Journal of Algebra 324 (2010) 1219–1228

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Wells exact sequence for the automorphism group of a group extension

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ARTICLE INFO

Article history:
Received 2 September 2008
Available online 11 June 2010
Communicated by Leonard L. Scott, Jr.

Keywords:
Automorphism
Group extension
Cohomology group
The Wells map

ABSTRACT

We obtain an explicit description of the Wells map for the automorphism group of a group extension in the full generality and investigate the dependency of this map on group extensions. Some applications are given.

1. Introduction

In recent years there has been considerable interest in the Wells sequence constructed in [8,10] for the automorphism group of a group extension, involved with automorphisms of group extensions, classifying spaces of finite groups, automorphism group rings of finite p-groups, and saturated fusion systems over 2-groups, etc. See [2,3,5–7], for example. However, this sequence contains a set map, known as *the Wells map*, which has not been well understood up to this point and consequently is hard to apply.

The present paper is a continuation of the first author' work [4] there we gave an explicit description of the Wells map in a special case. By developing some ideas on group actions due to Buckley [1], we are now able to prove that the Wells map is a derivation (or equivalently, a 1-cocycle) in the full generality. Further, this new description enables us to investigate the dependency of the Wells map

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[1] The author was supported by the NSF of Shanxi Province (No. 2008011011).
[2] The author was supported by the NSF of China (No. 10971054).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.04.034

on group extensions and to deduce some new applications for inducing automorphism pairs. It should be pointed out that the method used in the paper is more conceptual, not computational as in [4].

To state our results, we need to fix the following notation which will be used throughout this paper for convenience.

- Let Q and N be arbitrary groups and let $\chi : Q \to \operatorname{Out} N$ be a *fixed* group homomorphism which can be realized as the coupling of an extension of N by Q.
- $A = Z(N)$, the center of N regarded as a Q-module via χ.
- $\operatorname{Der}(Q, A)$, the group of derivations from Q into A. A map $\lambda : Q \to A$ is called a derivation whenever $(xy)^\lambda = (x^\lambda)^y y^\lambda$ for all $x, y \in Q$.
- $H = H^2(Q, A)$, the second cohomology group.
- $C = \operatorname{Comp}(\chi)$, the group of all *compatible* pairs of automorphisms $(\theta, \sigma) \in \operatorname{Aut} N \times \operatorname{Aut} Q$ for χ. Recall that an automorphism pair (θ, σ) is said to be compatible for χ (in the sense of Wells [10]) if θ and σ satisfy the equation

$$\bar{\theta}^{-1} x^\chi \bar{\theta} = \left(x^\sigma\right)^\chi$$

  where $x \in Q$ and $\bar{\theta} = \theta(\operatorname{Inn} N)$ denotes the image of θ in $\operatorname{Out} N$.
- $\operatorname{Ext}_\chi(Q, N)$, the set of equivalence classes $[\mathcal{E}]$ of χ*-extensions* $\mathcal{E}$. Here by a χ-extension, we mean a group extension $\mathcal{E} : N \rightarrowtail G \twoheadrightarrow Q$ with the coupling χ.
- $\operatorname{Aut} \mathcal{E}$, the automorphism group of an extension $\mathcal{E} : N \rightarrowtail G \twoheadrightarrow Q$, that is, the group of automorphisms of G that leave N invariant.
- $\Gamma = C \ltimes H$, the semidirect product under the natural action of C on H given in Section 2.

In addition, for any χ-extension $\mathcal{E} : N \rightarrowtail G \twoheadrightarrow Q$, we shall always write

$$\rho(\mathcal{E}) : \operatorname{Aut} \mathcal{E} \to C$$

for the restriction homomorphism, that is,

$$\gamma^{\rho(\mathcal{E})} = (\gamma|_N, \gamma|_Q) \in C$$

where $\gamma \in \operatorname{Aut} \mathcal{E}$. This homomorphism provides a means of studying $\operatorname{Aut} \mathcal{E}$ with $\operatorname{Aut} N$ and $\operatorname{Aut} Q$ under control. The crucial question on automorphisms of group extensions is to decide whether a given automorphism pair $c = (\theta, \sigma)$ is *inducible* from $\mathcal{E}$, that is, when c lies in the image $\operatorname{Im} \rho(\mathcal{E})$.

