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The center problem for a family of systems of differential equations having a nilpotent singular point ☆

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Abstract

We study the analytic system of differential equations in the plane

$$(\dot{x}, \dot{y})^t = \sum_{i=0}^{\infty} \mathbf{F}_{q-p+2is},$$

where $p, q \in \mathbb{N}$, $p \leqslant q$, $s = (n+1)p - q > 0$, $n \in \mathbb{N}$, and $\mathbf{F}_i = (P_i, Q_i)^t$ are quasi-homogeneous vector fields of type $\mathbf{t} = (p, q)$ and degree i, with $\mathbf{F}_{q-p} = (y, 0)^t$ and $Q_{q-p+2s}(1, 0) < 0$. The origin of this system is a nilpotent and monodromic isolated singular point. We prove for this system the existence of a Lyapunov function and we solve theoretically the center problem for such system. Finally, as an application of the theoretical procedure, we characterize the centers of several subfamilies.

Keywords: Centers; Nilpotent and monodromic singular point; Lyapunov function

1. Introduction

The characterization of the local phase portrait at an isolated singular point of a system of autonomous planar ordinary differential equations is a problem almost completely solved, see Arnold and Il'yashenko [6]. The only case that remains open is the case when the critical point is monodromic (the orbits move around the singular point). If the system is analytic, a monodromic point is either a center (i.e. a critical point with a punctured neighborhood filled with periodic orbits) or a focus (i.e. a critical point with a neighborhood where all the orbits are spirals which arrive at the equilibrium point in forward or backward time), see Écalle [10] and Il'yashenko [16]. The problem of distinguishing when a monodromic critical point is either a center or a focus is called *center problem*.

If the matrix of the associated linearized dynamical system at the singular point has no eigenvalues on the imaginary axes (hyperbolic fixed point) the orbit structure of the system near a critical point is qualitatively the same as the local orbit structure given by the associated linearized system, see Hartman [14] and Grobman [13].

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If the critical point is non-hyperbolic, three cases are distinguished:

(a) when the eigenvalues of the quoted matrix are imaginary with real part null, the critical point may be a center or a focus. This problem is known as the *Poincaré–Lyapunov center problem*,
(b) when the matrix of the linear part at the origin is not identically null but it has its eigenvalues equal to zero, by Andreev [4] we know what is the behavior of the solutions in a neighborhood of the singular point, except if it is a center or a focus (*nilpotent center problem*), and
(c) when the matrix of the linear part at the origin is identically null, the local phase portrait of the system can be seen in Brunella and Miari [7], except when the point is monodromic, which has been characterized by Medvedeva [19]. Distinguishing whether the point is a center or a focus (*degenerate center problem*) is a problem that remains open.

The Poincaré–Lyapunov center problem was theoretically solved by Poincaré [21] and Lyapunov [17], and the nilpotent center problem by Moussu [20] and Sadovskii [22]. Nevertheless, in practice, in spite of the efforts in the last years, given an analytic system with a monodromic point, it is very difficult to know if it is a focus or a center, even in the case of polynomial systems of a given degree. To understand the profound nature of this problem see [10].

In this paper, we are interested in the study of some families of nilpotent centers. An analytic system of differential equations in the plane having a nilpotent singular point, in some suitable coordinates, can be written as

$$\dot{x} = y + P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions without constant nor linear terms defined in a certain neighborhood of the origin.

There are only a few families of polynomial differential systems (1) whose centers are known. The center problem for the system (1), where $P(x, y) = P_{2n+1}(x, y)$ and $Q(x, y) = Q_{2n+1}(x, y)$ are homogeneous polynomials of degree $2n + 1$, was solved for $n = 1$ by Andreev, for $n = 2$ by Sadovskii and for $n = 3$ by Andreev et al., see [5]. In that paper, the authors also find the multiplicity of the focus for $n = 1, 2$ and 3. Sadovskii [22] finds the centers of the family (1) for $P(x, y) = P_2(x, y)$ and $Q(x, y) = Q_3(x, y)$. Gasull and Torregrosa [11], using the so-called Cherka's method, which consists in doing a change of variables that transforms (1) into a Liénard differential equation, characterize the centers of the system

$$\dot{x} = y + a_1xy + a_2x^{q+1}, \qquad \dot{y} = -x^{2q-1} + b_1y^2 + b_2x^q y + b_3x^{2q},$$

and

$$\dot{x} = y - \left(a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5\right), \qquad \dot{y} = -\left(b_3x^3 + b_4x^4 + b_5x^5\right),$$

where a_i, b_i, $i = 1, \dots, 5$, unknowns and $q \in \mathbb{N}$.

The same authors in [12] deal with systems (1) with $P(x, y) = P_{n+1}(x, y) + P_{n+2}(x, y) + \cdots$ and $Q(x, y) = -x^{2n-1} + Q_{n+1}(x, y) + Q_{n+2}(x, y) + \cdots$ where, in this case, the vector fields (P_k, Q_k) are quasi-homogeneous vector fields of type $(1, n)$ and degree k. They study the center problem of this family by writing its associated differential equation in the generalized polar coordinates introduced by Lyapunov [17,18], and compute the so-called *generalized Lyapunov constants* that give the stability of a singular point degenerated. In particular the center case is characterized by the vanishing of all the constants.

In [2,3], Álvarez and Gasull calculate the first two generalized Lyapunov constants of (1) and they solve the stability problem of several polynomial families.

In the present paper, fixed $p, q, n \in \mathbb{N}$ with $p \leqslant q$, we consider the system of differential equations in the plane whose origin is a nilpotent singular point

$$\dot{x} = y + \sum_{i=1}^{\infty} P_{q-p+2is}(x, y), \qquad \dot{y} = \sum_{i=1}^{\infty} Q_{q-p+2is}(x, y), \tag{2}$$

where $s = (n + 1)p - q > 0$ and $\mathbf{F}_i = (P_i, Q_i)^t$ is a vector field quasi-homogeneous of type (p, q) and degree i with $Q_{(2n+1)p-q}(1, 0) < 0$. That is, according to the degree, $\mathbf{F}_{q-p} = (y, 0)^t$ is the quasi-homogeneous component of minor degree, the second one is $\mathbf{F}_{(2n+1)p-q}$ which, among others, has the term $(0, -x^{2n+1})^t$, and the edges of its

Newton's polygon of the remaining components are parallel to the edge associated to $\mathbf{F}_{(2n+1)p-q}$. This class includes, among others, the nilpotent systems which are invariant to the change of variables $(x, y) \to (-x, -y)$. In particular, it includes the family $\dot{x} = y + X_{2n+1}(x, y)$, $\dot{y} = Y_{2n+1}(x, y)$ where X_{2n+1} and Y_{2n+1} are homogeneous polynomials of degree $2n + 1$ with $Y_{2n+1}(1, 0) < 0$ (case $p = q = 1$, $P_{2n}(x, y) = X_{2n+1}(x, y)$, $Q_{2n}(x, y) = Y_{2n+1}(x, y)$, $P_{2i} = Q_{2i} = 0$, $i > n$ in (2)).

