^4 + b_n^3 \tilde{\omega}_n^5). \]

By using (163), we obtain

\[ \Omega^T \left( \int_0^s S(s - s')(G_{2,n})_y(s') \, ds' \right) \]

\[ \leq C (\eta_T^2(Q_n))^2 (\eta_T^2(Q_n) + \eta_T^2(Q_n)) (\Omega^T(\tilde{\omega}_n))^2 + C (\eta_T^2(Q_n))(\eta_T^2(Q_n)) \]

\[ + \eta_T^2(Q_n) (\Omega^T(\tilde{\omega}_n))^3 + C (\eta_T^2(Q_n) + \eta_T^2(Q_n)) (\Omega^T(\tilde{\omega}_n))^4 + C (\Omega^T(\tilde{\omega}_n))^5. \]

Since

\[ \Omega^T(\tilde{\omega}_n) \leq \Omega^T(w_{2,n}) + \Omega^T \left( \mu_n^{1/2}(s) \left[ y_n \left( \frac{Q}{2} + y Q_y \right) + \delta_n Q_y \right] (s, \mu_n^{-1}(s)(y - \sigma_n(s))) \right) \]

\[ \leq \Omega^T(w_{2,n}) + C, \]

where \( C \) depends on \( T_0 \) but not on \( T \) (note that \( \forall s \in (-T_0, T_0), |y_n(s)| + |\delta_n(s)| \leq C_0 \)), we obtain:

\[ \Omega^T \left( \int_0^s S(s - s')(G_{2,n})_y(s') \, ds' \right) \leq C (1 + (\Omega^T(w_{2,n}))^5). \]

Therefore, gathering (i)–(iv), we obtain

\[ \Omega^T(z_n) \leq C |w_n(0) - w_0|_{L^2} + CT^{2/5} \Omega^T(z_n) + C b_n T^{2/5} \Omega^T(w_{2,n}) + C b_n T + C b_n \left( 1 + (\Omega^T(w_{2,n}))^5 \right). \]
Now, recall that \( w_{2,n} = z_n - \tilde{w}_1 \), where \( \tilde{w}_1(s, y) = w_1(s, y - s) \) is such that \( \Omega^{T_0}(\tilde{w}_1) \leq C_0 \) (see end of Step 2).

Therefore,

\[
\Omega^T(w_{2,n}) \leq \Omega^T(z_n) + \Omega^T(\tilde{w}_1) \leq \Omega^T(z_n) + \Omega^{T_0}(\tilde{w}_1) \leq \Omega^T(z_n) + C_0,
\]

and so

\[
\Omega^T(z_n) \leq C|w_n(0) - w_0|_{L^2} + CT^{2/5} \Omega^T(z_n) + C b_n T^{2/5} (\Omega^T(z_n) + C_0)
\]

\[
+ C b n T + C b n (C_0 + (\Omega^T(z_n))^5)
\]

\[
\leq C|w_n(0) - w_0|_{L^2} + CT^{2/5} \Omega^T(z_n) + C b_n + C b_n (\Omega^T(z_n))^5,
\]

where \( C \) depends on \( T_0 \) but not on \( T \) and \( n \).

Finally, choose \( T = \min(1, (2C)^{-5/2}) \), it follows that

\[
\frac{1}{2} \Omega^T(z_n) \leq C|w_n(0) - w_0|_{L^2} + C b_n + C b_n (\Omega^T(z_n))^5.
\]

By taking \( n \) large enough, this implies that

\[
\Omega^T(z_n) \leq 4(C|w_n(0) - w_0|_{L^2} + C b_n).
\]

Therefore, we have proved claim (166), and the proof of Lemma 22 is complete. \( \square \)

**Step 4.** Conclusion of the proof.

(i) We prove

\[
w_{1,n} \to w_1 \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}, L^2(\mathbb{R})),
\]

as \( n \to \infty \), where

\[
w_{1,n} = \frac{1}{\mu_n} \left[ \frac{\mu_n}{2} + y Q^y \right],
\]

and \( w_1 \) is the solution of (113) with initial data \( w_0 \neq 0 \).

Indeed, let \( T_0 > 0 \), it follows from (165) that

\[
\mu_n^{-1/2} w_{1,n}(s, \mu_n^{-1}(y - \sigma_n)) \to w_1(s, y - s) \quad \text{in} \quad L^\infty((-T_0, T_0), L^2(\mathbb{R})).
\]

By invariance of the \( L^2 \) norm by scaling and translation this is equivalent to

\[
|w_{1,n}(s, y) - \mu_n^{1/2} w_1(s, \mu_n(y - s) + \sigma_n)|_{L^\infty((-T_0, T_0), L^2)} \to 0
\]

as \( n \to \infty \).

Now, \( \forall s \in (-T_0, T_0) \), we have

\[
|\mu_n(s) - 1| + |\mu_n(s) - \sigma_n(s)| \leq C b_n.
\]

Thus, by the dominated convergence theorem, we have, \( \forall s \in (-T_0, T_0) \):

\[
|\mu_n^{1/2} w_1(s, \mu_n(y - s) + \sigma_n) - w_1(s, y)|_{L^2} \to 0.
\]
Since $w_1$ is continuous, thus uniformly continuous on $(-T_0, T_0)$, $\mu_n$ and $\sigma_n$ are uniformly continuous (with a constant independent of $n$). It is then a classical result that

\begin{equation}
(171) \quad \left| \mu_n^{1/2} w_1(s, \mu_n(y-s) + \sigma_n) - w_1(s, y) \right| \rightarrow 0 \quad \text{in } L^\infty((-T_0, T_0), L^2) \quad \text{as } n \rightarrow \infty.
\end{equation}

By (170) and (171), we obtain

$$w_{1,n} \rightarrow w_1 \quad \text{in } L^\infty((-T_0, T_0), L^2(\mathbb{R})), \quad \text{as } n \rightarrow \infty.$$ 

(ii) In order to prove that $w_n$ converges to a solution $w$ of (112), for certain functions $\alpha$ and $\beta$, we need only prove that $\alpha_n, \beta_n, \gamma_n$ and $\delta_n$ converge to some limit functions. This can be done by considering the expressions of $\alpha_n$ and $\beta_n$ in terms of $w_{1,n}$ instead of $w_n$.

Note that

$$\begin{align*}
\alpha_n &= \frac{1}{Q^4} \int L((Q^3)_y) w_n \\
&= \frac{4}{Q^4} \left( \int L((Q^3)_y) w_{1,n} + \gamma_n \int L((Q^3)_y) \left( \frac{Q}{2} + y Q_y \right) + \delta_n \int L((Q^3)_y) Q_y \right) \\
&= \frac{4}{Q^4} \int L((Q^3)_y) w_{1,n},
\end{align*}$$

by parity and since $L(Q_y) = 0$.