Now we give a new description of the Wells map in terms of group actions. Buckley [1, Theorem 1.1] first considered the natural action of the group C on the set $\operatorname{Ext}_\chi(Q, N)$, and proved that the image of $\rho(\mathcal{E})$ coincides with the stabilizer of $[\mathcal{E}]$ in C. In other words, he showed that (in our notation) an element $c \in C$ is inducible from a given χ-extension $\mathcal{E}$ if and only if $[\mathcal{E}]^c = [\mathcal{E}]$. However, in that paper Buckley did not consider the Wells map further. Actually, since the cohomology group H acts regularly on $\operatorname{Ext}_\chi(Q, N)$, as will be described explicitly in next section, it follows that there exists a unique element $h \in H$ such that $[\mathcal{E}]^c \cdot h = [\mathcal{E}]$. So, we have a set map $\omega(\mathcal{E}) : C \to H$ defined by the equation

$$[\mathcal{E}]^c \cdot c^{\omega(\mathcal{E})} = [\mathcal{E}]$$

for $c \in C$. This map has the property that c is inducible from $\mathcal{E}$ precisely when $c^{\omega(\mathcal{E})}$ is trivial. Of course, $\omega(\mathcal{E}) = \omega(\mathcal{E}')$ for equivalent extensions $\mathcal{E}$ and $\mathcal{E}'$. Consequently, the Wells sequence for a χ-extension $\mathcal{E}$, constructed in [8] or [10], can be reformulated as the following exact sequence

$$0 \to \operatorname{Der}(Q, A) \to \operatorname{Aut} \mathcal{E} \xrightarrow{\rho(\mathcal{E})} C \xrightarrow{\omega(\mathcal{E})} H.$$

Here we remark that the map $\omega(\mathcal{E}): C \to H$ defined above coincides with the original one defined in [8] or [10] after a routine calculation. So, for the sake of simplicity and clarity, we shall adopt $\omega(\mathcal{E})$ as the definition of the Wells map for a χ-extension $\mathcal{E}$ in this paper. We note that in Theorem 2.1 of [6] there is a similar set map $\epsilon: C \to H$ which works as the Wells map, defined by identifying $[\mathcal{E}]$ with an element of H. However it is not obvious that both maps are the same, since in general there is not a canonical correspondence between $\mathrm{Ext}_\chi(Q, N)$ and H (as a set).

The key to our approach is the study of the compatibility of three well-known group actions: the action on the set $\mathrm{Ext}_\chi(Q, N)$ of groups C and H respectively, and the action of C on H via automorphisms. Our first main result combines the actions of C and H on $\mathrm{Ext}_\chi(Q, N)$.

Theorem A. *With the above notation, there exists a group action of the semidirect product $\Gamma = CH$ on the set $\mathrm{Ext}_\chi(Q, N)$ defined by*

$$[\mathcal{E}] \cdot (ch) = [\mathcal{E}]^c \cdot h$$

for any χ-extension $\mathcal{E}$, $c \in C$ and $h \in H$. Furthermore, if $C(\mathcal{E})$ denotes the stabilizer of $[\mathcal{E}]$ in Γ, then $C(\mathcal{E})$ is a complement to H in Γ and the set $\{C(\mathcal{E}) \mid [\mathcal{E}] \in \mathrm{Ext}_\chi(Q, N)\}$ is a single conjugacy class of subgroups of Γ.

In the situation of Theorem A, since $C \cap C(\mathcal{E})$ is the stabilizer of $[\mathcal{E}]$ in C for each χ-extension $\mathcal{E}$, we may restate Theorem 1.1 of [1] as the following exact sequence

$$0 \to \mathrm{Der}(Q, A) \to \mathrm{Aut}\,\mathcal{E} \xrightarrow{\rho(\mathcal{E})} C \cap C(\mathcal{E}) \to 1$$

which indicates the dependency on extensions and provides a useful tool in the study of inducing automorphism pairs. To show the power of Theorem A, we shall give some of its applications. The first one is immediate, which covers [4, Theorem A].

Corollary B. *For any χ-extension $\mathcal{E}$, the Wells map $\omega(\mathcal{E})$ is a derivation from C into H under the natural action of C on H. Further, if $\mathcal{E}'$ is another χ-extension, then there exists an element $h \in H$ such that*

$$c^{\omega(\mathcal{E}')} = c^{\omega(\mathcal{E})} h \big(h^{-1}\big)^c$$

for all $c \in C$, that is, $\omega(\mathcal{E})$ and $\omega(\mathcal{E}')$ differ by an inner derivation.