The results obtained in this paper are the following: in the second section, we prove that for system (2) there exists a Lyapunov function of class $\mathcal{C}^\infty$ which can be formally expanded in the form $W = \frac{1}{2}y^2 + \sum_{l=1}^{\infty} W_{2q+2sl}$ where W_{2q+2sl} is a quasi-homogeneous function of type (p, q) and degree $2q + 2sl$, $l \geqslant 1$. This result allows us to solve theoretically the center problem for the system (2). Finally, as application, we characterize the centers of the family

$$\dot{x} = y + a_1x^5 + a_2x^2y + a_3x^7 + a_4x^4y + a_5xy^2,$$
$$\dot{y} = -x^7 + b_1x^4y - a_2xy^2 + b_3x^6y + b_4x^3y^2 + b_5y^3,$$

the centers of the families $(\dot{x}, \dot{y})^t = \mathbf{F}_2 + \mathbf{F}_i$ with $\mathbf{F}_2 = (y, 0)^t$, $\mathbf{F}_i$ quasi-homogeneous vector fields of type $\mathbf{t} = (1, i - 1)$ and degree i, with $i = 4, 6, 8$, and of the family $(\dot{x}, \dot{y})^t = \mathbf{F}_2 + \mathbf{F}_{52}$, where

$$\mathbf{F}_2 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \qquad \mathbf{F}_{52} = \begin{pmatrix} a_1x^{15}y^2 + a_2x^{10}y^5 + a_3x^5y^8 + a_4y^{11} \\ -x^{19} + b_1x^{14}y^3 + b_2x^9y^6 + b_3x^4y^9 \end{pmatrix}$$

are quasi-homogeneous vector fields of type $\mathbf{t} = (3, 5)$ and degree 2 and 52, respectively.

We find subfamilies of nilpotent centers which are neither a hamiltonian one, nor a time-reversible system (i.e. in this case, the system is not invariant neither to the change $(x, y, t) \to (-x, y, -t)$ nor to $(x, y, t) \to (x, -y, -t)$).

2. Main results

Note that it is usual to consider quasi-homogeneous expansions in the analysis of degenerated singularities (see Bruno [8] and Brunella and Miari [7], for instance). Recall that a function f of two variables is quasi-homogeneous of type $\mathbf{t} = (p, q)$ and degree k if $f(\varepsilon^p x, \varepsilon^q y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)$ is said quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+p}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+q}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

In what follows, given a function P and the vector fields $\mathbf{F} = (F_1, F_2)^t$, $\mathbf{G} = (G_1, G_2)^t$, the Lie derivative of P by $\mathbf{F}$ is defined by $L_{\mathbf{F}}P = \frac{\partial P}{\partial x}F_1 + \frac{\partial P}{\partial y}F_2$ and the wedge product of two vector fields by $\mathbf{F} \wedge \mathbf{G} = F_1G_2 - F_2G_1$.

The following result holds:

Lemma 1. *The following property are satisfied*:

(i) *If* $P \in \mathcal{P}_i^{\mathbf{t}}$ *and* $\mathbf{F} \in \mathcal{Q}_j^{\mathbf{t}}$, *then* $L_{\mathbf{F}}P \in \mathcal{P}_{i+j}^{\mathbf{t}}$.
(ii) *If* $\mathbf{F} \in \mathcal{Q}_j^{\mathbf{t}}$ *and* $\mathbf{G} \in \mathcal{Q}_k^{\mathbf{t}}$, *then* $\mathbf{F} \wedge \mathbf{G} \in \mathcal{P}_{j+k+p+q}^{\mathbf{t}}$.

Proof. (i) By differentiating we have that

$$L_{\mathbf{F}}P\big(\varepsilon^p x, \varepsilon^q y\big) = \varepsilon^{i-p}\frac{\partial P}{\partial x}(x, y)F_1\big(\varepsilon^p x, \varepsilon^q y\big) + \varepsilon^{i-q}\frac{\partial P}{\partial y}(x, y)F_2\big(\varepsilon^p x, \varepsilon^q y\big) = \varepsilon^{i+j}L_{\mathbf{F}}P(x, y).$$

(ii) This property is easily obtained. □

In [1], we can find more properties of the quasi-homogeneous functions and the quasi-homogeneous vector fields.

We consider the analytic system of differential equations

$$(\dot{x}, \dot{y})^t = \sum_{i=0}^{\infty} \mathbf{F}_{q-p+2is}, \tag{3}$$

where $p, q \in \mathbb{N}$, $p < q$ and without common factors, $s = (n + 1)p - q > 0$, $n \in \mathbb{N}$ and $\mathbf{F}_i = (P_i, Q_i)^t$ are quasi-homogeneous vector fields of type $\mathbf{t} = (p, q)$ and degree i, with $\mathbf{F}_{q-p} = (y, 0)^t$ and $Q_{q-p+2s}(1, 0) < 0$ (without loss of generality, we can assume $Q_{q-p+2s}(1,0)=-1$). In this system, this last condition implies that the germ is monodromic at O, see Andreev [4].

Note that if p or q is even, then the origin is a center of (3). Indeed, we assume, for instance, p is even then q will be odd (since p and q have no common factors), in that case $P_{q-p+2is}(x,-y)=-P_{q-p+2is}(x,y)$ and $Q_{q-p+2is}(x,-y)=Q_{q-p+2is}(x,y)$ since $q+2is$ is odd and $2q-p+2is$ is even. The system (3) is time-reversible, i.e. has symmetrical phase portrait with regard to a straight line passing through the origin ($y=0$, in this case), changing time direction. So, O is a center, since it is monodromic.

In what follows, we assume that p and q are odd.