Therefore, by (169),

$$\alpha_n(s) \rightarrow \frac{4}{Q^4} \int L((Q^3)_y) w_1(s) \equiv \alpha(s) \quad \text{in } C(\mathbb{R}).$$

In particular, we have

$$\gamma_n(s) \rightarrow \gamma(s) = \int_0^s \alpha(s') \, ds' \quad \text{in } C(\mathbb{R}).$$

Similarly,

$$\begin{align*}
\beta_n &= \frac{20}{Q^4} \int Q^3 Q^2_y w_n \\
&= \frac{2}{Q^4} \left( 20 \int Q^3 Q^2_y w_{1,n} + \gamma_n \int Q^3 Q^2_y \left( \frac{Q}{2} + y Q_y \right) + \delta_n \int Q^3 Q^2_y Q_y \right) \\
&= \frac{2}{Q^4} \left( 20 \int Q^3 Q^2_y w_{1,n} + \gamma_n \int Q^3 Q^2_y \left( \frac{Q}{2} + y Q_y \right) \right).
\end{align*}$$

In particular, for $s \in \mathbb{R},$

$$\beta_n(s) \rightarrow \frac{2}{Q^4} \left( 20 \int Q^3 Q^2_y w_1(s) + \gamma(s) \int Q^3 Q^2_y \left( \frac{Q}{2} + y Q_y \right) \right) \equiv \beta(s) \quad \text{in } C(\mathbb{R}).$$

Let $\delta(s) = \int_0^s (\beta(s') - 2\gamma(s')) \, ds'.$ We have $\delta_n \rightarrow \delta$ in $C(\mathbb{R}).$
Finally, set \( w = w_1 + \gamma\left(\frac{Q}{2} + yQ_y\right) + \delta Q_y \). Then as in Lemma 10,

\[
w_x - (Lw)_y = \alpha\left(\frac{Q}{2} + yQ_y\right) + \beta Q_y, \quad w(0) = w_0.
\]

Moreover,

\[
w_n = w_{1,n} + \gamma_n\left[\frac{Q}{2} + yQ_y\right] + \delta_n Q_y \rightarrow w_1 + \gamma\left[\frac{Q}{2} + yQ_y\right] + \delta Q_y = w,
\]

in \( L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R})) \), as \( n \to \infty \).

(iii) The orthogonality properties (H1′) and the uniform exponential decay property (101) are conserved through the passage to the limit. Finally, since \( w_0 = w_1(0) \neq 0 \), we have \( w(0) = w_1(0) \neq 0 \), and then \( w \neq 0 \) on \( \mathbb{R} \times \mathbb{R} \).

Hence, Lemma 11 is proved. \( \square \)

### Appendix C. Proof of Proposition 4

First, note that

\[ H(u, u) = -(Lu)_y, \]

so that

\[ H(u, v) = -\frac{1}{2}\left((y(Lu)_y - Lu - L(yu_y)), v\right). \]

We set:

\[ \overline{H}(u, v) = H(u, v) - \frac{1}{10}(Lu, v) = (\overline{L}_1 u, v), \]

where

\[
\overline{L}_1 u = L_1 u - \frac{1}{10} Lu \\
= -\frac{7}{5} u_{yy} + \frac{2}{5} u - 2Q^4 u + 10yQ^3 Q_y u \\
= -\frac{1}{2}\left( y(Lu)_y - \frac{4}{5} Lu - L(yu_y)\right).
\]

Now, we proceed in three steps to prove that under the assumptions of the proposition, we have \( \overline{H}(w, w) \geq 0 \).

We give a definition of the index of a bilinear form. Let \( B \) be a bilinear form on a vector space \( V \). Let us define the index of \( B \) on \( V \) by:

\[
\text{ind}_{V}(B) = \max\{k \in \mathbb{N} : \text{there exists a sub-space} \ P \ \text{of codimension} \ k \ \text{such that} \ B_{|P} \ \text{is positive definite}\}.
\]

Let \( H^1_e \) (respectively, \( H^1_o \)) denote the sub-space of even (respectively, odd) \( H^1 \) functions. Assume that \( H^1_e \) is \( B \)-orthogonal to \( H^1_o \). We say that \( B \) defined on \( H^1 \) has index \( i + j \) if \( \text{ind}_{H^1_e} = i \) and \( \text{ind}_{H^1_o} = j \).

**Step 1.** The index of \( \overline{H} \) is \( 1 + 1 \).
In this first step, we show that $H$ has index 1. This is done by comparing $H$ with a simpler quadratic form of classical type.

**Lemma 23 (Lower bound on $H$).** – For all $w \in H^1(\mathbb{R})$, we have

$$H(w, w) > \frac{7}{5}(\tilde{L}w, w) + \frac{1}{10}(w, w),$$

where

$$\tilde{L}w = -w_{xx} - 8Q^4w = -w_{xx} - 24\frac{w}{\text{ch}^2(2x)}.$$

**Proof.** – By the expression of $\tilde{L}1$, and integration by parts, we have

$$H(w, w) = \frac{7}{5}\int w_x^2 + \frac{2}{5}\int w^2 - 2\int Q^4w^2 + 10\int yQ^3Q_yw^2.$$

We claim that

$$\forall y \in \mathbb{R}, \quad \frac{2}{5} - 2Q^4 + 10yQ^3Q_y \geq -\frac{56}{5}Q^4 + \frac{1}{10}.$$  

(173)

Assuming this claim, we obtain

$$H(w, w) > \frac{7}{5}\left(\int w_x^2 - 8\int Q^4w^2\right) + \frac{1}{10}\int w^2 = \frac{7}{5}(\tilde{L}w, w) + \frac{1}{10}\int w^2,$$

which is the desired result.

Therefore, we need only prove (173) to complete the proof of the lemma. First note that (173) is equivalent to

$$\forall y \in \mathbb{R}, \quad 10yQ^3Q_y + \frac{3}{10} + \frac{46}{5}Q^4 \geq 0.$$  

Since

$$Q^4(y) = \frac{3}{\text{ch}^2(2y)} \quad \text{and} \quad Q_y(y) = -\frac{3\text{sh}(2y)}{\text{ch}^{3/2}(2y)},$$

it is equivalent to show that

$$\forall y \in \mathbb{R}, \quad 10y\frac{\text{sh}(2y)}{\text{ch}^2(2y)} \leq \frac{1}{10} + \frac{46}{5}\frac{1}{\text{ch}^2(2y)}.$$  

(174)

We set $z = 2y$, since $1/\text{ch}^2(z) = 1 - \text{th}^2(z)$, (174) is equivalent to

$$\forall z, \quad F(z) = 5z - \frac{1}{\text{th}(z)}\left(\frac{46}{5} + \frac{1}{10(1 - \text{th}^2(z))}\right) \leq 0.$$  

Let us denote $G(t) = \frac{46}{5t^2} + \frac{1}{10(1 - t^2)}$, so that

$$G'(t) = -\left\{\frac{46}{5t^2} + \frac{1}{10}\frac{1 - 3t^2}{t^2(1 - t^2)^2}\right\} = \frac{92(1 - t^2)^2 + (1 - 3t^2)}{10t^2(1 - t^2)^2}.$$  

Therefore, we have

$$F'(z) = 5 - (1 - \text{th}^2(z))G'(\text{th}^2(z)).$$
and so

\[ F'(z) = \frac{G_1(\text{th}^2(z))}{10\text{th}^2(z)(1 - \text{th}^2(z))}, \]

where

\[ G_1(u) = 50u(1 - u) + 92(1 - u)^2 + (1 - 3u) = 42u^2 - 137u + 93. \]

The second order polynomial \( G_1 \) has two distinct real positive roots: denoted by \( u_1 \) and \( u_2 \), with \( u_1 < u_2 \). Since \( G_1(0) = 93 > 0 \), and \( G_1(1) = -2 < 0 \), we have \( 0 < u_1 < 1 \) and \( u_2 > 1 \). Moreover

\[ u_1 = \frac{137 - \sqrt{3145}}{84} \approx 0.963. \]

Therefore, the function \( F \) attains its global maximum at the point

\[ z_1 = \text{ath}(\sqrt{u_1}) = \frac{1}{2} \ln \left( \frac{1 + \sqrt{u_1}}{1 - \sqrt{u_1}} \right). \]