Note that in the situation of Corollary B, the Wells map $\omega(\mathcal{E})$ defines a unique element $[\omega(\mathcal{E})] \in H^1(C, H)$, which we call *the associated cohomology element* with the coupling χ and denote by $[\chi]$. We shall exhibit an example in Section 4 to show that the group C does not coincide with $C(\mathcal{E})$ in Γ for any χ-extension $\mathcal{E}$ in general. However, the cohomology element $[\chi]$ can be used to obtain a criterion. It might be interesting to mention that $[\chi] \in H^1(C, H)$ can be thought of as the obstruction to every compatible pair being inducible in some extension. Compare this with $[\chi] \in H^3(Q, A)$, the obstruction to the existence of some extension with coupling χ.

Corollary C. *The following statements are equivalent*:

(a) $[\chi]$ *vanishes in* $H^1(C, H)$.
(b) $C = C(\mathcal{E})$ *for some* χ*-extension* $\mathcal{E}$.
(c) *There exists a* χ*-extension* $\mathcal{E}$ *such that* $\rho(\mathcal{E}): \mathrm{Aut}\,\mathcal{E} \to C$ *is surjective.*

We mention that Corollary C applies when χ is trivial or when N is abelian. Moreover, we let $s(\chi)$ denote the number of equivalent classes of splitting χ-extensions and call $s(\chi)$ the *splitting index* of χ. Then we shall prove that $[\chi]$ vanishes whenever $s(\chi) = 1$, which covers the case where N is abelian and hence extends [7, Lemma 1.2].

Next, we shall deal with an interesting case where $[\chi]$ vanishes. For convenience we call an element $c \in C$ is *absolutely inducible* for χ if c is inducible from each χ-extension.

Theorem D. *If $[\chi]$ vanishes, then an element $c \in C$ is absolutely inducible if and only if c acts trivially on H.*

By definition, it is easy to verify that such pairs $(\theta, 1_Q)$ must act trivially on H, where $\theta \in C_{\operatorname{Aut} N}(Q^\chi, A)$, the group of those automorphisms of N that act trivially on Q^χ and on A. Moreover, the extension constructed in Section 4 shows that the condition $[\chi] = 0$ in Theorem D cannot be removed.

Finally, we shall give an application of Theorem A to finite groups.

Theorem E. *Assume that both N and Q are finite groups. Let $c \in C$. If $(o(c), |A|) = 1$, then c is absolutely inducible if and only if c acts trivially on H, in which case there is an automorphism $\varphi \in \operatorname{Aut}\mathcal{E}$ for each χ-extension $\mathcal{E}$ such that φ induces c and the orders $o(\varphi)$ and $o(c)$ are equal.*

This extends [4, Theorem B] except for the uniqueness. Again Theorem E is not true without the coprimeness condition, as indicated by the same example in Section 4.

Most notation used in this paper will be standard, see [9], for example.

2. Group actions and Theorem A

In this section, we shall review some known results of group actions and then prove Theorem A.

2.1. The action of H on $\operatorname{Ext}_\chi(Q, N)$

It is well known that the second cohomology group $H = H^2(Q, A)$ acts regularly on the set $\operatorname{Ext}_\chi(Q, N)$, see [8,9] for example. We shall give an explicit description for this regular action, which will be crucial in the proof of Theorem A. The following is adopted from [8, Section 2] with a minor change in notation.

Let $\mathcal{E} : N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ be a χ-extension. Recall that a map $\lambda : Q \to G$ is called a *transversal function* for $\mathcal{E}$ if $\lambda\pi = 1$ and $1^\lambda = 1$. For an element x in Q, denote by x^ξ the automorphism of N induced by x^λ by conjugation in G and we obtain a map $\xi : Q \to \operatorname{Aut} N$ satisfying

$$a^{x^\xi} = \left(x^\lambda\right)^{-1} a x^\lambda$$

for all $a \in N$ and $x \in Q$. Also, if x and y are elements of Q, then $x^\lambda y^\lambda$ and $(xy)^\lambda$ differ by an element of N. Thus we have a map $\alpha : Q \times Q \to N$ such that

$$x^\lambda y^\lambda = (xy)^\lambda (x, y)\alpha.$$

Then (ξ, α) will be referred to as an *associated pair* for $\mathcal{E}$. Moreover, let

$$G(\xi, \alpha) = \left\{(x, a) \;\middle|\; x \in Q,\ a \in N\right\},$$

equipped with the binary operation

$$(x, a)(y, b) = \left(xy, (x, y)\alpha \cdot a^{y^\xi} \cdot b\right).$$

Then $G(\xi, \alpha)$ is a group. We write

$$\mathcal{E}(\xi, \alpha) : N \rightarrowtail G(\xi, \alpha) \twoheadrightarrow Q$$

for the corresponding extension. It is routine to check that the transversal function $x \mapsto (x, 1)$ gives rise to the functions ξ and α as an associated pair for $\mathcal{E}(\xi, \alpha)$.