For all $k\geqslant 1$, we define the linear operator (see Lemma 1):

$$\begin{aligned}\ell_k:\mathcal{P}_k^{\mathbf{t}}&\to\mathcal{P}_{k+q-p}^{\mathbf{t}},\\ U_k&\to L_{\mathbf{F}_{q-p}}U_k.\end{aligned}$$

It is easy to prove that if k can be expressed as $k=k_3pq+k_2q+k_1p$ with $0\leqslant k_1<q$, $0\leqslant k_2<p$, then the set $B_k^{\mathbf{t}}=\{x^{qi+k_1}y^{p(k_3-i)+k_2},\ 0\leqslant i\leqslant k_3\}$ is a base of $\mathcal{P}_k^{\mathbf{t}}$. Otherwise, $\mathcal{P}_k^{\mathbf{t}}=\{0\}$.

We prove the following result, which we will use later on.

Lemma 2. *For $k\geqslant 1$, $k=k_3pq+k_2q+k_1p$ with $0\leqslant k_1<q$, $0\leqslant k_2<p$, $k_3\geqslant -1$, it holds*:

(i) *If $k_2=p-1$, a complementary subspace to the range* (*co-range*) *of ℓ_k is* $\operatorname{Cor}(\ell_k)=\operatorname{span}\{x^{q(k_3+1)+k_1-1}\}$. *Otherwise,* $\operatorname{Cor}(\ell_k)=\{0\}$.
(ii) *If $k_1=0$, the kernel of the linear operator ℓ_k is* $\operatorname{Ker}(\ell_k)=\operatorname{span}\{y^{pk_3+k_2}\}$. *Otherwise,* $\operatorname{Ker}(\ell_k)=\{0\}$.

Proof. Let $U_k=\sum_{i=0}^{k_3}\alpha_i^{(k)}x^{qi+k_1}y^{p(k_3-i)+k_2}\in\mathcal{P}_k^{\mathbf{t}}$.

We first assume that $k_1>0$. If $k_2<p-1$, for $k_3=-1$ it has $\mathcal{P}_{k+q-p}^{\mathbf{t}}=\mathcal{P}_k^{\mathbf{t}}=\{0\}$, and if $k_3>-1$ a basis of $\mathcal{P}_{k+q-p}^{\mathbf{t}}$ is

$$B_{k+q-p}^{\mathbf{t}}=\{x^{qi+k_1-1}y^{p(k_3-i)+k_2+1},\ 0\leqslant i\leqslant k_3\},$$

that is $\dim(\mathcal{P}_k^{\mathbf{t}})=\dim(\mathcal{P}_{k+q-p}^{\mathbf{t}})=k_3+1$. Moreover,

$$\ell_k(U_k)=\frac{\partial U_k}{\partial x}(x,y)y=\sum_{i=0}^{k_3}\alpha_i^{(k)}(qi+k_1)x^{qi+k_1-1}y^{p(k_3-i)+k_2+1}.$$

So, it deduces that $\operatorname{Ker}(\ell_k)=\{0\}$ and $\operatorname{Cor}(\ell_k)=\{0\}$.

If $k_2=p-1$, a basis of $\mathcal{P}_{k+q-p}^{\mathbf{t}}$ is

$$B_{k+q-p}^{\mathbf{t}}=\{x^{qi+k_1-1}y^{p(k_3+1-i)},\ 0\leqslant i\leqslant k_3+1\},$$

therefore $\dim(\mathcal{P}_k^{\mathbf{t}})=k_3+1$ and $\dim(\mathcal{P}_{k+q-p}^{\mathbf{t}})=k_3+2$. The operator ℓ_k has the form

$$\ell_k(U_k)=\frac{\partial U_k}{\partial x}(x,y)y=\sum_{i=0}^{k_3}\alpha_i^{(k)}(qi+k_1)x^{qi+k_1-1}y^{p(k_3+1-i)}.$$

Therefore, in such a case, it has that

$$\operatorname{Ker}(\ell_k)=\{0\},\qquad \operatorname{Cor}(\ell_k)=\operatorname{span}\{x^{q(k_3+1)+k_1-1}\}.$$

For $k_1=0$, we have $k_3>-1$ since otherwise $k+q-p<0$. Using an argument similar to that given above, it has that

$$\operatorname{Ker}(\ell_k)=\operatorname{span}\{y^{pk_3+k_2}\},\quad \operatorname{Cor}(\ell_k)=\{0\}\quad \text{if } k_2<p-1,$$

and

$$\operatorname{Ker}(\ell_k)=\operatorname{span}\{y^{p(k_3+1)-1}\},\quad \operatorname{Cor}(\ell_k)=\operatorname{span}\{x^{q(k_3+1)-1}\}\quad \text{if } k_2=p-1.\qquad\square$$

Now we prove our main result.

Theorem 2.1. *For system* (3) *with p and q odd, there exists W a $\mathcal{C}^\infty$-function in a neighborhood of the origin whose $2(q+Ns)$-quasi-homogeneous jet of type $\mathbf{t}=(p,q)$ at origin, $N\geqslant 0$, is*

$$\mathcal{J}^{2(q+Ns)}W=\sum_{l=0}^{N}W_{2(q+sl)}$$

where $W_{2(q+sl)}$ is a quasi-homogeneous polynomial of type (p,q) and degree $2(q+sl)$, $l\geqslant 0$, with $W_{2q}(x,y)=\frac{1}{2}y^2$ and $W_{2(q+s)}(1,0)=\frac{1}{2(q+s)}$ such that the derivative of W along the trajectories of the system (3) *has the form*

$$\dot{W}=x^{2m}\sum_{i=1}^{\infty}f_i x^{2is}+\tau(x,y)$$

where $m\in\mathbb{N}$ and f_i, $i\geqslant 1$, are polynomials in the coefficients of the right-hand sides of (3) *and τ is a flat function at the origin.*

Proof. We first consider a formal series

$$U=\sum_{l=0}^{\infty}U_{2(q+sl)}$$

with $U_{2q}(x,y)=\frac{1}{2}y^2$, where $U_{2(q+sl)}$ is a quasi-homogeneous function of type (p,q) and degree $2(q+sl)$, $l\geqslant 0$.

