Strong Convergence of Averaged Approximants for Asymptotically Nonexpansive Mappings in Banach Spaces

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Let $C$ be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let $T$ be an asymptotically nonexpansive mapping from $C$ into itself such that the set $F(T)$ of fixed points of $T$ is nonempty. In this paper, we show that $F(T)$ is a sunny, nonexpansive retract of $C$. Using this result, we discuss the strong convergence of the sequence $(x_n)$ defined by $x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T_j x_n$ for $n = 0, 1, 2, \ldots$, where $x \in C$ and $\{\alpha_n\}$ is a real sequence in $(0, 1]$.

1. INTRODUCTION

Let $C$ be a subset of a Banach space. A mapping $T$ from $C$ into $E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. A mapping $T$ from $C$ into itself is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ such that $\lim_{n \to \infty} k_n = 1$ and $\|T^n x - T^ny\| \leq k_n \|x - y\|$ for each $x, y \in C$ and $n = 0, 1, 2, \ldots$.

Let $C$ be a closed, convex subset of a Banach space $E$. Let $T$ be a nonexpansive mapping from $C$ into itself such that the set $F(T)$ of fixed points of $T$ is nonempty, let $x$ be an element of $C$ and for each $t$ with $0 < t < 1$, let $x_t$ be the unique point of $C$ which satisfies $x_t = tx + (1 - t) Tx_t$. Browder

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[2] showed that \( \{x_n\} \) converges strongly to the element of \( F(T) \) which is nearest to \( x \) in \( F(T) \) as \( t \downarrow 0 \) in the case when \( E \) is a Hilbert space. Reich [8] extended Browder’s result to the case when \( E \) is a uniformly smooth Banach space and he showed that \( F(T) \) is a sunny, nonexpansive retract of \( C \), i.e., there exists a nonexpansive retraction \( P \) from \( C \) onto \( F(T) \) such that \( P(x + t(x - Px)) = Px \) for each \( x \in C \) and \( t \geq 0 \) with \( Px + t(x - Px) \in C \). Recently, using an idea of Browder [2], Shimizu and Takahashi [10] studied the convergence of another approximating sequence for an asymptotically nonexpansive mapping. Let \( T \) be an asymptotically nonexpansive mapping with Lipschitz constants \( \{k_n\} \) such that the set \( F(T) \) of fixed points of \( T \) is nonempty. Let \( 0 < a < 1 \), let \( b_n = 1/n \sum_{j=0}^{n} (|1 - k_j| + e^{-j}) \) and let \( a_n = \frac{b_n - 1}{b_n - 1 + a} \) for \( n = 1, 2, \ldots \). Let \( x \) be an element of \( C \) and let \( x_n \) be the unique point of \( C \) which satisfies \( x_n = a_n x + (1 - a_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \) for \( n = 1, 2, \ldots \). They showed that \( \{x_n\} \) converges strongly to the element of \( F(T) \) which is nearest to \( x \) in \( F(T) \) in the case when \( E \) is a Hilbert space.

In this paper, we extend Shimizu and Takahashi’s result to a Banach space. For an asymptotically nonexpansive mapping \( T \), we show that the set \( F(T) \) of fixed points of \( T \) is a sunny, nonexpansive retract of \( C \) and the sequence \( \{x_n\} \) defined above converges strongly to an element of \( F(T) \).

Our results are the following:

**Theorem 1.** Let \( C \) be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let \( T \) be an asymptotically nonexpansive mapping from \( C \) into itself such that the set \( F(T) \) of fixed points of \( T \) is nonempty. Then \( F(T) \) is a sunny, nonexpansive retract of \( C \).

**Theorem 2.** Let \( C \) be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let \( T \) be an asymptotically nonexpansive mapping from \( C \) into itself with Lipschitz constants \( \{k_n\} \) such that the set \( F(T) \) of fixed points of \( T \) is nonempty and let \( P \) be the sunny, nonexpansive retraction from \( C \) onto \( F(T) \). Let \( \{a_n\} \) be a real sequence such that

\[
0 < a_n \leq 1, \quad \lim_{n \to \infty} a_n = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n - 1}{a_n} < 1,
\]

where \( b_n = 1/(n+1) \sum_{j=0}^{n} k_j \) for \( n = 0, 1, \ldots \). Let \( x \) be an element of \( C \) and let \( x_n \) be the unique point of \( C \) which satisfies

\[
x_n = a_n x + (1 - a_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n \tag{1.1}
\]
for } n \geq N_0, \text{ where } N_0 \text{ is a sufficiently large natural number. Then } \{x_n\} \text{ converges strongly to } P_x.