After these preparations we can now describe the natural action of H on $\mathrm{Ext}_\chi(Q, N)$. The following result is a direct consequence of (2.12) in [8], which is of fundamental importance in the theory of group extensions.

Lemma 2.1. *Let (ξ, α) be an associated pair of functions for a χ-extension $\mathcal{E}$. Then $\mathcal{E}$ is equivalent to the constructed extension $\mathcal{E}(\xi, \alpha)$ and $(\xi, \alpha\beta)$ is also an associated pair of functions for a χ-extension where $\beta \in Z^2(Q, A)$. Furthermore, the operation*

$$[\mathcal{E}] \cdot [\beta] = [\mathcal{E}(\xi, \alpha)] \cdot [\beta] = [\mathcal{E}(\xi, \alpha\beta)]$$

will give rise to the regular action of H on $\mathrm{Ext}_\chi(Q, N)$.

We mention that in the above lemma the product $\alpha\beta$ is defined by

$$(x, y)(\alpha\beta) = (x, y)\alpha(x, y)\beta$$

for all $x, y \in Q$, as H is written multiplicatively in this paper.

2.2. The action of C on $\mathrm{Ext}_\chi(Q, N)$

For a χ-extension $\mathcal{E} : N \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ and a pair $c = (\theta, \sigma) \in C$, Buckley [1] defined the action of c on $\mathcal{E}$ as (in our notation)

$$\mathcal{E}^c : N \overset{\theta^{-1}\iota}{\rightarrowtail} G \overset{\pi\sigma}{\twoheadrightarrow} Q.$$

Lemma 2.2. *With the above notation, the following statements hold*:

(a) *$\mathcal{E}^c$ is also a χ-extension.*
(b) *If $\mathcal{E}'$ is a χ-extension, then $[\mathcal{E}] = [\mathcal{E}']$ if and only if $[\mathcal{E}^c] = [(\mathcal{E}')^c]$. This induces a natural action of C on $\mathrm{Ext}_\chi(Q, N)$ by setting $[\mathcal{E}]^c = [\mathcal{E}^c]$.*
(c) *If (ξ, α) is an associated pair of functions for $\mathcal{E}$ with respect to a transversal function $\lambda : Q \to G$, then the associated functions for $\mathcal{E}^c$ with respect to the transversal function $\sigma^{-1}\lambda$ are ξ^c and α^c, where*

$$x^{\xi^c} = \theta^{-1}\big(x^{\sigma^{-1}}\big)^{\xi}\theta \quad \text{and} \quad (x, y)\alpha^c = \big(\big(x^{\sigma^{-1}}, y^{\sigma^{-1}}\big)\alpha\big)^{\theta}$$

for all $x, y \in Q$. In particular, the constructed extensions $\mathcal{E}(\xi^c, \alpha^c)$ and $\mathcal{E}(\xi, \alpha)^c$ are equivalent.

Proof. All these are standard facts and the proof is straightforward, see [1,8] for the details. □

Furthermore, Buckley [1] proved that the pair $c = (\theta, \sigma)$ is inducible from $\mathcal{E}$ if and only if $\mathcal{E}^c$ is equivalent to $\mathcal{E}$, that is, $[\mathcal{E}]^c = [\mathcal{E}^c] = [\mathcal{E}]$. So, the image of $\rho(\mathcal{E}) : \mathrm{Aut}\,\mathcal{E} \to C$ can be described as the stabilizer of $[\mathcal{E}]$ in C.

Now, assume that the above χ-extension $\mathcal{E} : N \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ splits, that is, there exists a homomorphism $\lambda : Q \to G$ such that $\lambda\pi = 1_Q$. Let $\lambda' = \sigma^{-1}\lambda$. Since σ is an automorphism of Q, $\lambda' : Q \to G$ is a homomorphism. Clearly $\lambda'(\pi\sigma) = 1_Q$, which implies that $\mathcal{E}^c$ also splits. We mention that such two splitting extensions $\mathcal{E}^c$ and $\mathcal{E}$ need not be equivalent in general, as indicated by the example constructed in Section 4.

2.3. The action of C on H

This action is also known, see [8], and we review some related results to establish our notation. For any $c = (\theta, \sigma) \in C$ and $\alpha \in Z^2(Q, A)$, define α^c by the following rule

$$(x, y)\alpha^c = \big(\big(x^{\sigma^{-1}}, y^{\sigma^{-1}}\big)\alpha\big)^\theta$$

for $x, y \in Q$. We may verify that $\alpha^c \in Z^2(Q, A)$ and that the coboundaries $B^2(Q, A)$ are setwise invariant under this action. This induces the desired action of C on H by setting $[\alpha]^c = [\alpha^c]$.