At this point, the value of \( F \) is

\[ F(z_1) = \frac{5}{2} \ln \left( \frac{1 + \sqrt{u_1}}{1 - \sqrt{u_1}} \right) - \frac{1}{\sqrt{u_1}} \left( \frac{46}{5} + \frac{1}{10(1 - u_1)} \right). \]

We have

\[ \frac{1}{\sqrt{u_1}} \left( \frac{46}{5} + \frac{1}{10(1 - u_1)} \right) \approx 12.15. \]

and

\[ \frac{5}{2} \ln \left( \frac{1 + \sqrt{u_1}}{1 - \sqrt{u_1}} \right) \approx 11.68, \]

which proves that \( F(z) \leq F(z_1) < 0 \), for all \( z \in \mathbb{R} \). Therefore (173) is proved.

\[ \square \]

**Lemma 24** (Structure of \( \tilde{L} \)). – The self-adjoint operator \( \tilde{L} \) in \( L^2 \) defined by (172) has exactly two negative eigenvalues \(-16 \) and \(-4 \), respectively associated to the eigenfunctions \( Q^4 \) and \( QQ_y \). Moreover, if \( w \in H^1(\mathbb{R}) \) is such that

\[ (w, Q^4) = (w, QQ_y) = 0, \]

then

\[ (\tilde{L}w, w) \geq 0. \]

**Proof.** – The fact that \( \tilde{L}Q^4 = -16Q^4 \) and \( \tilde{L}(QQ_y) = -4QQ_y \) follows from simple direct calculations. Indeed, we have

\[ \tilde{L}Q^4 = -4(Q^3Q_y) - 8Q^8 = -12Q^2Q_y^2 - 4Q^3Q_{yy} - 8Q^8. \]

Since \( Q_y^2 = Q^2 - \frac{1}{2}Q^6 \) and \( Q_{yy} = Q - Q^5 \), we obtain

\[ \tilde{L}Q^4 = -12Q^4 + 4Q^8 - 4Q^4 + 4Q^8 - 8Q^8 = -16Q^4. \]
On the other hand, we have

\[ \tilde{L}Q_y = -(Q_y^2 + Q_{yy})_y - 8Q^5Q_y = -\left(2Q^2 - \frac{4}{3}Q^6\right)_y - 8Q^5Q_y \]

\[ = -4Qy + 8Q^5Q_y - 8Q^5Q_y = -4Qy. \]

According to Titchmarsh [22], Ch. IV, §4.19, the third eigenvalue is 0 and there is a continuous spectrum from 0 to +∞.

Using Schechter [20], Ch. 8, Lemma 7.10, and arguing as in [14], Lemma 2, we obtain (175). This concludes the proof of Lemma 24.

It follows that \( \tilde{L} + 1/14 \) has two negative eigenvalues and a third eigenvalue 1/14 strictly positive. By (175), it follows that \( \tilde{L} + 1/14 \) has index 1 + 1. By Lemma 23, we have

\[ \overline{H}(w, w) \geq \frac{7}{5} \left( \langle \tilde{L}, w \rangle + \frac{1}{14} \langle w, w \rangle \right). \]

Therefore, \( \overline{H} \) has at most index 1 + 1. In fact, let us show that \( \overline{H} \) has exactly index 1 + 1.

**Lemma 25 (The index of \( \overline{H} \) is 1 + 1).** – We have:

(i) \( \overline{H}(Q^3, Q^3) < 0, \overline{H}(Q_y, Q_y) = 0 \)

and

\[ \text{ind}_{\overline{H}} \overline{H} = 1 + 1. \]

(ii) The kernel of \( \overline{L}_1 \) is \( \{0\} \).

(iii) The operator \( \overline{L}_1 \) has exactly two negative eigenvalues \( \lambda_1, \lambda_2 \), respectively associated to the eigenfunctions \( \psi_1, \psi_2 \). Moreover, \( \langle \overline{L}_1 w, w \rangle \) is coercive on \( \text{span}(\psi_1, \psi_2)^\perp \).

Note that \([\cdot]^\perp \) denotes the orthogonal in the \( L^2 \) sense.

**Proof.** – First, we prove (176). We have \( \overline{\Pi}(Q^3, Q^3) = -\int (LQ^3)_y(yQ^3) - \frac{1}{10}(LQ^3, Q^3). \)

Note that

\[ LQ^3 = -3(Q^2Q_y)_x + Q^3 - 5Q^7 = -6Q^2x - 3Q^2Q_{yy} + Q^3 - 5Q^7 \]

\[ = -6Q\left(\frac{Q^2 - \frac{1}{3}Q^6}{x}\right) - 3Q^2(Q - Q^5) + Q^3 - 5Q^7 = -8Q^3. \]

by using \( Q_{yy} + Q^5 = Q \) and \( Q^2 + \frac{1}{3}Q^6 = Q^2 \). Thus, we obtain

\[ \overline{\Pi}(Q^3, Q^3) = 8\int (Q^3)_y(yQ^3) + \frac{8}{10}\int Q^6 = -4\int Q^6 + \frac{4}{5}\int Q^6 = -\frac{16}{5}\int Q^6 < 0. \]

On the other hand,

\[ \overline{\Pi}(Q_y, Q_y) = -\int (LQ_y)_y(yQ_y) - \frac{1}{10}(LQ_y, Q_y), \]

since \( LQ_y = 0 \), we obtain \( \overline{\Pi}(Q_y, Q_y) = 0 \). Note that this follows from the fact that \( Q_y \) is a solution of (118).
Observe now that we have $L_1 Q_y \neq 0$. Indeed,

$$L_1 Q_y = -\frac{1}{2} (y L(Q_y))_y - \frac{4}{2} L Q_y - L(y Q_{yy}) = \frac{1}{2} L(y Q_{yy}),$$

and $L(y Q_{yy}) \neq 0$ since the spectrum of $L$ is exactly $\text{span}(Q_y)$ (see [14], Lemma 2).

This implies in particular that $Q_y$ is not a critical point of $L_1$, and since $H(Q_y, Q_y) = 0$, it follows that $\text{ind}_{H_1} = 1 + 1$.

Now, we prove the property about the spectrum of $L_1$. Suppose that there exists $\chi \in H^1(\mathbb{R})$, such that $L_1 \chi = 0$. Write $\chi = \chi_e + \chi_o$, where $\chi_e \in H^1_\text{e}$ and $\chi_o \in H^1_\text{o}$. We still have $L_1 \chi_e = L_1 \chi_o = 0$, since $L_1 \chi_e$ is even (respectively, $L_1 \chi_o$ is odd). We decompose $\chi_e = a Q^3 + \chi_1$, where $H(Q^3, \chi_1) = 0$. Next, we have $0 = (\overline{L_1 \chi_e}, \chi_1) = \overline{H} (a Q^3 + \chi_1, \chi_1) = \overline{H} (\chi_1, \chi_1)$. Since $\text{ind}_{H_1} = 1$, we have $\chi_1 = 0$, and then $a = 0$, so that $\chi_e = 0$. Arguing in the same way and using $L_1 Q_y \neq 0$, we find $\chi_o = 0$.