From Lemma 1, the quasi-homogeneous of type (p,q) expansion of the derivative of U along the trajectories of the system (3) is given by

$$\begin{aligned}\dot{U}&=L_{\mathbf{F}_{q-p}}U+\sum_{i=1}^{\infty}L_{\mathbf{F}_{q-p+2si}}U\\&=\sum_{l=1}^{\infty}L_{\mathbf{F}_{q-p}}U_{2(q+sl)}+\sum_{l=0}^{\infty}\sum_{i=1}^{\infty}L_{\mathbf{F}_{q-p+2si}}U_{2(q+sl)}\\&=\sum_{l=1}^{\infty}\left[L_{\mathbf{F}_{q-p}}U_{2(q+sl)}+\sum_{j=1}^{l}L_{\mathbf{F}_{q-p+2sj}}U_{2(q+sl-sj)}\right]\\&=\sum_{l=1}^{\infty}\left[\ell_{2(q+sl)}(U_{2(q+sl)})-A_{3q-p+2sl}\right]\end{aligned}$$

being $A_{3q-p+2sl}=-\sum_{j=1}^{l}L_{\mathbf{F}_{q-p+2sj}}U_{2(q+sl-sj)}\in\mathcal{P}^{\mathbf{t}}_{3q-p+2sl}$. By breaking down $A_{3q-p+2sl}=R_{3q-p+2sl}+C_{3q-p+2sl}$, where

$$R_{3q-p+2sl}\in\operatorname{Range}(\ell_{2(q+sl)})\quad\text{and}\quad C_{3q-p+2sl}\in\operatorname{Cor}(\ell_{2(q+sl)}),$$

and by choosing $U_{2(q+sl)}$ such that $\ell_{2(q+sl)}(U_{2(q+sl)})=R_{3q-p+2sl}$, for all $l\geqslant 1$, from Lemma 2, it is possible by means of a recursive procedure to obtain U such that the quasi-homogeneous terms of the derivative of U along the trajectories of (3) are of the form $x^{\tilde{m}}$ with $\tilde{m}p=3q-p+2sl$. As $3q-p+2sl=[2(n+1)l-1]p+(3-2l)q$ and p and q have no common factors, it must be $2l-3$ multiple of p, that is $2l=(2k-1)p+3$. So, $3q-p+2sl=[3n+2+(2k-1)s]p$. Concretely, if $k_0=\min\{k\in\mathbb{Z},\ 3n+2+(2k-1)s>0\}$ we have that

$$\dot{U}=\sum_{i=0}^{\infty}f_i x^{3n+2+[2(k_0+i)-1]s}=x^{3n+2+(2k_0-1)s}\sum_{i=0}^{\infty}f_i x^{2is}.$$

Moreover, as p and q are odd, it is easy to prove that $3n+2+(2k_0-1)s$ is even, since n is even (odd) if and only if s is even (odd).

Also, $A_{3q-p+2s}=-L_{\mathbf{F}_{q-p+2s}}U_{2q}=-yQ_{q-p+2s}(x,y)\in\operatorname{Range}(\ell_{2(q+s)})$, therefore, $f_0=0$.

We now see that $U_{2(q+s)}(1,0) = \frac{1}{2(n+1)}$. As $2(q+s) = 2(n+1)p$, the polynomial $U_{2(q+s)}$ has the form $U_{2(q+s)}(x,y) = \alpha_{2(n+1)}^{(2(q+s))} x^{2(n+1)} + \cdots$ and as $\ell_{2(q+s)}(U_{2(q+s)}) = A_{3q-p+2s}$, we have that

$$\frac{\partial U_{2(q+s)}}{\partial x}(x,y)y + yQ_{q-p+2s}(x,y) = 0,$$

thus, $\frac{\partial U_{2(q+s)}}{\partial x}(x,y) + Q_{q-p+2s}(x,y) = 0$. For $(x,y) = (1,0)$, get

$$2(n+1)\alpha_{2(n+1)}^{2(q+s)} = -Q_{q-p+2s}(1,0) = 1.$$

Thereby, $\alpha_{2(n+1)}^{2(q+s)} = \frac{1}{2(n+1)}$.

Lastly, from the Borel's Lemma, see Hartman [15], there exists W a $\mathcal{C}^\infty$-function in a neighborhood of the origin such that $\mathcal{J}^n W = \mathcal{J}^n U$, for all $n \geqslant 1$; thus, the result is proved. □

Note that, in general, from Lemma 2, the terms $U_{2(q+sl)}$, $l \geqslant 0$, are not unique and, as consequence, the constants $f_i, i \geqslant 1$, are not unique either. Nevertheless, by imposing that $U_{2(q+sl)}(1,0) = 0$, for all $l \geqslant 0$, it arrives at the uniqueness of the formal series U and of the constants f_i.

Throughout the following, the f_i are referred to as the *focus quantities* of the singular point O of the system (3). The above recursive procedure will allow us to compute the first quantities focus of a family given.

Lemma 3. *In the conditions of Theorem* 2.1*, the locus of points satisfying* $W(x,y) = C =$ constant *are closed curves for different values of* $C > 0$ *encircling* O *with* $W(O) = 0$ *and* $W(x,y) > 0$*,* $(x,y) \neq O$*, in a neighborhood of the origin.*

Proof. It is enough to prove that the origin of the hamiltonian system

$$\dot{x} = \frac{\partial W}{\partial y}(x,y), \qquad \dot{y} = -\frac{\partial W}{\partial x}(x,y) \tag{4}$$

is a center.

The quasi-homogeneous principal part of the system (4) is $(y, -x^{2n+1})^t$, see Brunella and Miari [7], which does not have curves that arrive at O with defined direction. From Andreev [4], by using a quasi-homogeneous blow up, the system (4) is monodromic and as it is hamiltonian, it follows that O is a center. As a consequence, the curves $W(x,y) = C > 0$, are closed and fill a punctured neighborhood of the origin. □

Theorem 2.2. *In the conditions of Theorem* 2.1*, the origin is a center of* (3) *if and only if* $f_i = 0$*, for all* $i \geqslant 1$.

Proof. If there exists $M > 0$ such that $f_i = 0$, $1 \leqslant i \leqslant M-1$, and $f_M \neq 0$, the $\mathcal{C}^\infty$-function W verifies $\dot{W} = f_M x^{2(m+M)s} + O(x^{2(m+M)s})$. So, there exists a neighborhood of the origin where $\dot{W}$ does not change its sign, and from Lemma 3 the curves $W(x,y) =$ constant are closed; therefore W is a Lyapunov function, thus, O is a focus. Concretely, if $f_M < 0$, O is asymptotically stable, otherwise O is asymptotically unstable.