By the above definition, it is clear that if $\theta \in C_{\mathrm{Aut}\,N}(Q^\chi, A)$, that is, if θ centralizes Q^χ and A, then the pair $(\theta, 1_Q)$ lies in C and acts trivially on $Z^2(Q, A)$ and hence on $H^2(Q, A)$, as mentioned in the Introduction.

Now, we are ready to prove Theorem A in the Introduction which we restate here for convenience.

Theorem 2.3. *There exists a group action of the semidirect product $\Gamma = CH$ on the set $\mathrm{Ext}_\chi(Q, N)$ defined by*

$$[\mathcal{E}] \cdot (ch) = [\mathcal{E}]^c \cdot h$$

for any χ-extension $\mathcal{E}$, $c \in C$ and $h \in H$. Further, let $C(\mathcal{E})$ denote the stabilizer of $[\mathcal{E}]$ in Γ. Then $C(\mathcal{E})$ is a complement to H in Γ and the set $\{C(\mathcal{E}) \mid [\mathcal{E}] \in \mathrm{Ext}_\chi(Q, N)\}$ is a single conjugacy class of subgroups of Γ.

Proof. To prove Γ acts on $\mathrm{Ext}_\chi(Q, N)$ in the manner, it suffices to verify that $([\mathcal{E}] \cdot h)^c = [\mathcal{E}]^c \cdot h^c$ for any χ-extension $\mathcal{E}$, $c \in C$ and $h \in H$.

Choose an associated pair of functions (ξ, α) for $\mathcal{E}$. By Lemma 2.2, we know that (ξ^c, α^c) is an associated pair of functions for $\mathcal{E}^c$ and the constructed extensions $\mathcal{E}(\xi^c, \alpha^c)$ and $\mathcal{E}(\xi, \alpha)^c$ are equivalent. For any $h \in H$, we may write $h = [\beta]$ for some $\beta \in Z^2(Q, A)$. Then by Lemma 2.1, we have

$$\begin{aligned}
\big([\mathcal{E}] \cdot h\big)^c &= \big(\big[\mathcal{E}(\xi, \alpha)\big] \cdot [\beta]\big)^c \\
&= \big[\mathcal{E}(\xi, \alpha\beta)\big]^c \\
&= \big[\mathcal{E}\big(\xi^c, \alpha^c\beta^c\big)\big] \\
&= \big[\mathcal{E}\big(\xi^c, \alpha^c\big)\big] \cdot \big[\beta^c\big] \\
&= \big[\mathcal{E}(\xi, \alpha)\big]^c \cdot [\beta]^c \\
&= [\mathcal{E}]^c \cdot h^c
\end{aligned}$$

which proves that Γ acts on $\mathrm{Ext}_\chi(Q, N)$ in the desired manner.

Fix a χ-extension $\mathcal{E}$. Since H acts regularly on $\mathrm{Ext}_\chi(Q, N)$ and $C(\mathcal{E})$ is the stabilizer of $[\mathcal{E}]$ in Γ by definition, it easily follows that $\Gamma = C(\mathcal{E})H$ and $C(\mathcal{E}) \cap H = 1$. Hence $C(\mathcal{E})$ is a complement to H in Γ. For each χ-extension $\mathcal{E}'$, we may write $[\mathcal{E}'] = [\mathcal{E}] \cdot h$ for some $h \in H$. Then $C(\mathcal{E}') = C(\mathcal{E})^h$ and the result follows. □

3. Applications

In this section, as applications of Theorem A we shall prove Corollaries B and C and Theorems D and E from the Introduction.

Proof of Corollary B. In the situation of Theorem A, it is well known that complements of H in Γ correspond to derivations from C to H, see [9, 11.1.2]. Let $C(\mathcal{E})$ correspond to $\lambda \in \mathrm{Der}(C, H)$. Then $C(\mathcal{E}) = \{cc^\lambda \mid c \in C\}$. Since $C(\mathcal{E})$ is the stabilizer of $[\mathcal{E}]$ in Γ, we have $[\mathcal{E}]^c \cdot c^\lambda = [\mathcal{E}] \cdot (cc^\lambda) = [\mathcal{E}]$. By the definition of $\omega(\mathcal{E})$ introduced in the Introduction, we have $\omega(\mathcal{E}) = \lambda$, a derivation from C into H.