From standard variational arguments, there are $\psi_1, \psi_2 \in H^1(\mathbb{R})$, and $\lambda_1, \lambda_2 < 0$, such that

$$L_1(\psi_1) = \lambda_1 \psi_1, \quad L_1(\psi_2) = \lambda_2 \psi_2.$$ 

Moreover, we have

$$\langle \overline{L_1} \psi, \psi \rangle \geq 0 \quad \text{if } (\psi, \psi) = 0.$$ 

Finally, let us prove that $(w, w) \mapsto \langle L_1 w, w \rangle$ is coercive on $[\text{span}(\psi_1, \psi_2)]^\perp$. In fact, we prove by contradiction that if $w \in [\text{span}(\psi_1, \psi_2)]^\perp$, then

$$\langle L_1 w, w \rangle \geq \frac{1}{20} (w, w).$$

Indeed, otherwise, there would be $w_0 \in [\text{span}(\psi_1, \psi_2)]^\perp$ such that

$$\langle L_1 w_0, w_0 \rangle < \frac{1}{20} (w_0, w_0).$$

Thus, on $\text{span}(\psi_1, \psi_2, w_0)$, we have

$$\langle L_1 w, w \rangle \leq \frac{1}{20} (w, w).$$

From Lemma 23, we thus have

$$\langle L_1 w, w \rangle \leq \frac{5}{7} \left( \frac{1}{20} - \frac{1}{10} \right) (w, w) = \frac{1}{28} (w, w),$$

on $\text{span}(\psi_1, \psi_2, w_0)$, which is contradiction with Lemma 24. \(\square\)

**Step 2.** Positivity property on $H^1_\text{e}$.

We show that if $w \in H^1_\text{e}$ is such that $(w, Q) = 0$, then $\overline{H}(w, w) \geq 0$.

**Lemma 26.** There exists a unique function $\phi_1 \in H^1_\text{e}(\mathbb{R})$ such that $L_1 \phi_1 = Q$. Furthermore,

$$\frac{(Q^3, Q)}{\overline{H}(Q^3, Q)} < (\phi_1, Q) < 0.$$
Remark. – The above inequality is checked numerically.

Proof. – The existence of \( \phi_1 \) is a consequence of Lemma 25. Indeed, for any \( \chi \in H^1 \), we have \( \chi = \chi_0 + a_1 \psi_1 + a_2 \psi_2 \), where \( \chi_0 \in [\text{span}(\varphi_1, \varphi_2)]^\perp \). Since \( L_1 \) is coercive on \( [\text{span}(\varphi_1, \varphi_2)]^\perp \), from Lax–Milgram theorem, there exists \( \phi_0 \) such that \( L_1 \phi_0 = \chi_0 \), and then \( \phi = \phi_0 + a_1 \psi_1 + a_2 \psi_2 \) is such that \( L_1 \phi = \chi \).

We denote by \( \phi_1 \) the unique function such that \( L_1 \phi_1 = \chi \). Since \( \phi \) is even, and by the expression of \( L_1 \), the function \( \phi_1 \) is also even.

We find numerically the function \( \phi_1 \) which satisfies:

\[
L_1 \phi_1 = -\frac{7}{5} \phi_{1yy} + \frac{2}{5} \phi_1 - 2 Q^3 \phi_1 + 10 y Q^3 Q_3 \phi_1 = Q.
\]

Next, we evaluate numerically the value

\[
\int Q \phi_1 \simeq -0.485.
\]

On the other hand, we have

\[
\int Q^4 = 3 \int \frac{dx}{\cosh(\sqrt{2} x)} = \frac{3}{2} \int \frac{dy}{\cosh(\sqrt{2} y)} = \frac{3}{2} \left[ \tanh(\sqrt{2}) \right]_{-\infty}^{\infty} = 3,
\]

and

\[
\int Q^6 = \frac{3}{2} \int Q^2 = \frac{3\sqrt{3}}{2} \int \frac{dx}{\cosh(\sqrt{2} x)} = \frac{3\sqrt{3}}{2} \int_{\infty}^{\infty} \frac{dz}{z^2 + 1} = \frac{3\sqrt{3}}{2} \left[ \tanh(z) \right]_{0}^{\infty} = \frac{3\pi \sqrt{3}}{4}.
\]

Therefore,

\[
\frac{(Q^3, Q^3)^2}{\overline{H}(Q^3, Q^3)} = -\frac{45}{12\pi \sqrt{3}} \simeq -0.689,
\]

and (177) is satisfied. \( \Box \)

Lemma 27. – If \( w \in H^1_0(\mathbf{R}) \) satisfies \( (w, Q) = 0 \) then \( \overline{H}(w, w) \geq 0 \).

Proof. – The proof is divided in several steps. First, we consider the plane \( P_1 \) spanned by \( Q^3 \) and \( \phi_1 \), and we show that \( \overline{H} \), restricted to \( P_1 \), is not degenerate.

Next, we define \( P_1^\perp \) the orthogonal of \( P_1 \) in \( H^1_0(\mathbf{R}) \) for the quadratic form \( \overline{H} \). By an index argument, we show that \( \overline{H} \) is nonnegative on \( P_1^\perp \).

Finally, we show that if \( w \in P_1 \), with \( w \perp Q \), then \( \overline{H}(w, w) \geq 0 \).

(i) Let \( P = \text{span}(Q^3, \phi_1) \). Note that since \( \overline{H}(u, v) = (L_1 u, v) \), and \( L_1 \phi_1 = Q \), we have

\[
\frac{\overline{H}(Q^3, Q^3)}{\overline{H}(Q^3, \phi_1)} = \frac{\overline{H}(Q^3, \phi_1)}{\overline{H}(\phi_1, \phi_1)} = \frac{\overline{H}(Q^3, Q^3)}{(Q^3, Q)} = \left( Q^3, Q \right) (Q^3, Q) - (Q^3, Q)^2 < 0
\]

by (177), so that \( \overline{H} \) restricted to \( P_1 \) is not degenerate. It follows that \( H^1_0(\mathbf{R}) = P_1 \oplus P_1^\perp \).

(ii) Since the index of \( \overline{H} \) in \( H^1_0 \) is 1, and \( \overline{H}(Q^3, Q^3) < 0 \), it follows that \( \overline{H} \geq 0 \) on \( P_1^\perp \).
(iii) Let \( w \in P_1 \), \( w \neq 0 \), be such that \( (w, Q) = 0 \). We have
\[
w = \alpha Q^3 + \beta \phi_1.
\]
Since \( (w, Q) = 0 \), we have \( \beta \neq 0 \), and
\[
\frac{\alpha}{\beta} = -\frac{(Q, \phi_1)}{(Q, Q)} = -\frac{(Q, \phi_1)}{H(Q^3, \phi_1)}.
\]
It follows that
\[
\frac{H(w, w)}{\beta^2} = \left(\frac{\alpha}{\beta}\right)^2 \frac{H(Q^3, Q^3)}{H(Q^3, \phi_1)} + 2 \left(\frac{\alpha}{\beta}\right) \frac{H(Q^3, \phi_1)}{H(Q^3, \phi_1)} + \frac{H(\phi_1, \phi_1)}{H(Q^3, \phi_1)}
\]
\[= -\left(\frac{Q, \phi_1}{H(Q^3, \phi_1)}\right) - \frac{H(Q^3, Q^3)}{H(Q^3, \phi_1)^2} (Q, \phi_1) + 1.
\]
By (177), we thus obtain \( H(w, w) > 0 \).

Now, let \( w \) be any even function such that \( (w, Q) = 0 \). Then \( w = w_1 + w_2 \), where \( w_2 \in P_1^\perp \) and \( w_1 \in P_1 \). Since \( 0 = H(w_2, \phi_1) = (w_2, T_1 \phi_1) = (w_2, Q) = 0 \), and \( (w, Q) = 0 \), we also have \( (w_1, Q) = 0 \), and so \( H(w_1, w_1) \geq 0 \).