On the other hand, if $f_i = 0$, for all $i \geqslant 1$, then O is a focus of infinite order. In the case of nilpotent monodromic fields, there exists a Poincaré map which is analytic, see Lyapunov [18]. Concretely, we can choose a section transversal to the field and a parametrization of this one, such that the Poincaré map is analytic. Therefore, a focus of infinity order is a center. Consequently, if $f_i = 0$, for all $i \geqslant 1$, it follows that O is a center. □

3. Nilpotent centers of several families of polynomial system

In this section, we have computed the first focus quantities of some subfamilies of the system (3) by means of the recursive procedure developed in Theorem 2.1. These have the form

$$f_1 = \alpha_1 g_1, \quad f_i = \alpha_i g_i + \sum_{j=1}^{i-1} \beta_{i,j} f_j, \quad i \geqslant 2,$$

with α_i positive constants and $\beta_{i,j}$ polynomials in the coefficients of the right-hand sides of (3).

Furthermore, in order to obtain a simpler expression of the focus quantities of (3), we emphasize the following decomposition that we will use later.

Lemma 4. *For each* $\mathbf{t}=(p,q)$*, given* $\mathbf{F}_k=(P_k,Q_k)^t\in\mathcal{Q}_k^{\mathbf{t}}$*, there exist* $h_k\in\mathcal{P}_{k+p+q}^{\mathbf{t}}$ *and* $\mu_k\in\mathcal{P}_k^{\mathbf{t}}$ *such that*

$$\mathbf{F}_k=\frac{1}{k+p+q}(\mathbf{X}_{h_k}+\mu_k\mathbf{D_t}), \tag{5}$$

with $h_k=\mathbf{F}_k\wedge\mathbf{D_t}$ *and* $\mu_k=\operatorname{div}(\mathbf{F}_k)$*, where it denotes* $\mathbf{D_t}=(px,qy)^t$ *and* $\mathbf{X}_{h_k}=(\frac{\partial h_k}{\partial y}(x,y),-\frac{\partial h_k}{\partial x}(x,y))^t$.

Furthermore, a such decomposition is unique.

Proof. It is straightforward to show that

$$\frac{\partial h_k}{\partial y}(x,y)+px\operatorname{div}(\mathbf{F}_k)=\left(px\frac{\partial P_k}{\partial x}(x,y)+qy\frac{\partial P_k}{\partial y}(x,y)\right)+qP_k,$$
$$-\frac{\partial h_k}{\partial x}(x,y)+qy\operatorname{div}(\mathbf{F}_k)=\left(px\frac{\partial Q_k}{\partial x}(x,y)+qy\frac{\partial Q_k}{\partial y}(x,y)\right)+pQ_k.$$

As $P_k\in\mathcal{P}_{k+p}^{\mathbf{t}}$ and $Q_k\in\mathcal{P}_{k+q}^{\mathbf{t}}$, from Euler's Theorem for quasi-homogeneous polynomial it follows the first part.

We see the second part. For any $\tilde{h}_k\in\mathcal{P}_{k+p+q}^{\mathbf{t}}$, $\tilde{\mu}_k\in\mathcal{P}_k^{\mathbf{t}}$ it holds

$$\operatorname{div}(\mathbf{X}_{\tilde{h}_k})=0,$$
$$\operatorname{div}(\tilde{\mu}_k\mathbf{D_t})=L_{\mathbf{D_t}}\tilde{\mu}_k+(p+q)\tilde{\mu}_k=(k+p+q)\tilde{\mu}_k.$$

Therefore, if $\tilde{h}_k$, $\tilde{\mu}_k$ verify (5), it has that

$$\operatorname{div}(\mathbf{F}_k)=\frac{1}{k+p+q}\big(\operatorname{div}(\mathbf{X}_{\tilde{h}_k})+\operatorname{div}(\tilde{\mu}_k\mathbf{D_t})\big)=\tilde{\mu}_k,$$
$$\mathbf{F}_k\wedge\mathbf{D_t}=\frac{1}{k+p+q}\mathbf{X}_{\tilde{h}_k}\wedge\mathbf{D_t}=\frac{1}{k+p+q}L_{\mathbf{D_t}}\tilde{h}_k=\tilde{h}_k. \qquad \square$$

We show several applications of our research. Firstly, we study the problem of center of the family

$$\dot{x}=y+a_1x^5+a_2x^2y+a_3x^7+a_4x^4y+a_5xy^2,$$
$$\dot{y}=-x^7+b_1x^4y-a_2xy^2+b_3x^6y+b_4x^3y^2+b_5y^3. \tag{6}$$

This system is a subfamily of (3) given by

$$(\dot{x},\dot{y})^t=\mathbf{F}_2+\mathbf{F}_4+\mathbf{F}_6,$$

with $\mathbf{F}_i\in\mathcal{Q}_i^{\mathbf{t}}$, $i=2,4,6$, $\mathbf{t}=(1,3)$, and

$$\mathbf{F}_2=\begin{pmatrix}y\\0\end{pmatrix},\qquad \mathbf{F}_4=\begin{pmatrix}a_1x^5+a_2x^2y\\-x^7+b_1x^4y+b_2xy^2\end{pmatrix},$$
$$\mathbf{F}_6=\begin{pmatrix}a_3x^7+a_4x^4y+a_5xy^2\\b_6x^9+b_3x^6y+b_4x^3y^2+b_5y^3\end{pmatrix}$$

with $b_6=0$ and $b_2=-a_2$.

The following result characterizes the centers of the system (6).

Theorem 3.1. *The origin of the system* (6) *is a center if and only if one of the following three series is satisfied*:

(i) $5a_1+b_1=7a_3+b_3=2a_4+b_4=a_5+3b_5=0$ (*hamiltonian system*).
(ii) $a_1=a_3=a_5=b_1=b_3=b_5=0$ (*time-reversible system*).
(iii) $a_2=4a_1^2$, $b_1=-a_1$, $b_5=a_1b_4$, $a_5=a_1(4a_4-b_4)$.

Moreover, each one of them has a local analytic first integral.