Furthermore, if $\mathcal{E}'$ is another χ-extension, then we may write $[\mathcal{E}'] = [\mathcal{E}] \cdot h$ for some $h \in H$ by the regular action of H on $\mathrm{Ext}_\chi(Q, N)$. Since $C(\mathcal{E})$ is the stabilizer of $[\mathcal{E}]$ in Γ, it follows that $C(\mathcal{E}') = C(\mathcal{E})^h$ and $c^{\omega(\mathcal{E}')} = c^{\omega(\mathcal{E})} h (h^{-1})^c$ for all $c \in C$. Hence $c^{\omega(\mathcal{E})}$ and $c^{\omega(\mathcal{E}')}$ differ by an inner derivation. This completes the proof. □

As mentioned in the Introduction, we let $[\chi] = [\omega(\mathcal{E})] \in H^1(C, H)$, the cohomology element associated with the coupling χ. An interesting question is to decide when $[\chi]$ vanishes.

Proof of Corollary C. For any χ-extension $\mathcal{E}$, we have seen in the proof of Corollary B that the complement $C(\mathcal{E})$ of H in Γ corresponds to the Wells map $\omega(\mathcal{E}) \in \mathrm{Der}(Q, A)$. So, the set of all complements $C(\mathcal{E})$ correspond to the associated cohomology element $[\chi] = [\omega(\mathcal{E})]$ of $H^1(C, H)$. It follows from [9, 11.1.3] that $[\chi]$ vanishes if and only if C lies in the set of all complements $C(\mathcal{E})$, that is, C has the form $C(\mathcal{E})$ for some χ-extension $\mathcal{E}$. This proves the equivalence of statements (a) and (b).

The statements (b) and (c) are clearly equivalent, since the image of $\rho(\mathcal{E})$ is $C \cap C(\mathcal{E})$. The proof is complete. □

Note that if χ is trivial, then $C = \mathrm{Aut}\,N \times \mathrm{Aut}\,Q$ and the direct product extension $\mathcal{E}_0 : N \rightarrowtail Q \times N \twoheadrightarrow Q$ clearly has the trivial coupling χ. In this case the restriction map $\rho(\mathcal{E}_0) : \mathrm{Aut}\,\mathcal{E}_0 \to C$ must be surjective, which implies that the associated cohomology element $[\chi]$ is also trivial by Corollary C.

Also, if N is abelian, or more generally, if $s(\chi) = 1$ (that is, all the splitting χ-extensions are equivalent), then we have $[\chi] = 0$. To see this, let $\mathcal{E}_0$ be a χ-extension which splits. Then $\mathcal{E}_0^c$ also splits for any $c \in C$ (see the final paragraph in Section 2.2), which implies that $\mathcal{E}_0^c$ and $\mathcal{E}_0$ are equivalent. So, each element of C is inducible from the χ-extension $\mathcal{E}_0$ and hence $C \subseteq C(\mathcal{E}_0)$. Since both C and $C(\mathcal{E}_0)$ are complements of H in Γ, we may deduce that $C = C(\mathcal{E}_0)$ and $[\chi] = 0$ by Corollary C.

Now, we turn to the case where $[\chi]$ vanishes.

Proof of Theorem D. Since $[\chi]$ vanishes in $H^1(C, H)$, we conclude from Corollary C that $C = C(\mathcal{E})$ for some χ-extension $\mathcal{E}$. By Theorem A, we see that an element $c \in C$ is absolutely inducible if and only if c lies in each conjugate of C in Γ, or equivalently $c \in C^h$ for all $h \in H$. Note that the intersection of all C^h coincides with the centralizer of H in C and the result follows. □

Proof of Theorem E. If c is absolutely inducible, then $c \in C(\mathcal{E})$ for all χ-extension $\mathcal{E}$. By Theorem A, all complements $C(\mathcal{E})$ form a conjugacy class of subgroups of Γ, which implies that c acts trivially on H.

Conversely, assume that c acts trivially on H. For each χ-extension $\mathcal{E}$, it follows from Corollary B that the Wells map $\omega(\mathcal{E})$ which, when restricted to $\langle c \rangle$, is a group homomorphism. It is well known that the exponent of H must divide the exponent of A. So, c and H have coprime orders, which forces the image $c^{\omega(\mathcal{E})}$ must be trivial and hence c is inducible from $\mathcal{E}$.