Since by (ii), we have \( H(w_2, w_2) \geq 0 \), and since \( H(w_1, w_2) = 0 \), by definition, we obtain that \( H(w, w) \geq 0 \), and the lemma is proved. \( \square \)

**Step 3. Positivity property on \( H_0^1 \).**

We show that if \( w \in H_0^1 \) is such that \( (w, y(Q + yQ_y)) = 0 \), then \( H(w, w) \geq 0 \). The proof is similar to Step 2.

**Lemma 28.** There exists a unique function \( \phi_2 \in H_0^1(\mathbb{R}) \) such that \( T_1 \phi_2 = y\left(\frac{Q}{2} + yQ_y\right) \).

Furthermore,
\[
(\phi_2, y\left(\frac{Q}{2} + yQ_y\right)) < 0.
\]

**Proof.** The existence and uniqueness of \( \phi_2 \) follows from the same argument as in the proof of Lemma 26. The function \( \phi_2 \) is odd.

We find numerically the solution \( \phi_2 \) of the equation:
\[
T_1 \phi_2 = -\frac{7}{5} \phi_{2yy} + \frac{2}{5} \phi_2 - 2Q^4 \phi_2 + 10yQ^3 Q_y \phi_2 = y\left(\frac{Q}{2} + yQ_y\right).
\]

Next, we evaluate numerically
\[
\int y\left(\frac{Q}{2} + yQ_y\right) \phi_2 \simeq -0.255.
\]

Therefore, (178) is proved. \( \square \)

**Lemma 29.** If \( w \in H_0^1(\mathbb{R}) \) satisfies \( (w, y(Q + yQ_y)) = 0 \), then \( H(w, w) \geq 0 \).

**Proof.** The proof is similar to the one of Lemma 27.

(i) First, if we define \( P_2 = \text{span}(Q_y, \phi_2) \), then \( H \) is not degenerate on \( P_2 \) since
\[
\frac{H(Q_y, Q_y)}{H(\phi_2, \phi_2)} = -\frac{H(Q_y, \phi_2)}{H(\phi_2, \phi_2)} = -(Q_y, T_1 \phi_2)^2 = -(Q_y, y\left(\frac{Q}{2} + yQ_y\right))^2.
\]
and

$$\mu_0 = \left( Q_y, y \left( \frac{Q}{2} + y Q_y \right) \right) \neq 0,$$

see the proof of Lemma 14.

(ii) As in the proof of Lemma 27, $$\overline{H}$$ is nonnegative on $$P_2^\perp$$, where $$P_2^\perp$$ is the orthogonal of $$P_2$$ in $$H_1^0$$, with respect to the quadratic form $$\overline{H}$$.

(iii) If $$w \in P_2$$, $$w \neq 0$$ is such that $$(w, y(\frac{Q}{2} + y Q_y)) = 0$$, then

$$w = \alpha Q_y + \beta y \left( \frac{Q}{2} + y Q_y \right),$$

with $$\beta \neq 0$$, and

$$\frac{\alpha}{\beta} = -\frac{(y(\frac{Q}{2} + y Q_y), \phi_2)}{(Q_y, y(\frac{Q}{2} + y Q_y))} = -\frac{(y(\frac{Q}{2} + y Q_y), \phi_2)}{\overline{H}(Q_y, \phi_2)}.$$

It follows that:

$$\frac{\overline{H}(w, w)}{\beta^2} = \left( \frac{\alpha}{\beta} \right)^2 \overline{H}(Q_y, Q_y) + 2 \left( \frac{\alpha}{\beta} \right) \overline{H}(Q_y, \phi_2) + \overline{H}(\phi_2, \phi_2) = -\overline{H}(\phi_2, \phi_2) \geq 0,$$

by Lemma 28 (recall that $$\overline{H}(Q_y, Q_y) = 0$$). □

This concludes the proof of Proposition 4.

**Appendix D. Proof of Lemma 17**

First, we show the following lemma for the critical KdV equation, following the proof of a similar lemma for the critical Schrödinger equation, given in Glangetas and Merle [5]. This lemma implies the desired result.

**Lemma 30 (Reduction to the critical KdV equation).** Assume that there exists a sequence $$t_n \to +\infty$$ and $$\tilde{u}_0 \in H^1(\mathbb{R})$$ such that

$$u(t_n, x(t_n) + \cdot) \to \tilde{u}_0 \quad \text{in } H^1(\mathbb{R});$$

then, if $$\tilde{u}$$ is the solution of

$$\tilde{u}_t + \tilde{u}_{xxx} + (\tilde{u}^5)_x = 0,$$

with initial value $$\tilde{u}(0) = \tilde{u}_0$$, we have

$$\forall t \in \mathbb{R}, \quad u(t_n + t, x(t_n) + \cdot) \to \tilde{u}(t, \cdot) \quad \text{in } H^1(\mathbb{R}),$$

$$\forall t \in \mathbb{R}, \quad u(t_n + t, x(t_n) + \cdot) \to \tilde{u} \quad \text{in } C([-t, t], L^2_{\text{loc}}(\mathbb{R})).$$

Assume that Lemma 30 is proved and let us conclude the proof of Lemma 17. We recall that

$$\varepsilon(s_n) \to \tilde{\varepsilon}_0 \quad \text{in } H^1 \quad \text{and} \quad \lambda(s_n) \to \tilde{\lambda}_0.$$
We want to show that

\[(182) \quad \forall s \in \mathbb{R}, \quad \varepsilon(s_n + s) \to \tilde{\varepsilon}(s) \quad \text{in } H^1(\mathbb{R}) \text{ as } n \to +\infty, \]

where for all \( s \in \mathbb{R} \), \( \tilde{\varepsilon}(s) \) is a solution of (18) with \( \tilde{\varepsilon}(0) = \tilde{\varepsilon}_0, \lambda = \tilde{\lambda}, \) and \( x = \tilde{x} \), where \( \lambda \) and \( \tilde{x} \) are defined so that \( \tilde{\varepsilon} \) satisfies \( (\tilde{\varepsilon}, Q^1) = (\tilde{\varepsilon}, Q_\lambda) = 0. \)

We have

\[ u(s_n, x(s_n) + x) = \lambda^{-1/2}(s_n)(Q + \varepsilon)(s_n, \lambda^{-1}(s_n)x). \]

Since \( \varepsilon(s_n) \to \tilde{\varepsilon}_0 \) in \( H^1 \),

\[ u(s_n, x(s_n) + x) \to \tilde{u}_0 = \tilde{\lambda}^{-1/2}(Q + \tilde{\varepsilon}_0)(\tilde{\lambda}_0^{-1}x) \quad \text{in } H^1. \]

By Lemma 30,

\[ (183) \quad \forall s \in \mathbb{R}, \quad u(s_n + s, x_n + \cdot) \to \tilde{u}(s, \cdot) \quad \text{in } H^1, \]

\[ (184) \quad \forall s \in \mathbb{R}, \quad u(s_n + s, x_n + \cdot) \to \tilde{u} \quad \text{in } C([-s_1, s_1], L^2_{\text{loc}}(\mathbb{R})) \]

where \( \tilde{u} \) is the solution of (179) with initial value \( \tilde{u}(0) = \tilde{u}_0. \)