Remark. The inequality } \lim_{n \to \infty} \frac{a_n}{b_n} < 1 \text{ implies that there exists a natural number } N_0 \text{ such that } (1-a_n)b_n < 1 \text{ for } n \geq N_0. \text{ So for } n \geq N_0, \text{ there exists the unique point } x_n \text{ of } C \text{ which satisfies (1.1), since the mapping } T_n \text{ from } C \text{ into itself defined by } T_n u = a_n v + (1-a_n) 1/(n+1) \sum_{j=0}^n T_j v \text{ satisfies } \|T_n u - T_n v\| \leq (1-a_n) b_n \|u-v\| \text{ for each } u, v \in C.

In the case when } T \text{ is nonexpansive, we have the following:

Theorem 3. Let } C \text{ be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let } T \text{ be a nonexpansive mapping from } C \text{ into itself such that the set } F(T) \text{ of fixed points of } T \text{ is nonempty and let } p \text{ be the sunny, nonexpansive retraction from } C \text{ onto } F(T). \text{ Let } \{a_n\} \text{ be a real sequence such that } 0 < a_n \leq 1 \text{ and } a_n \to 0. \text{ Let } x \text{ be an element of } C \text{ and let } x_n \text{ be the unique point of } C \text{ which satisfies (1.1) for } n = 0, 1, \ldots. \text{ Then } \{x_n\} \text{ converges strongly to } P_x.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, all vector spaces are real and we denote by } \mathbb{N}, \text{ the set of all nonnegative integers. For a real number } a, \text{ we also denote } \max\{a, 0\} \text{ by } (a)_+. \text{ We denote by } \mathcal{A}^n, \text{ the set } \{\lambda = (\lambda_0, \ldots, \lambda_n); \lambda_0 \geq 0, \sum_{j=0}^n \lambda_j = 1\} \text{ for } n \in \mathbb{N}. \text{ For a subset } C \text{ of a Banach space, we denote by } \text{co } C, \text{ the convex hull of } C.

Let } E \text{ be a Banach space and let } r > 0. \text{ We denote by } B_r, \text{ the closed ball in } E \text{ with center } 0 \text{ and radius } r. \text{ } E \text{ is said to be uniformly convex if for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } \|x+y\|/2 \leq 1 - \delta \text{ for each } x, y \in B_1 \text{ with } \|x-y\| \geq \epsilon. \text{ Let } C \text{ be a subset of } E \text{, let } T \text{ be a mapping from } C \text{ into } E \text{ and let } \varepsilon > 0. \text{ By } F(T) \text{ and } F_J(T), \text{ we mean the sets } \{x \in C : x = Tx\} \text{ and } \{x \in C : \|x-Tx\| \leq \epsilon\}, \text{ respectively. Let } k \geq 0. \text{ We denote by } \text{Lip}(C, k), \text{ the set of all mappings from } C \text{ into } E \text{ satisfying } \|Tx-Ty\| \leq k \|x-y\| \text{ for each } x, y \in C. \text{ We remark that } \text{Lip}(C, 1) \text{ is the set of all nonexpansive mappings from } C \text{ into } E. \text{ The following is a useful proposition due to Bruck [5]:}

Proposition 1. Let } C \text{ be a closed, convex subset of a uniformly convex Banach space. Then for each } R > 0, \text{ there exists a strictly increasing, convex, continuous function } \gamma: [0, \infty) \to [0, \infty) \text{ such that } \gamma(0) = 0 \text{ and}

\[
\gamma \left( \left\| \sum_{j=0}^n \lambda_j x_j - \sum_{j=0}^n \lambda_j T x_j \right\| \right) \leq \max_{0 \leq j < k \leq n} (\|x_j - x_k\| - \|T x_j - T x_k\|)
\]

for all } n \in \mathbb{N}, \lambda \in \mathcal{A}^n, x_0, \ldots, x_n \in C \cap B_R, \text{ and } T \in \text{Lip}(C, 1).
Let $\mu$ be a continuous, linear functional on $l^{\infty}$ and let $(a_0, a_1, \ldots) \in l^{\infty}$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. We call $\mu$ a Banach limit [1] when $\mu$ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu(a_{n+1}) = \mu(a_n)$ for each $(a_0, a_1, \ldots) \in l^{\infty}$.