Finally, assume that $\mathcal{E}$ is a χ-extension from which c is inducible. Then c lies in the image of the restriction $\rho(\mathcal{E}) : \mathrm{Aut}\,\mathcal{E} \to C$. Let K denote the kernel of $\rho(\mathcal{E})$. Then K is isomorphic to $\mathrm{Der}(Q, A)$ which has exponent dividing $|A|$. Thus c and K have coprime orders, which implies that there is some $\varphi \in \mathrm{Aut}\,\mathcal{E}$ such that φ induces c and $o(\varphi) = o(c)$. The proof is now complete. □

4. Examples

In the final section, we shall construct two finite groups N, Q, and a nontrivial group homomorphism $\chi : Q \to \mathrm{Out}\,N$ with the following properties:

(i) C acts trivially on H but transitively on $\mathrm{Ext}_\chi(Q, N)$.
(ii) $C \neq C(\mathcal{E})$ for all χ-extensions $\mathcal{E}$, or equivalently, $\rho(\mathcal{E}) : \mathrm{Aut}\,\mathcal{E} \to C$ is not surjective for each χ-extension $\mathcal{E}$.
(iii) Each χ-extension splits and $|\mathrm{Ext}_\chi(Q, N)| = |H| > 1$.

(iv) For each χ-extension $\mathcal{E}$, the corresponding Wells sequence can be strengthened as the following exact sequence of groups and homomorphisms:

$$0 \to \mathrm{Der}(Q, A) \to \mathrm{Aut}\,\mathcal{E} \xrightarrow{\rho(\mathcal{E})} C \xrightarrow{\omega(\mathcal{E})} H \to 1.$$

In this case by (ii) and Corollary C, we see that the associated cohomology element $[\chi]$ with χ cannot vanish in $H^1(C, H)$. Fix a χ-extension $\mathcal{E}$. Then by (ii) again there exists an element c of C such that $c \notin \mathrm{Im}\,\rho(\mathcal{E})$, which means that c is not absolutely inducible. However, the element c acts trivially on H by (i). This proves that the conditions $[\chi] = 0$ and $(o(c), |A|) = 1$ in Theorem D and Theorem E, respectively, cannot be removed.

Actually, we shall take the above group N to be the generalized quaternion group $Q_{2^{n+1}}$ for $n \geqslant 3$. So the following facts about automorphisms of $Q_{2^{n+1}}$ will be needed.

Lemma 4.1. *For $n \geqslant 3$, let*

$$Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ a^b = a^{-1} \rangle.$$

Then the following statements hold:

(a) *Each automorphism θ of $Q_{2^{n+1}}$ can be described as $a^\theta = a^i, b^\theta = ba^j$ for $(i, j) \in U(\mathbb{Z}_{2^n}) \times \mathbb{Z}_{2^n}$. In particular,* $\mathrm{Aut}\, Q_{2^{n+1}} \cong U(\mathbb{Z}_{2^n}) \ltimes \mathbb{Z}_{2^n}$, *the holomorph of the cyclic group of order 2^n.*
(b) *The automorphism θ in* (a) *is inner if and only if the corresponding pair (i, j) satisfies $i = \pm 1$ and $2 \mid j$.*
(c) $\mathrm{Out}\, Q_{2^{n+1}}$ *is abelian. More precisely,* $\mathrm{Out}\, Q_{2^{n+1}} \cong \mathbb{Z}_{2^{n-2}} \oplus \mathbb{Z}_2$.

Proof. The proof is a routine computation. □

Now, our example can be constructed as follows. Fix an integer $n \geqslant 3$ and let $N = Q_{2^{n+1}}$ be the generalized quaternion group with the presentation

$$N = \langle a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ a^b = a^{-1} \rangle.$$

Let $Q = \{1, x\}$ be a cyclic group of order 2. Take an automorphism τ of N of order 2 defined by $a^\tau = a^{1+2^{n-1}}$ and $b^\tau = b$. Define a group homomorphism $\chi : Q \to \mathrm{Out}\, N$ by setting $x^\chi = \tau(\mathrm{Inn}\, N)$. Then, by Lemma 4.1 we know that χ is nontrivial, and we shall in turn verify the following assertions.

- *C acts trivially on H.*

In fact, since $\mathrm{Out}\, N$ is abelian (by Lemma 4.1) and $\mathrm{Aut}\, Q$ is trivial, it follows that the group C turns out to be $\mathrm{Aut}\, N \times \{1_Q\}$. This implies that C acts trivially on H, as $\mathrm{Aut}\, N$ clearly centralize the center A of N.

- $|\mathrm{Ext}_\chi(Q, N)| = |H| = 2$.

Note that both Q and A are cyclic of order 2 and Q acts trivially on A via χ. The result follows.

- *C acts transitively on* $\mathrm{Ext}_\chi(Q, N)$.