In particular, let \( \tilde{\lambda}(s), \tilde{x}(s) \) and \( \tilde{\varepsilon}(s) \) be such that

\[ \tilde{\varepsilon}(x, x) = \tilde{\lambda}^{1/2}(x - Q)(\tilde{\lambda}(x)(x + \tilde{x}(s))) \]

satisfies

\[ (185) \quad \forall s \in \mathbb{R}, \quad (\xi(s), Q^1) = (\tilde{\varepsilon}(s), Q_\lambda) = 0. \]

Then, by (183) and the relations (68), (69) for \( \tilde{\lambda}, \tilde{x} \) (see Part A), we have:

\[ (186) \quad \forall s \in \mathbb{R}, \quad \lambda(s_n + s) \to \tilde{\lambda}(s), \quad x(s_n + s) \to \tilde{x}(s), \quad \text{in } C([-s_1, s_1], \mathbb{R}), \text{ as } n \to +\infty. \]

Therefore, \( \forall s \in \mathbb{R}, \)

\[ \varepsilon(s_n + s, \cdot) \to \tilde{\lambda}^{1/2}(s_n + s)(u - Q)(s_n + \lambda(s_n + s)(\cdot + x(s_n + s))) \]

\[ \to \tilde{\lambda}^{1/2}(x - Q)(\tilde{\lambda}(s)(\cdot + \tilde{x}(s))) = \tilde{\varepsilon}(s, \cdot) \]

in \( H^1 \) as \( n \to \infty. \) Thus Lemma 17 is proved.

**Proof of Lemma 30.** – The key of the proof is the fact that the Cauchy problem for equation (1) is well posed in \( L^2 \) and a local viriel identity of the type (131) (Part C, Lemma 16). Let \( M \) be such that

\[ \forall t \geq 0, \quad |u(t)|_{H^1} \leq M. \]

Note that it suffices to prove (180) on an interval \([-t_0, t_0]\), with \( t_0 = t_0(M) > 0 \), then Lemma 30 is obtained by iteration in time. We will use the norms and the estimates introduced in [11] to solve the local Cauchy problem in \( H^1(\mathbb{R}) \) for equation (179).

Since \( t_n \to +\infty \), we may assume that \( \forall n \in \mathbb{N}, t_n \geq 1. \) For \( t \in [-1, 1], \) we set:

\[ \forall s \in \mathbb{R}, \quad x_n = x(t_n), \quad u_n(t, x) = u(t+n, x_n + x). \]

The proof follows the same steps as the proof of Proposition 2.2 in [5].
**Step 1.** Decomposition of $u_n(t)$ in compact and noncompact parts.

Since

$$
\int u_n^2(0) \leq M^2, \quad u_n(0, \cdot) = u(t_n, x(t_n) + \cdot) \to \tilde{u}_0 \quad \text{in} \ L^2_{\text{loc}}(\mathbb{R}),
$$

we can write

$$
u_n(0) = u_{1,n}(0) + u_{2,n}(0),
$$

where

$$
u_{1,n}(0) \to \tilde{u}_0 \quad \text{in} \ L^2 \quad \text{as} \ n \to +\infty
$$

and

$$
u_{2,n}(0, x) = 0, \quad \text{if} \ |x| \leq 2\rho_n,
$$

with $\rho_n \to +\infty$ as $n \to +\infty$.

Next, we set $z_n(0) = u_{1,n}(0) - \tilde{u}_0$; we have $u_n(0) = \tilde{u}_0 + z_n(0) + u_{2,n}(0)$, with

$$
\int z_n^2(0) \leq \frac{c}{n},
$$

$$
|\tilde{u}_0|_{H^1}, \ |u_{1,n}(0)|_{H^1}, \ |z_n(0)|_{H^1}, \ |u_{2,n}(0)|_{H^1} \leq K_0.
$$

We then consider the solutions $\tilde{u}(t), z_n(t), u_{2,n}(t)$ of (179), with respective initial values $\tilde{u}_0$, $z_n(0)$, $u_{2,n}(0)$.

Finally, we define the interaction term $R_n$ by

$$
R_n(t) = u_n(t) - (\tilde{u}(t) + z_n(t) + u_{2,n}(t)).
$$

In Step 2, we estimate for $t$ small, $z_n(t), u_{2,n}(t)$ and $\tilde{u}(t)$. Then, in Step 3, we consider $R_n$, estimating the interaction between $u_{1,n}$ and $u_{2,n}$, and we conclude the proof.

**Step 2.** Stability in time of the properties of $z_n(t), u_{2,n}(t)$ and $\tilde{u}(t)$.

Recall that in Appendix B, we have defined the following norms. For $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $T > 0$,

$$
\eta_1^T(\xi) = \sup_{t \in (-T,T)} |\xi(t)|_{L^2}, \quad \eta_2^T(\xi) = |\xi|_{L^5 L^{5/2}}, \quad \eta_3^T(\xi) = |\xi|_{L^8 L^{8/3}}.
$$

$$
\Omega^T(\xi) = \max_{j = 1,2,3} \eta_j^T(\xi).
$$

The norm $\Omega^T$ was introduced in [11] to solve the local Cauchy problem (179) in $L^2$. When solving the local Cauchy problem in $H^1$, we need to consider

$$
\tilde{\Omega}^T(\xi) = \max \{ \eta_j^T(\xi_1), \eta_j^T(D_t^{1/3} \xi), \eta_j^T(\xi) : j = 2,3 \}.
$$

By the local well posedness result of [11] (see Corollary 2.11), there exists $t_1 > 0$, and $K_1 > 0$ such that if $z(t)$, solution of (179) satisfies $|z(0)|_{H^1} \leq K_0$, then

$$
\sup_{t \in [-t_1,t_1]} |z(t)|_{H^1} + \tilde{\Omega}^T(z) \leq K_1.
$$
In particular
\[
\sup_{t \in [-t_1, t_1]} \left( |\vec{u}(t)|_{H^1}, |u_n(t)|_{H^1}, |z_n(t)|_{H^1}, |u_{2,n}(t)|_{H^1} \right),
\]
(187)
\[\vec{\Omega}^1(\vec{u}), \vec{\Omega}^1(u_n), \vec{\Omega}^1(z_n), \vec{\Omega}^1(u_{2,n}) \leq K_1.\]

Now, we show the following lemma:

L E M M A 31. – There exists \( n_0 \in \mathbb{N} \) such that, \( \forall n \geq n_0, \)

(i)
\[(\Omega^1(z_n))^2 \leq C \left( \int |z_n|^2(0) \right) \leq \frac{C}{n},\]

(ii)
\[\forall t \in [-t_1, t_1], \quad \int_{|x| < \rho_n} (u_{2,n}(t,x))^2 \, dx \leq \frac{C}{\rho_n}.\]

(iii)
\[\int_{|x| < \rho_n} \left( \int_{-t_1}^{t_1} (u_{2,n}(t,x))^{10} \, dt \right)^{1/2} \, dx \leq \frac{C}{\rho_n^{1/4}}.\]

(iv)
\[\sup_{|x| \leq \rho_n^{1/4}} \left( \int_{-t_1}^{t_1} ((u_{2,n})_x(t,x))^2 \, dt \right)^{1/2} \leq \frac{C}{\rho_n^{1/12}}.\]

P r o o f of L e m m a 31. – (i) From the proof of Theorem 2.8 in [11] (well-posedness of the global Cauchy problem for (179) in \( L^2 \)), we have, \( \Omega^1(z_n) \leq C |z_n(0)|_{L^2} \), and then the result follows from (186).