For a Banach limit, we know that

$$\lim_{n \to \infty} a_n \leq \mu_n(a_n) \leq \lim_{n \to \infty} a_n \quad \text{for all} \quad (a_0, a_1, \ldots) \in l^{\infty}. \quad (2.1)$$

We also know the following from Lemma in [11] and its proof; see also [9, pp. 314-315]:

**Proposition 2.** Let $C$ be a closed, convex subset of a uniformly convex Banach space $E$. Let $\{x_n\}$ be a bounded sequence of $E$, let $\mu$ be a Banach limit and let $g$ be a real valued function on $C$ defined by

$$g(y) = \mu_n \|x_n - y\|^2 \quad \text{for each} \quad y \in C.$$ 

Then $g$ is continuous and convex, and $g$ satisfies $\lim_{\|y\| \to \infty} g(y) = \infty$. Moreover, for each $R > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$g\left(\frac{y + z}{2}\right) \leq g(y) + \frac{g(z)}{2} - \delta$$

for all $y, z \in C \cap B_R$ with $\|y - z\| \geq \varepsilon$.

Let $E^*$ be the topological dual of $E$. The value of $y \in E^*$ at $x \in E$ will be denoted by $\langle x, y \rangle$. We also denote by $J$, the duality mapping from $E$ into $2^{E^*}$, i.e.,

$$Jx = \{y \in E^* : \langle x, y \rangle = \|x\|^2 = \|y\|^2\} \quad \text{for each} \quad x \in E.$$ 

Let $U = \{x \in E : \|x\| = 1\}$. $E$ is said to be smooth if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.2) exists uniformly for $x \in U$. $E$ is said to be uniformly smooth if the limit (2.2) exists uniformly for $x, y \in U$. It is well known that if $E$ is smooth then the duality mapping is single-valued and norm to weak star continuous. In the case when the norm of $E$ is uniformly Gâteaux differentiable, we know the following from [12, Lemma 1]; see also [6, p. 586]:
Proposition 3. Let $C$ be a convex subset of a Banach space $E$ whose norm is uniformly Gâteaux differentiable. Let $\{x_n\}$ be a bounded subset of $E$, let $z$ be a point of $C$ and let $\mu$ be a Banach limit. Then

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n \langle y - z, J(x_n - z) \rangle \leq 0 \quad \text{for all } y \in C.$$

Let $C$ be a convex subset of $E$, let $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$, i.e., $Px = x$ for each $x \in K$. A retraction $P$ is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If the sunny retraction $P$ is also nonexpansive, then $K$ is said to be a sunny, nonexpansive retract of $C$. Concerning sunny, nonexpansive retractions, we know the following [3, 7]:

Proposition 4. Let $C$ be a convex subset of a smooth Banach space, let $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$. Then $P$ is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px) \rangle \leq 0 \quad \text{for all } x \in C \text{ and } y \in K.$$

Hence there is at most one sunny, nonexpansive retraction from $C$ onto $K$.

3. PROOF OF THEOREMS

To prove Lemmas 1, 2, 3 below, we use the methods employed in [4, 5].

Lemma 1. Let $C$ be a closed, convex subset of a uniformly convex Banach space. Then for each $R > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that

$$(\text{co}(F_e(T) \cap B_\varepsilon) + B_\eta) \cap C \subset F_e(T)$$

for all $T \in \text{Lip}(C, 1 + \eta)$.