Proof. Taking $\mathbf{t} = (1, 4)$ (that is, the type of the vector field $(y, -x^7)^t$, i.e. the quasi-homogeneous principal part of the system (6)) and applying Lemma 4 degree to degree, the system (6) comes given by $(\dot{x}, \dot{y})^t = \mathbf{X}_h + \mu \mathbf{D_t}$, being h the defined positive function

$$h(x, y) = \frac{1}{8}\left(x^8 + 4y^2\right) - \frac{1}{9}c_1 x^5 y - \frac{1}{10}c_2 x^2 y^2 - \frac{1}{11}c_3 x^7 y - \frac{1}{12}c_4 x^4 y^2 - \frac{1}{13}c_5 x y^3,$$

and

$$\mu(x, y) = \frac{1}{9}d_1 x^4 + \frac{1}{11}d_3 x^6 + \frac{1}{12}d_4 x^3 y + \frac{1}{13}d_5 y^2,$$

where

$$\begin{aligned}
&c_1 = b_1 - 4a_1, \quad d_1 = 5a_1 + b_1, \quad c_2 = -5a_2,\\
&c_3 = b_3 - 4a_3, \quad d_3 = 7a_3 + b_3, \quad c_4 = b_4 - 4a_4, \quad d_4 = 4a_4 + 2b_4,\\
&c_5 = b_5 - 4a_5 \quad d_5 = a_5 + 3b_5.
\end{aligned}$$

The first nine constants g_i, $i = 1, \ldots, 9$, have the form

$$\begin{aligned}
&g_1 = d_1, \quad g_2 = d_3, \quad g_3 = d_5 + \frac{12}{13}c_1 d_4,\\
&g_4 = d_4\left[c_3 + 2c_1\left(c_2 + 2c_1^2\right)\right],\\
&g_5 = d_4\left[c_5 + 4c_1\left(c_4 + \frac{1}{13}d_4\right) - \frac{100}{3}c_1^3\left(c_2 + 2c_1^2\right)\right],\\
&g_6 = -d_4 c_1\left(c_2 + 2c_1^2\right)\left[c_4 + \frac{1}{2}d_4 - \frac{62}{3}c_1^2\left(c_2 + 2c_1^2\right)\right],\\
&g_7 = d_4 c_1\left(c_2 + 2c_1^2\right)\left[d_4 - \frac{24}{5}c_2^2 - \frac{748}{15}c_2 c_1^2 - \frac{1408}{15}c_1^4\right],\\
&g_8 = -d_4 c_1\left(c_2 + 2c_1^2\right)\left(774c_2^2 - 4681c_2 c_1^2 + 6641c_1^4\right),\\
&g_9 = -d_4 c_1\left(c_2 + 2c_1^2\right)\left(381374c_2 + 859813c_1^2\right).
\end{aligned}$$

First, we suppose that $d_4 = 0$. Imposing $g_1 = g_2 = g_3 = 0$, we have that $d_1 = d_3 = d_5 = 0$, i.e. $5a_1 + b_1 = 7a_3 + b_3 = 2a_4 + b_4 = a_5 + 3b_5 = 0$.

In this case, (6) is a hamiltonian system whose Hamilton's function is

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{8}x^8 + a_1 x^5 y + \frac{1}{2}a_2 x^2 y^2 + a_3 x^7 y + \frac{1}{2}a_4 x^4 y^2 - b_5 x y^3$$

and therefore, O is a center.

In particular, H is a local analytic first integral defined at the origin.

If we suppose that $c_1 = 0$ and $d_4 \neq 0$, from $g_i = 0$, $i = 1, \ldots, 5$, it successively has that $d_1 = d_3 = d_5 = c_3 = c_5 = 0$, i.e. $a_1 = a_5 = a_3 = b_1 = b_3 = b_5 = 0$. Thereby, the singular point O is a center, since the direction field of the system (6) is symmetric around $y = 0$, i.e. the system is invariant to the change $(x, y, t) \to (x, -y, -t)$. Therefore, the system has a local analytic first integral, see Chavarriga et al. [9].

Finally, we suppose that $c_1 d_4 \neq 0$. If g_i are zero, $i = 1, 2, 3$ then it arrives at $d_1 = d_3 = 0$, $d_5 = -\frac{12}{13}c_1 d_4$.

If $c_2 = -2c_1^2$, from $g_4 = 0$, we obtain $c_3 = 0$, and from $g_5 = 0$, it follows that $c_5 = -4c_1(c_4 + \frac{1}{13}d_4)$. Therefore, by substituting we have

$$a_2 = 4a_1^2, \quad b_1 = -a_1, \quad b_5 = a_1 b_4, \quad a_5 = a_1(4a_4 - b_4).$$

In this case, the system (6) has the form

$$\begin{aligned}
&\dot{x} = y + \frac{1}{4}\left(g(x, y) - b_4 y^2\right)\frac{\partial g}{\partial y}(x, y) + a_4 y g(x, y),\\
&\dot{y} = \frac{1}{4}\left(b_4 y^2 - g(x, y)\right)\frac{\partial g}{\partial x}(x, y),
\end{aligned} \tag{7}$$

where $g(x,y)=x^4+4a_1xy$. Making the change $u=g(x,y)$, $v=y^2$, $d\tau=y\frac{\partial g}{\partial x}(x,y)\,dt$, the system (6) is transformed into

$$\frac{du}{d\tau}=1+a_4u, \qquad \frac{dv}{d\tau}=\frac{1}{2}(b_4v-u). \tag{8}$$

The origin is a regular point of (8). As a consequence of flow box theorem, the system (8) has an analytic first integral $\phi(u,v)=cte$ at the origin. So, $\Phi(x,y)=\phi(g(x,y),y^2)=cte$ is an analytic first integral of (7) on $\mathcal{N}_0\setminus\mathcal{N}$ where $\mathcal{N}_0$ is a neighborhood of the origin and $\mathcal{N}$ the set of null measure $\{y=0\}\cup\{g_x=0\}$.

The function Φ can be prolonged analytically in a neighborhood of the O and its derivative on the trajectories of (7) is zero in a neighborhood of the origin, therefore (7) has an analytic first integral at O, it follows that O is a center.