(ii) We use a Viriel type identity for (179) as in [5] for the Schrödinger equation. Consider \( \gamma : [0, +\infty) \rightarrow [0, 1] \) a smooth function satisfying
\[\gamma(r) = 1, \quad \text{for} \ 0 \leq r \leq 1, \quad \gamma(r) = 0, \quad \text{for} \ r \geq 2.\]

By formula (129), we have, for \( t \in [-t_1, t_1], \)
\[
\frac{d}{dt} \int \gamma \left( \frac{|x|}{\rho_n} \right) (u_{2,n}(t,x))^2 \, dx
\]
\[= -\frac{3}{\rho_n} \int \gamma' \left( \frac{|x|}{\rho_n} \right) (u_{2,n}(t,x))^2 \, dx
\]
\[+ \frac{1}{\rho_n} \int \gamma'' \left( \frac{|x|}{\rho_n} \right) (u_{2,n}(t,x))^2 \, dx + \frac{5}{6\rho_n} \int \gamma' \left( \frac{|x|}{\rho_n} \right) (u_{2,n}(t,x))^6 \, dx.\]

For \( n \) large enough so that \( \rho_n \geq 1 \), and using the Gagliardo–Nirenberg inequality, we obtain, \( \forall t \in [-t_1, t_1]. \)
\[
\left| \frac{d}{dt} \int \gamma \left( \frac{|x|}{\rho_n} \right) \left( u_{2,n}(t,x) \right)^2 dx \right| \\
\leq \frac{C}{\rho_n} \left( |\gamma'|_{L^\infty} \left( u_{2,n} \right)^2_{L^2} + |\gamma'''|_{L^\infty} \left( u_{2,n} \right)^2_{L^2} + |\gamma''|_{L^\infty} \left( u_{2,n} \right)^2_{L^2} \right) \leq \frac{C}{\rho_n}.
\]

Since \( u_{2,n}(0,x) = 0 \) for \( x \leq 2\rho_n \), we have \( \int \gamma \left( \frac{|x|}{\rho_n} \right) \left( u_{2,n}(0,x) \right)^2 dx = 0 \), and thus, \( \forall t \in [-t_1, t_1] \),
\[
\int_{|x|<\rho_n} \left( u_{2,n}(t,x) \right)^2 dx \leq \int \gamma \left( \frac{|x|}{\rho_n} \right) \left( u_{2,n}(t,x) \right)^2 dx \leq \frac{C}{\rho_n}.
\]

(iii) By Cauchy–Schwartz inequality, we have:
\[
\int_{|x|<\sqrt{\rho_n}} \left( \int_{-t_1}^{t_1} \left( u_{2,n} \right)^{10} \right)^{1/2} dx \leq \rho_n^{1/4} \left( \int_{|x|<\rho_n} \int_{-t_1}^{t_1} \left( u_{2,n} \right)^{10} dx \right)^{1/2}
\]
\[
\leq C \rho_n^{1/4} K_4 \sup_{t \in [-t_1, t_1]} \left( \int_{|x|<\rho_n} \left( u_{2,n}(t) \right)^2 \right)^{1/2} \leq \frac{C}{\rho_n^{1/4}}.
\]

Thus, (iii) is proved.

(iv) Let \( F(x) = \int_{-t_1}^{t_1} \left( u_{2,n}(t,x) \right)^2 dx \). First, from Sobolev inequality (Gagliardo–Nirenberg inequality and cut-off), we have:
\[
\int_{|x|<\rho_n^{1/4}/2} F^2(x) dx \leq \int_{|x|<\rho_n^{1/4}/2} \left( \int_{-t_1}^{t_1} (u_{2,n})_x(x) \right)^2 \leq C \rho_n^{-3/4} \int_{|x|<\rho_n^{1/4}/2} \left( (u_{2,n})_{xx} \right)^2 + C \rho_n^{3/4} \int_{|x|<\rho_n^{1/4}} \left( u_{2,n} \right)^2
\]
\[
\leq C \rho_n^{-1/2} \sup_{x \in \mathbb{R}} \int_{-t_1}^{t_1} \left( (u_{2,n})_{xx}(x) \right)^2 + C \rho_n^{3/4} \sup_{t \in (-t_1, t_1)} \int_{|x|<\rho_n^{1/4}} \left( u_{2,n} \right)^2.
\]

By (187) and (ii), we obtain
\[
\int_{|x|<\rho_n^{1/4}/2} F^2(x) dx \leq \frac{C}{\rho_n^{1/4}}.
\]

Next, by Cauchy–Schwartz inequality:
\[
|F'(x)| = \frac{1}{F(x)} \left| \int_{-t_1}^{t_1} (u_{2,n})_x(u_{2,n})_{xx} \right| \leq \left( \sup_{x \in \mathbb{R}} \int_{-t_1}^{t_1} \left( (u_{2,n})_{xx} \right)^2 \right)^{1/2}.
\]

Therefore, by (187), we have \( \sup_{t \in \mathbb{R}} |F'(x)| \leq C \).
Now, from Sobolev inequality,

$$\sup_{|x| \leq \rho_n^{1/4}/2} |F(x)| \leq C \left( \sup_{|x| \leq \rho_n^{1/4}/2} |F'(x)| \right)^{1/3} \left( \int_{|x| \leq \rho_n^{1/4}/2} F^2 \right)^{1/3} \leq \frac{C}{\rho_n^{1/12}}.$$ 

Therefore, (iv) is proved.

**Step 3.** Estimate of the nonlinear interaction term \( R_n(t) \).

We now claim that there exists \( t_2 \in (0, t_1) \) such that for all \( \delta > 0 \), there exists an integer \( n_0(\delta) \) such that

\[
\forall n > n_0(\delta), \quad \Omega^{1/2}(R_n) \leq \delta.
\]

This concludes the proof of Lemma 30. Indeed, claim (188) means

\[
\lim_{n \to \infty} \sup_{t \in [-t_2, t_2]} \| R_n(t) \|_{L^2} = 0,
\]

and then \( \forall t \in [-t_2, t_2] \), \( R_n(t) \to 0 \) in \( H^1 \) as \( n \to \infty \).

By Lemma 31(i)--(iii), we have, for \( t \in [-t_2, t_2] \), \( z_n(t) \to 0 \), and \( u_{2,n}(t) \to 0 \) in \( H^1 \) as \( n \to \infty \), so that finally \( u_n(t) \to \tilde{u}(t) \) in \( H^1(\mathbb{R}) \). Similarly, \( u_n \to \tilde{u} \) in \( C([-t_2, t_2], L^2_{\text{loc}}(\mathbb{R})) \). By iteration in time, the proof of Lemma 30 is complete.