Proof. Let $R > 0$. Then there exists a function $\gamma$ which satisfies the conditions in Proposition 1. Let $\varepsilon > 0$. Choose $\eta > 0$ such that $(3 + \eta) \eta + (1 + \eta) \gamma^{-1}(2(1 + R) \eta) \leq \varepsilon$. Let $T \in \text{Lip}(C, 1 + \eta)$. Pick $\lambda \in A^n$, $x_0, \ldots, x_n \in F_e(T) \cap B_\varepsilon$ and $y \in B_\eta$ such that \(\sum_{i=0}^n \lambda_i x_i + y \in C\). Since \(1/(1 + \eta)\) $T \in \text{Lip}(C, 1)$, we have

$$\sum_{i=0}^n \lambda_i x_i + y \in C.$$

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\[
\gamma \left( \frac{1}{1 + \eta} \right) \left\| T \left( \sum_{i=0}^{n} \lambda_i x_i \right) - \sum_{i=0}^{n} \lambda_i T x_i \right\|
\]
\[
\leq \max_{0 \leq i < j \leq n} \left( \|x_i - x_j\| - \frac{1}{1 + \eta} \|T x_i - T x_j\| \right)
\]
\[
\leq \max_{0 \leq i < j \leq n} \left( \|x_i - T x_j\| + \|x_j - T x_i\| + \frac{\eta}{1 + \eta} \|T x_i - T x_j\| \right)
\]
\[
\leq 2(1 + R) \eta.
\]

Hence we get
\[
\left\| \left( \sum_{i=0}^{n} \lambda_i x_i + y \right) - T \left( \sum_{i=0}^{n} \lambda_i x_i + y \right) \right\|
\]
\[
\leq \|y\| + \left\| \sum_{i=0}^{n} \lambda_i x_i - \sum_{i=0}^{n} \lambda_i T x_i \right\|
\]
\[
+ \left\| \sum_{i=0}^{n} \lambda_i T x_i - T \left( \sum_{i=0}^{n} \lambda_i x_i \right) \right\| + \left\| \left( \sum_{i=0}^{n} \lambda_i x_i \right) - T \left( \sum_{i=0}^{n} \lambda_i x_i + y \right) \right\|
\]
\[
\leq (3 + \eta) \eta + (1 + \eta) \gamma^{-1} (2(1 + R) \eta) \leq \varepsilon.
\]

**Lemma 2.** Let \( C \) be a closed, convex subset of a uniformly convex Banach space. Then for each \( p \in \mathbb{N} \), \( R > 0 \) and \( \varepsilon > 0 \), there exist \( \eta > 0 \) and \( N \in \mathbb{N} \) such that for each pair \( T \in \text{Lip}(C, 1 + \eta) \) and \( \{ x_{j,n} : n \in \mathbb{N}, j = 0, ..., p \} \subset C \cap B_R \) satisfying
\[
\frac{1}{n+1} \sum_{i=0}^{n} \|x_{j,n+1} - T x_{j,i}\| \leq \eta \quad \text{for all} \quad n \geq N \quad \text{and} \quad j = 0, ..., p,
\]
there holds
\[
\frac{1}{n+1} \sum_{i=0}^{n} \left\| \sum_{j=0}^{n} \lambda_j x_{j,n+1} - T \left( \sum_{j=0}^{n} \lambda_j x_{j,i} \right) \right\| \leq \varepsilon
\]
\[
\text{for all} \quad n \geq N \quad \text{and} \quad \lambda \in \Delta^p.
\]

**Proof.** Let \( R > 0 \). Then there exists a function \( \gamma \) which satisfies the conditions in Proposition 1. Let \( p \in \mathbb{N} \) and let \( \varepsilon > 0 \). Then there exist \( \eta > 0 \) and \( N \in \mathbb{N} \) satisfying
\[
\eta + (1 + \eta) \gamma^{-1} \left( \frac{p(p+1)}{2} \left( \frac{2R}{N+1} + 2(1 + R) \eta \right) \right) \leq \varepsilon.
\]
Pick $T \in \text{Lip}(C, 1 + \eta)$ and $\{x_{j,i} : i \in \mathbb{N}, j = 0, ..., p \} \subset C \cap B_{R}$ satisfying (3.1). Let $n \gg N$ and $\lambda \in A^p$. Since

$$- \frac{1}{1 + \eta} \|Tx_{j,i} - Tx_{k,i}\| \leq - \|x_{j,i+1} - x_{k,i+1}\| + \|x_{j,i+1} - Tx_{j,i}\|$$