Let $G = \langle \tau \rangle \ltimes N$ be the semidirect product. We claim that the restriction

$$\rho : C_{\mathrm{Aut}\, G}(G/N) \to \mathrm{Aut}\, N$$

is not surjective, where $C_{\mathrm{Aut}\, G}(G/N)$ denotes the group of those automorphisms of G that act trivially on G/N.

To see this, let $\theta \in \operatorname{Aut} N$. By Lemma 4.1, we may write

$$a^\theta = a^i \quad \text{and} \quad b^\theta = ba^j$$

for some $(i, j) \in U(\mathbb{Z}_{2^n}) \times \mathbb{Z}_{2^n}$. Since $\tau \in G$ has order 2, it easily follows that $\theta \in \operatorname{Im} \rho$ if and only if there exists an element $s \in N$ such that $s^\tau = s^{-1}$ and the commutator $[\tau, \theta]$ is an inner automorphism of N induced by s via conjugation (see [10, Lemma 1] for example). From this we may deduce that $\theta \in \operatorname{Im} \rho$ if and only if j is even. Hence $|\operatorname{Aut} N : \operatorname{Im} \rho| = 2$ and ρ cannot be surjective, as claimed.

Now, let $\mathcal{E}_0 : N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ denote the semidirect product extension with $\tau^\pi = x$. Then $\mathcal{E}_0$ has the coupling χ and $\operatorname{Aut} \mathcal{E}_0 = C_{\operatorname{Aut} G}(G/N)$, as $\operatorname{Aut} Q$ is trivial. Clearly $C = \operatorname{Aut} N \times \{1_Q\}$. We deduce that $\rho(\mathcal{E}_0) : \operatorname{Aut} \mathcal{E}_0 \to C$ cannot be surjective and hence the Wells map $\omega(\mathcal{E}_0) : C \to H$ is not trivial. But C acts trivially on H and $|H| = 2$, it follows from Corollary B that $\omega(\mathcal{E}_0)$ must be a surjective homomorphism. Therefore, we have the following exact sequence of groups and homomorphisms:

$$0 \to \operatorname{Der}(Q, A) \to \operatorname{Aut} \mathcal{E}_0 \xrightarrow{\rho(\mathcal{E}_0)} C \xrightarrow{\omega(\mathcal{E}_0)} H \to 1. \tag{1}$$

Furthermore, since $\operatorname{Im} \rho(\mathcal{E}_0) = C \cap C(\mathcal{E}_0) \neq C$, we conclude that C cannot leave $[\mathcal{E}_0]$ invariant. This, along with $|\operatorname{Ext}_\chi(Q, N)| = 2$, implies that C acts transitively on $\operatorname{Ext}_\chi(Q, N)$.

- *For any χ-extension $\mathcal{E}$, $\rho(\mathcal{E}) : \operatorname{Aut} \mathcal{E} \to C$ cannot be surjective and the above sequence* (1) *also holds for $\mathcal{E}$.*

From the transitivity of C on $\operatorname{Ext}_\chi(Q, N)$, we see that C does not fix $[\mathcal{E}]$ invariant and hence $C \neq C \cap C(\mathcal{E})$. The result easily follows.

- *All χ-extensions split.*

Let $\tau' \in \operatorname{Aut} N$ defined by $a^{\tau'} = a^{-1+2^{n-1}}$ and $b^{\tau'} = b$. Then $o(\tau') = 2$. Let $\mathcal{E}' : N \rightarrowtail Q \ltimes_{\tau'} N \twoheadrightarrow Q$ be the semidirect product extension. It is easy to verify that $\mathcal{E}'$ is a χ-extension but $[\mathcal{E}'] \neq [\mathcal{E}_0]$. Since $|\operatorname{Ext}_\chi(Q, N)| = 2$, it follows that $\mathcal{E}_0$ and $\mathcal{E}'$ are the full representatives of all χ-extensions. This proves that all χ-extensions split, as wanted.

We conclude this paper with a remark. Actually, the above sequence (1) also provides a counterexample to Theorem B of [4] to show that there condition (c) cannot be removed. Since then each automorphism of $\mathcal{E}_0$ clearly centralizes A and $\operatorname{Out} N$ is abelian, it is easy to verify that this sequence becomes into the following exact sequence of groups and homomorphisms:

$$0 \to \operatorname{Der}(Q, A) \to C_{\operatorname{Aut} G}(Q, A) \to C_{\operatorname{Aut} N}\big(Q^\chi, A\big) \to H \to 1$$

which implies that some element of $C_{\operatorname{Aut} N}(Q^\chi, A)$ cannot be extended to G with trivial action on Q.

Acknowledgments

We would like to deeply thank the referee for his valuable comments and suggestions that helped us improve this paper.

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