**Proof of claim (188).** – Let \( t_2 \in (0, t_1) \) to be chosen later. The function \( R_n \) satisfies the following equation:

\[
(R_n)_t + (R_n)_{xxx} + (u_n^5 - (\tilde{u}^5 + z_n^5 + u_{2,n}^5))_x = 0,
\]

\[
R_n(0) = 0.
\]

Since \( u_n = R_n + \tilde{u} + z_n + u_{2,n} \), we obtain

\[
u_n^5 - (\tilde{u}^5 + z_n^5 + u_{2,n}^5) = (R_n + (\tilde{u} + z_n + u_{2,n}))^5 - (\tilde{u}^5 + z_n^5 + u_{2,n}^5) = V_n R_n + F_n,
\]

where

\[
V_n = R_n^4 + 5 R_n^3 (\tilde{u} + z_n + u_{2,n}) + 10 R_n^2 (\tilde{u} + z_n + u_{2,n})^2 + 10 R_n (\tilde{u} + z_n + u_{2,n})^3 + 5 (\tilde{u} + z_n + u_{2,n})^4,
\]

and

\[
F_n = (\tilde{u} + z_n + u_{2,n})^5 - (\tilde{u}^5 + z_n^5 + u_{2,n}^5).
\]

Recall that \( S(t) \) denotes the linear Airy group (see Step 2 of the proof of Proposition 1). We have

\[
R_n(t) = A_n(t) + B_n(t),
\]

where

\[
A_n(t) = \int_0^t S(t-s)(V_n R_n)_x(s) \, ds, \quad B_n(t) = \int_0^t S(t-s)(F_n)_x(s) \, ds.
\]

**Estimate on \( A_n(t) \) for \( t \) small.** By using (163), we have:
\[ \Omega^{\delta_2}(A_n) \leq C \left( \eta^{\delta_2}_2(\tilde{u} + z_n + u_{2,n})^3 + \eta^{\delta_2}_3(\tilde{u} + z_n + u_{2,n}) \right) \left( \eta^{\delta_2}_2(\tilde{u} + z_n + u_{2,n})^3 + \eta^{\delta_2}_3(\tilde{u} + z_n + u_{2,n}) \right) \]

By (5.19) in [11], for some \(\alpha > 0\), we have
\[ \forall \nu, \quad \eta^{\delta_2}_2(\nu) \leq \nu^{\alpha} \Omega^{\delta_2}(\nu). \]

Therefore,
\[ \Omega^{\delta_2}(A_n) \leq C \ell_2^{\delta_2} \Omega^{\delta_2}(R_n) \]

By (187), this implies
\[ \Omega^{\delta_2}(A_n) \leq C \ell_2^{\delta_2} \Omega^{\delta_2}(R_n). \]

By choosing \(\ell_2 > 0\) such that \(C \ell_2^{\delta_2} < 1/2\), we obtain
\[ \Omega^{\delta_2}(R_n) \leq \Omega^{\delta_2}(A_n) + \Omega^{\delta_2}(B_n) \leq 1/2 \Omega^{\delta_2}(R_n) + \Omega^{\delta_2}(B_n), \]

and so
\[ \Omega^{\delta_2}(R_n) \leq 2 \Omega^{\delta_2}(B_n). \]

**Estimate on \(B_n\).** We claim that for all \(\delta > 0\), there exists \(n_0(\delta)\) such that
\[ \forall n \geq n_0(\delta), \quad \Omega^{\delta_2}(B_n) \leq \delta^{\frac{\alpha}{2}}. \]

By (3.7), (3.8) and (3.12) in [11], we have
\[ n^{\delta_2}_1(B_n) + n^{\delta_2}_2(B_n) + n^{\delta_2}_3(B_n) \leq C |F_n|_{L^1_t L^2_x} + C \left( |F_n|_{L^{s/4}_{x} L^{10/9}_t} \right). \]

Note that in the expression of \(F_n\), there are only crossed terms and Lemma 31 implies the result. Let us check, for example, the estimate for the terms
\[ \left| \tilde{u}^2 u_{2,n} |_{L^1_t L^2_x}, \quad |(u_{2,n}) x \tilde{u}^4 |_{L^{s/4}_{x} L^{10/9}_t}, \quad |(z_n) x \tilde{u}^4 |_{L^{s/4}_{x} L^{10/9}_t}. \]

Let \(\varepsilon_0 > 0\). Since \(|\tilde{u}|_{L^1_t L^{10/9}_x} < \infty\), there exists \(R_{\varepsilon_0} > 0\) such that
\[ \int_{|x| > R_{\varepsilon_0}} \left( \int_{-t_1}^{t_1} |\tilde{u}(t,x)|^2 dt \right)^{1/2} \leq \varepsilon_0. \]

Therefore, we have, for \(n\) large enough such that \(\sqrt{p_n} \geq R_{\varepsilon_0}, \)
\[ \left| \tilde{u}^3 u_{2,n} |_{L^1_t L^2_x} \right| \leq \int_{|x| > R_{\varepsilon_0}} \left( \int_{-t_2}^{t_2} \tilde{u}^6 u_{2,n}^4 dt \right)^{1/2} + \int_{|x| < \sqrt{p_n}} \left( \int_{-t_2}^{t_2} \tilde{u}^6 u_{2,n}^4 dt \right)^{1/2} \]
\begin{align*}
&\leq \int_{|x|> R_0} \left( \int_{-t_2}^{t_2} \tilde u_{10}^{10} \right)^{3/10} \left( \int_{-t_2}^{t_2} u_{2,n}^{10} \right)^{1/5} + \eta_5^3 (\tilde u) \left( \int_{|x|< \sqrt{\rho_n}} \left( \int_{-t_2}^{t_2} u_{2,n}^{10} \right)^{1/2} \right) 2^{5/10} \\
&\leq \left( \int_{|x|> R_0} \left( \int_{-t_2}^{t_2} \tilde u_{10}^{10} \right)^{3/10} \right)^{5/4} \left( \eta_2^{12} (u_{2,n}) \right)^2 + \frac{C}{\rho_n^{10/10}} \leq C \varepsilon_0^{3/5} + \frac{C}{\rho_n^{1/10}} \leq \delta',
\end{align*}

for \( n \) large.

If now \( n \) is large enough so that \( \rho_n^{1/4} \geq \frac{R_0}{4} \), we have:

\( |(u_{2,n})_x \tilde u^{41/4}_{L_{t/2}^{10/9}} \rho_n^{10/9} \)

\begin{align*}
&\leq \int_{|x|> R_0} \left( \int_{-t_2}^{t_2} \left( |(u_{2,n})_x|^{10/9} |\tilde u|^{40/9} \right) \right)^{9/8} \left( \int_{|x|< \rho_n^{1/4}/4} \left( |(u_{2,n})_x|^{10/9} |\tilde u|^{40/9} \right) \right)^{9/8} \\
&\leq \left( \int_{|x|> R_0} \left( \int_{-t_2}^{t_2} \tilde u^{10} \right)^{1/2} \right) \eta_5^2 (u_{2,n}) + \left( \eta_2^{12} (\tilde u) \right)^2 \sup_{|x| \leq \rho_n^{1/4}/4} \left( \int_{-t_2}^{t_2} (u_{2,n})_x (x) \right)^2 \\
&\leq C \varepsilon_0 + \frac{C}{\rho_n^{1/12}} \leq \delta'
\end{align*}

for \( n \) large.

Moreover,

\begin{equation}
(190) \quad \left( \begin{array}{c} |(z_n)_x \tilde u^{41/4}_{L_{t/2}^{10/9}} \rho_n^{10/9} \right| \left( z_n \right)_{x} \right)_{L_{t}^{1} L_{t/2}^{10}} \leq \left( \left| \tilde u^{41/4}_{L_{t}^{1} L_{t/2}^{10}} \right| \left( z_n \right)_{x} \right)_{L_{t}^{\infty} L_{t/2}^{10}} \leq \left( \eta_2^{12} (\tilde u) \right)^4 \Omega_4 (z_n) \leq \frac{C}{\sqrt{n}}.
\end{equation}

Finally, by (189), (190), we have

\[ \forall n \geq n_0 (\delta), \quad \Omega_4 (B_n) \leq \frac{\delta}{2}, \quad \text{and} \quad \Omega_4 (R_n) \leq \delta. \]

Thus, the proof of claim (188) is complete. \( \square \)

REFERENCES