we get

$$\gamma \left( \frac{1}{n + 1} \sum_{i=0}^{n} \frac{1}{1 + \eta} \left( \sum_{j=0}^{p} \lambda_j T x_{j,i} - T \left( \sum_{j=0}^{p} \lambda_j x_{j,i} \right) \right) \right) \leq \frac{1}{n + 1} \sum_{i=0}^{n} \gamma \left( \sum_{j=0}^{p} \lambda_j T x_{j,i} - T \left( \sum_{j=0}^{p} \lambda_j x_{j,i} \right) \right)$$

$$\leq \frac{1}{n + 1} \sum_{i=0}^{n} \max_{0 \leq j < k \leq p} \left( \|x_{j,i} - x_{k,i}\| - \frac{1}{1 + \eta} \|Tx_{j,i} - Tx_{k,i}\| \right)$$

$$\leq \frac{1}{n + 1} \sum_{i=0}^{n} \sum_{0 \leq j < k \leq p} \left( \|x_{j,i} - x_{k,i}\| - \frac{1}{1 + \eta} \|Tx_{j,i} - Tx_{k,i}\| \right)$$

$$\leq \sum_{0 \leq j < k \leq p} \frac{\|x_{j,0} - x_{k,0}\| - \|x_{j,n+1} - x_{k,n+1}\|}{n + 1} + 2(1 + R) \eta$$

So we obtain

$$\frac{1}{n + 1} \sum_{i=0}^{n} \left( \sum_{j=0}^{p} \lambda_j x_{j,i+1} - Tx_{j,i+1} \right) \leq \sum_{j=0}^{p} \lambda_j \left( \frac{1}{n + 1} \sum_{i=0}^{n} \|x_{j,i+1} - Tx_{j,i}\| \right)$$

$$+ \frac{1}{n + 1} \sum_{i=0}^{n} \left( \sum_{j=0}^{p} \lambda_j Tx_{j,i} - T \left( \sum_{j=0}^{p} \lambda_j x_{j,i} \right) \right)$$

$$\leq \eta + (1 + \eta) \gamma^{-1} \left( \frac{p(p + 1)}{2} \left( \frac{2R}{N + 1} + 2(1 + R) \eta \right) \right) \leq \varepsilon.$$  

The following is crucial to the proof of our theorems:
Theorem 3. Let $C$ be a closed, convex subset of a uniformly convex Banach space. Then for each $r > 0$, $R \geq r$ and $\varepsilon > 0$, there exist $\eta > 0$ and $N \in \mathbb{N}$ such that for each $l \in \mathbb{N}$ and for each mapping $T$ from $C$ into itself satisfying $\sup \{ \| T^n x \| : n \in \mathbb{N}, x \in C \cap B_r \} \leq R$ and $T^l \in \text{Lip}(C, 1 + \eta)$, there holds

$$\left\| \frac{1}{m+1} \sum_{i=0}^{m} T^i x - \frac{1}{m+1} \sum_{i=0}^{m} T^i x \right\| \leq \varepsilon$$

for all $m \geq N$ and $x \in C \cap B_r$. Especially, for each $r > 0$ and for each asymptotically nonexpansive mapping $T$ from $C$ into itself with $F(T) \neq \emptyset$,

$$\lim_{l \to \infty} \lim_{m \to \infty} \sup_{x \in C \cap B_r} \left\| \frac{1}{m+1} \sum_{i=0}^{m} T^i x - \frac{1}{m+1} \sum_{i=0}^{m} T^i x \right\| = 0.$$

Proof. Let $r > 0$, let $R \geq r$ and let $\varepsilon > 0$. By Lemma 1, there exist $\eta > 0$ and $\xi > 0$ such that

$$(\text{co}(F(S) \cap B_R) + B_R) \cap C \subset F(S)$$

for all $S \in \text{Lip}(C, 1 + \eta)$ and

$$(\text{co}(F(S) \cap B_R) + B_\xi) \cap C \subset F(S)$$

for all $S \in \text{Lip}(C, 1 + \xi)$.