Lastly, if $(c_2+2c_1^2)c_1d_4\neq 0$, g_8 and g_9 are not zero simultaneously. Therefore, the origin of system (6) is a focus. □

We now give necessary and sufficient conditions for what the origin of the families

$$(\dot{x},\dot{y})^t=\mathbf{F}_2+\mathbf{F}_i, \quad i=4,6,8,$$

with $\mathbf{F}_2=(y,0)^t\in\mathcal{Q}_2^{\mathbf{t}}$, $\mathbf{F}_i\in\mathcal{Q}_i^{\mathbf{t}}$, $i=4,6,8$, $\mathbf{t}=(1,i-1)$, be a center.

In the first two, we prove that O is a center if and only if the system is either hamiltonian or time-reversible. Nevertheless, in the third family there are centers which have got an analytic first integral and they are not hamiltonian nor time-reversible system.

Theorem 3.2. *The origin of the system*

$$\begin{aligned}\dot{x}&=y+a_1x^5+a_2x^2y,\\ \dot{y}&=-x^7+b_1x^4y+b_2xy^2\end{aligned} \tag{9}$$

is a center if and only if one of the following two series is satisfied:

(i) $b_1+5a_1=a_2+b_2=0$ (*hamiltonian system*).
(ii) $a_1=b_1=0$ (*time-reversible system*).

Moreover, each one of them has a local analytic first integral.

Proof. As $g_1=b_1+5a_1$ and $g_2=(b_1-4a_1)(a_2+b_2)$, from the vanishing of g_1 and g_2 the assertion follows.

On the other hand, if (i) holds, its hamiltonian is, in particular, an analytic first integral at O and if (ii) holds, the system is time-reversible under the change $(x,y,t)\to(x,-y,-t)$, thereby, the system has a local analytic first integral, see Chavarriga et al. [9]. □

Theorem 3.3. *The origin of the system*

$$\begin{aligned}\dot{x}&=y+a_1x^7+a_2x^4y+a_3xy^2,\\ \dot{y}&=-x^9+b_1x^6y+b_2x^3y^2+b_3y^3\end{aligned} \tag{10}$$

is a center if and only if one of the following two series is satisfied:

(i) $b_1+7a_1=2a_2+b_2=a_3+3b_3=0$ (*hamiltonian system*).
(ii) $a_1=b_1=a_3=b_3=0$ (*time-reversible system*).

Moreover, each one of them has a local analytic first integral.

Proof. Applying Lemma 4 for (10) with $\mathbf{t}=(1,5)$, degree to degree, the system (10) has the form $(\dot{x},\dot{y})^t=\mathbf{X}_h+\mu\mathbf{D}_{\mathbf{t}}$, being

$$h(x,y)=\frac{1}{10}\left(x^{10}+5y^2\right)-\frac{1}{12}(b_1-5a_1)x^7y-\frac{1}{14}(b_2-5a_2)x^4y^2-\frac{1}{16}(b_3-5a_3)xy^3,$$
$$\mu(x,y)=\frac{1}{12}(b_1+7a_1)x^6+\frac{1}{14}(4a_2+2b_2)x^3y+\frac{1}{16}(a_3+3b_3)y^2.$$

Therefore, it is convenient to replace a_1, a_2, a_3, b_1, b_2 and b_3 by c_1, c_2, c_3, d_1, d_2 and d_3 where

$$c_1=b_1-5a_1,\quad c_2=b_2-5a_2,\quad c_3=b_3-5a_3,$$
$$d_1=b_1+7a_1,\quad d_2=4a_2+2b_2,\quad d_3=a_3+3b_3.$$

It has that

$$g_1=d_1,\quad g_2=c_1d_2+12d_3,$$
$$g_3=d_2\left(70c_1^3+45c_1d_2+288c_1c_2+756c_3\right),\quad g_4=-c_1^5d_2.$$

From $g_i=0$, $i=1,\ldots,4$, follows the statement. □

Theorem 3.4. *The origin of the system*

$$\dot{x}=y+a_1x^9+a_2x^6y+a_3x^3y^2+a_4y^3,$$
$$\dot{y}=-x^{11}+b_1x^8y+b_2x^5y^2+b_3x^2y^3 \tag{11}$$

is a center if and only if one of the following three series is satisfied:

(i) $9a_1+b_1=b_2+3a_2=a_3+b_3=0$ (*hamiltonian system*).
(ii) $a_1=b_1=a_3=b_3=0$ (*time-reversible system*).
(iii) $b_1=-9a_1, a_3=-a_1(b_2-6a_2+54a_1^2), b_3=3a_1(b_2+18a_1^2)$.

Moreover, each one of them has a local analytic first integral.

Proof. Taking $\mathbf{t}=(1,6)$ and applying Lemma 4 for (11), this system has the form $(\dot{x},\dot{y})^t=\mathbf{X}_h+\mu\mathbf{D_t}$, being

$$h(x,y)=\frac{1}{12}\left(x^{12}+6y^2\right)-\frac{1}{15}c_1x^9y-\frac{1}{18}c_2x^6y^2-\frac{1}{21}c_3x^3y^3-\frac{1}{24}c_4y^4,$$
$$\mu(x,y)=\frac{1}{15}d_1x^8+\frac{1}{18}d_2x^5y+\frac{1}{21}d_3x^2y^2,$$

where the new coefficients that appear are

$$c_1=b_1-6a_1,\quad c_2=b_2-6a_2,\quad c_3=b_3-6a_3,\quad c_4=-6a_4,$$
$$d_1=b_1+9a_1,\quad d_2=6a_2+2b_2,\quad d_3=3a_3+3b_3,$$

the expressions of the first focus quantities are

$$g_1=d_1,\quad g_2=c_1d_2+5d_3,$$
$$g_3=d_2\left(42c_1^3+25c_1d_2+175c_1c_2+375c_3\right).$$

If $d_2=0$. For that $g_1=g_2=g_3=0$, it must be $d_1=d_2=d_3=0$, i.e. $9a_1+b_1=b_2+3a_2=a_3+b_3=0$. In this case, the system (11) is hamiltonian system and O is a center. In particular, its hamiltonian is a local analytic first integral defined at the origin.

We suppose that $c_1=0$, but $d_2\neq 0$. From $g_i=0$, $i=1,2,3$, it has that $d_1=c_1=d_3=c_3=0$, i.e. $a_1=b_1=a_3=b_3=0$. The singular point O is a center, since the system is time-reversible under the change $(x,y,t)\to(x,-y,-t)$. Also, by Chavarriga et al. [9] the system has a local analytic first integral.