Choose $r > 0$ and $p \in \mathbb{N}$ such that $Rr \leq \xi/3$, $r \leq \xi$ and $2R((p+1) \leq r^2/2$. By Lemma 2, there exist $\eta > 0$ and $N \in \mathbb{N}$ such that for each $S \in \text{Lip}(C, 1 + \eta)$ and $\{ x_{j,n} : n \in \mathbb{N}, j = 0, ..., p \} \subset C \cap B_R$ satisfying

$$\frac{1}{n+1} \sum_{i=0}^{n} |x_{j+1,i} - Sx_{j,i}| \leq \eta \quad \text{for all} \quad n \geq N \quad \text{and} \quad j = 0, ..., p,$$

there holds

$$\frac{1}{n+1} \sum_{i=0}^{n} \left| \sum_{j=0}^{p} \lambda_j x_{j+1,i} - S \left( \sum_{j=0}^{p} \lambda_j x_{j,i} \right) \right| \leq \frac{r^2}{2}$$

for all $n \geq N$ and $\lambda \in A^p$.

We may assume $\eta \leq \xi$ and $PR/(N+1) \leq \xi/3$. Let $l \in \mathbb{N}$ and let $T$ be a mapping from $C$ into itself satisfying $\sup \{ \| T^n x \| : n \in \mathbb{N}, x \in C \cap B_r \} \leq R$ and $T^l \in \text{Lip}(C, 1 + \eta)$. We may assume $l \neq 0$. Let $x \in C \cap B_r$. Set $y^*_n = T^{*n} x$ for $n \in \mathbb{N}$ and $q = 0, ..., l-1$. We remark from the hypothesis of $T$ that $\| y^*_n \| \leq R$ for $n \in \mathbb{N}$ and $q = 0, ..., l-1$. Let $n_l = 1/(p+1) \sum_{i=0}^{p} y^*_j$, for $i \in \mathbb{N}$ and $q = 0, 1, ..., l-1$. Let $n \geq N$ and let $q \in \{ 0, 1, ..., l-1 \}$. Since $y^*_{j+1,i} = T^j y^*_i$ for $j = 0, 1, ..., p$, we get
\[
\frac{1}{n+1} \sum_{i=0}^{n} (w_i^q - T^i w_i^q) \leq \frac{1}{n+1} \sum_{i=0}^{n} (w_i^q - w_{i+1}^q) + \frac{1}{n+1} \sum_{i=0}^{n} (w_{i+1}^q - T^i w_i^q) \\
\leq 2R \frac{\tau^2}{p+1} \leq \tau^2.
\]

Set \( A_q^* = \{ i \in \{0, \ldots, n\} : ||w_i^q - T^i w_i^q|| \geq \tau \} \) and \( B_q^* = \{0, \ldots, n\} \setminus A_q^* \). Then we have \( \#A_q^* \leq \tau \), where \( \#A_q^* \) is the cardinality of the set \( A_q^* \). Since

\[
\left| \frac{1}{n+1} \sum_{i=0}^{n} y_i^q - \frac{1}{n+1} \sum_{i=0}^{n} w_i^q \right| \\
\leq \left| \frac{1}{n+1} \sum_{i=0}^{n} y_i^q - \frac{1}{n+1} \sum_{i=0}^{n} w_i^q \right| + \left| \frac{1}{n+1} \sum_{i=B_q^*}^{n} w_i^q - \frac{1}{\#A_q^*} \sum_{i=A_q^*}^{n} w_i^q \right| \\
\leq \frac{2R}{n+1} \frac{\tau^2}{p+1} \leq \frac{\tau^2}{p+1},
\]

we have

\[
\left| \frac{1}{n+1} \sum_{i=0}^{n} y_i^q - \frac{1}{\#B_q^*} \sum_{i=B_q^*}^{n} w_i^q \right| \\
\leq \left| \frac{1}{n+1} \sum_{i=0}^{n} y_i^q - \frac{1}{n+1} \sum_{i=0}^{n} w_i^q \right| + \left| \frac{1}{n+1} \sum_{i=B_q^*}^{n} w_i^q - \frac{1}{\#B_q^*} \sum_{i=B_q^*}^{n} w_i^q \right| \\
\leq \frac{2R}{n+1} \frac{\tau^2}{p+1} \leq \frac{\tau^2}{p+1}.
\]

So by \( 1/\#B_q^* \sum_{i=B_q^*}^{n} w_i^q \in \text{co } F_q(T^i) \cap B_R \), we get

\[
\frac{1}{n+1} \sum_{i=0}^{n} y_i^q \in \text{co } (F_q(T^i) \cap B_R) \cap C \subset F_q(T^i)
\]

for all \( n \geq N \) and \( q = 0, 1, \ldots, l-1 \). Let \( m \geq h(N+1) \). Choose \( n \in \mathbb{N} \) and \( s \in \{0, \ldots, l-2\} \) such that \( m = h(n+1) + s \). Then \( n \geq N \). Hence we obtain

\[
\frac{1}{m+1} \sum_{i=0}^{m} T^i x = \frac{n+2}{m+1} \sum_{q=0}^{n} \left( \frac{1}{n+2} \sum_{i=0}^{n+1} y_i^q \right) + \frac{n+1}{m+1} \sum_{q=s+1}^{l-1} \left( \frac{1}{n+1} \sum_{i=0}^{n} y_i^q \right) \\
\in \text{co } (F_q(T^i) \cap B_R) \cap C \subset F_q(T^i)
\]

for all \( m \geq h(N+1) \) and \( x \in C \cap B_r \).  

In the rest of this section, we assume that $C$, $T$, $\{k_n\}$, $\{a_n\}$, $\{b_n\}$, $x$ and $\{x_n\}$ are as in Theorem 2, we set $a = \lim_n (b_n - 1)/a_n$ and we set $x_n = x$ for $n = 0, 1, \ldots, N_0 - 1$.

**Lemma 4.** Let $\{x_n\}$ be a subsequence of $\{x_n\}$ and let $\mu$ be a Banach limit. Then there exists the unique element $z$ of $C$ satisfying
\[
\mu_i \|x_n - z\|^2 = \min_{y \in C} \mu_i \|x_n - y\|^2
\]
and the point $z$ is a fixed point of $T$.

**Proof.** From Proposition 2, it is easy to see that there exists the unique element $z$ of $C$ satisfying (3.2). If we can show $\lim_n T^{n} z = z$, then $z$ is a fixed point of $T$.

Suppose $\lim_n T^{n} z \neq z$. Then there exists $\varepsilon > 0$ such that for each $m \in \mathbb{N}$, there exists $l \geq m$ satisfying $\|T^l z - z\| > \varepsilon$. Set $R = \sup\{\|T^m z\| : m \in \mathbb{N}\}$. By Proposition 2, there exists $\delta > 0$ such that
\[
\mu_i \left\| x_n - \frac{x + y}{2} \right\|^2 \leq \frac{1}{2} \left( \mu_i \|x_n - x\|^2 + \mu_i \|x_n - y\|^2 \right) - \delta
\]
for all $x, y \in C \cap B_R$ with $\|x - y\| > \varepsilon$. By the property of $\varepsilon$, $\lim_n k_i \leq 1$ and Lemma 3, there also exists $l \geq m$ satisfying $\|T^l z - z\| > \varepsilon$, $(k_i^2 - 1)\mu_i \|x_n - z\|^2 < \delta$ and $\mu_i \|x_n - T^l z\|^2 < \mu_i \|T^l x_n - T^l z\|^2 + \delta$. From (3.3), we have
\[
\mu_i \left\| x_n - \frac{T^l z + z}{2} \right\|^2 \leq \frac{1}{2} \left( \mu_i \|x_n - T^l z\|^2 + \mu_i \|x_n - z\|^2 \right) - \delta
\]
\[
= \mu_i \|x_n - z\|^2 + \frac{1}{2} (k_i^2 - 1)\mu_i \|x_n - z\|^2 - \delta
\]
\[
< \mu_i \|x_n - z\|^2.
\]
So we get a contradiction. This completes the proof.

**Lemma 5.**
\[
\langle x_n - x, J(x_n - z) \rangle \leq \frac{(b_n - 1) + a_n}{a_n} \|x_n - z\|^2
\]
for all $n \geq N_0$ and $z \in F(T)$.