Now, we suppose that $c_1d_2\neq 0$. Referring to the expressions given above for the first focus quantities, for a center we have $d_1=0$, $d_3=-\frac{1}{5}c_1d_2$ and $c_3=-\frac{1}{375}(42c_1^3+25c_1d_2+175c_1c_2)$.

This implies that $b_1=-9a_1, a_3=-a_1(b_2-6a_2+54a_1^2), b_3=3a_1(b_2+18a_1^2)$. Taking $\lambda_1=b_2+18a_1^2, \lambda_2=6a_1^2-a_2$, in this case, the system (11) has the form

$$\dot{x} = y + \frac{1}{6}g(x,y)\frac{\partial g}{\partial y}(x,y) + \frac{1}{3}(\lambda_1 - 3\lambda_2)g(x,y)y + (a_4 - \lambda_1\lambda_2)y^3,$$
$$\dot{y} = -\frac{1}{6}g(x,y)\frac{\partial g}{\partial x}(x,y),$$

where $g(x,y) = x^6 + 6a_1x^3y - \lambda_1 y^2$. The change $u = g(x,y)$, $v = y^2$, $d\tau = y\frac{\partial g}{\partial x}(x,y)\,dt$, reduces the system (11) to the form

$$\frac{du}{d\tau} = 1 + \frac{1}{3}(\lambda_1 - 3\lambda_2)u + (a_4 - \lambda_1\lambda_2)v, \qquad \frac{dv}{d\tau} = -\frac{1}{3}u.$$

This system has an analytic first integral in a neighborhood of the origin. By passing this integral to the variables x and y and reasoning in the same way that in Theorem 3.1 we obtain an analytic first integral of system (11) in a neighborhood of O. It follows that the singular point O is a center. □

Finally, we find the centers of the family

$$(\dot{x}, \dot{y})^t = \mathbf{F}_2 + \mathbf{F}_{6n-2}, \quad 1 \leqslant n \leqslant 9, \tag{12}$$

with $\mathbf{t} = (3,5)$ and where $\mathbf{F}_2 = (y,0)^t \in \mathcal{Q}_2^{\mathbf{t}}$ and $\mathbf{F}_{6n-2} \in \mathcal{Q}_{6n-2}^{\mathbf{t}}$.

For $1 \leqslant n \leqslant 8$, the vanishing of the first focus quantities leads us to hamiltonian or time-reversible systems.

For $n = 9$, we have the system

$$(\dot{x}, \dot{y})^t = \mathbf{F}_2 + \mathbf{F}_{52}, \tag{13}$$

with

$$\mathbf{F}_2 = \begin{pmatrix} y \\ 0 \end{pmatrix}, \qquad \mathbf{F}_{52} = \begin{pmatrix} a_1x^{15}y^2 + a_2x^{10}y^5 + a_3x^5y^8 + a_4y^{11} \\ -x^{19} + b_1x^{14}y^3 + b_2x^9y^6 + b_3x^4y^9 \end{pmatrix}.$$

In this case, we also find nontrivial centers (neither hamiltonian nor time-reversible system).

Theorem 3.5. *The origin of system* (13) *is a center if and only if one of the following three series is satisfied*:

(i) $5a_1 + b_1 = 3b_2 + 5a_2 = 5a_3 + 9b_3 = 0$ (*hamiltonian system*).
(ii) $a_1 = b_1 = a_3 = b_3 = 0$ (*time-reversible system*).
(iii) $10773a_3 + 2a_1(-150530a_2 + 17447b_2 + 26600a_1^2) = 0$, $b_1 + 5a_1 = 0$, *and* $96957b_3 - 10a_1(-138560a_2 + 24629b_2 + 26600a_1^2) = 0$.

Moreover, each one of them has a local analytic first integral.

Proof. Taking $\mathbf{t} = (1,9)$ and applying, degree to degree, Lemma 4 for (13), we have that

$$h(x,y) = \frac{1}{20}\left(x^{20} + 10y^2\right) - \frac{1}{45}c_1x^{15}y^3 - \frac{1}{70}c_2x^{10}y^6 - \frac{1}{95}c_3x^5y^9 - \frac{1}{10}c_4y^{12},$$
$$\mu(x,y) = \frac{1}{45}d_1x^{14}y^2 + \frac{1}{70}d_2x^9y^5 + \frac{1}{95}d_3x^4y^8,$$

where the new coefficients are

$$c_1 = b_1 - 5a_1, \quad c_2 = b_2 - 10a_2, \quad c_3 = b_3 - 10a_3, \quad c_4 = -10a_4,$$
$$d_1 = 15a_1 + 3b_1, \quad d_2 = 10a_2 + 6b_2, \quad d_3 = 5a_3 + 9b_3.$$

The expressions of the first focus quantities are

$$g_1 = d_1, \quad g_2 = c_1d_2 + 9d_3,$$
$$g_3 = d_2\left[5103c_3 + \left(266c_1^2 + 405d_2 + 15395c_2\right)c_1\right].$$

If $d_2 = 0$. So that $g_1 = g_2 = g_3 = 0$, it must be $d_1 = d_2 = d_3 = 0$, i.e. it holds (i). The system (13) is hamiltonian system whose Hamilton's function is $h(x,y)$ and O is a center.

We suppose that $c_1 = 0$ but $d_2 \neq 0$. From $g_i = 0$, $i = 1, 2, 3$, it has that $d_1 = c_1 = d_3 = c_3 = 0$, i.e. $a_1 = b_1 = a_3 = b_3 = 0$. Thereby, the singular point O is a center, since the system is time-reversible.

Now, we suppose that $c_1 d_2 \neq 0$. Referring to the expressions given above for the first focus quantities, for a center we have $d_1 = 0$, $d_3 = -\frac{1}{9} c_1 d_2$ and $c_3 = -\frac{1}{5103}(266 c_1^2 + 405 d_2 + 15395 c_2) c_1$.

Substituting, it has (iii). In this case, by means of similar reasoning that in Theorem 3.1, making the change $u = g(x, y)$, $v = y^2$, $d\tau = y \frac{\partial g}{\partial x}(x, y)\, dt$, with

$$g(x, y) = x^{10} + \frac{20}{9} a_1 x^5 y^3 - \left(b_2 + \frac{200}{81} a_1^2\right) y^6,$$

it arrives at that the singular point O is a center. □

The centers found that belong to (3) have a local analytic first integral (hamiltonian and time-reversible system, therein). We conjecture that all the centers of (3) have this property.

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