**Proof.** Let $n \geq N_0$ and let $z \in F(T)$. Since $a_n(x_n - x) = (1 - a_n)(1/(n + 1) \sum_{j=0}^n T^j x_n - x_n)$ and $z \in F(T)$, we get
\[
\langle x_n - x, J(x_n - z) \rangle = \frac{1 - a_n}{a_n} \left( \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n - x_n, J(x_n - z) \right)
\]
\[
= \frac{1 - a_n}{a_n} \left( \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n - \frac{1}{n+1} \sum_{j=0}^{n} T^j z, J(x_n - z) \right)
+ \langle z - x_n, J(x_n - z) \rangle
\]
\[
\leq \frac{1 - a_n}{a_n} \left( \frac{1}{n+1} \sum_{j=0}^{n} k_j \|x_n - z\|^2 - \|x_n - z\|^2 \right)
\]
\[
\leq \frac{(b_n - 1)}{a_n} \|x_n - z\|^2.
\]

**Lemma 6.** Each subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) contains a subsequence of \( \{x_{n_k}\} \) converging strongly to an element of \( F(T) \).

**Proof.** Let \( \{x_{n_k}\} \) be a subsequence of \( \{x_n\} \) and let \( \mu \) be a Banach limit. There exists \( z \in F(T) \) satisfying (3.2). By Lemma 5, we get \( \mu, \langle x_n - x, J(x_n - z) \rangle \leq (a.+ \mu, \|x_n - z\|^2 \right)^2 \). This inequality and Proposition 3 yield

\[
\mu, \|x_n - z\|^2 \leq \frac{1}{1 - (a_+) \mu, \langle x - z, J(x_n - z) \rangle \leq 0.
\]

By (2.1), there exists a subsequence of \( \{x_{n_k}\} \) converging strongly to \( z \).

Now we can prove our theorems.

**Proof of Theorem 1.** Taking, for example,

\[
a_n = \begin{cases} 
\frac{1}{n+1} & \text{if } b_n \leq 1, \\
\sqrt{b_n - 1} & \text{if } 1 < b_n \leq 2, \\
1 & \text{if } 2 < b_n,
\end{cases}
\]

we may assume \( a \leq 0 \) only in this proof. First we shall show that \( \{x_{n_k}\} \) converges strongly to an element of \( F(T) \). By Lemma 6, we know that each subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) contains a subsequence of \( \{x_{n_k}\} \) converging strongly to an element of \( F(T) \). Let \( \{x_{n_k}\} \) and \( \{x_{m_k}\} \) be subsequences of \( \{x_n\} \) converging strongly to elements \( y \) and \( z \) of \( F(T) \), respectively. We shall show \( y = z \). From Lemma 5, we have \( \langle x_{n_k} - x, J(x_{n_k} - z) \rangle \leq (b_n - 1)_+ / a_n \|x_{n_k} - z\|^2 \). So we get \( \langle y - x, J(y - z) \rangle \leq 0 \). By the same argument, we have \( \langle z - x, J(z - y) \rangle \leq 0 \). Adding these inequalities, we get \( \|y - z\|^2 \leq 0 \),
i.e., \( y = z \). So \( \{ x_n \} \) converges strongly to an element of \( F(T) \). Hence we can define a mapping \( P \) from \( C \) onto \( F(T) \) by \( Px = \lim_{n \to \infty} x_n \), since \( x \) is an arbitrary point of \( C \). By the argument above, we have \( \langle x - Px, J(z - Px) \rangle \leq 0 \) for all \( x \in C \) and \( z \in F(T) \). Therefore \( P \) is the sunny, nonexpansive retraction by Proposition 4.

**Proof of Theorem 2.** Let \( \{ x_n \} \) be a subsequence of \( \{ x_n \} \) converging strongly to an element \( y \) of \( F(T) \). We shall show \( y = Px \). By Lemma 5, we have \( \langle x_n - x, J(x_n - Px) \rangle \leq (b_n - 1) \| x_n - Px \|^2 \). So we get \( \langle y - x, J(y - Px) \rangle \leq (a_n) + \| y - Px \|^2 \). Hence we obtain

\[
(1 - (a_n)) \| y - Px \|^2 \leq \langle x - Px, J(y - Px) \rangle \leq 0
\]

by Proposition 4. From \( a < 1 \), we have \( y = Px \). Hence by Lemma 6, \( \{ x_n \} \) converges strongly to \( Px \).

**Proof of Theorem 3.** Since \( T \) is nonexpansive, we have \( k_n = 1 \) for all \( n \in \mathbb{N} \) and hence \( \lim \limits_{n \to \infty} (b_n - 1)/a_n = 0 < 1 \). So we obtain the desired result by Theorem 2.

REFERENCES