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On the stability in terms of two measures for perturbed impulsive integro-differential equations ☆

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Abstract

This paper establishes several stability criteria for perturbed impulsive integro-differential equations with fixed moments of impulsive effect. By using a new comparison theorem, which connects the solutions of perturbed system and the unperturbed one, some sufficient conditions for the stability in terms of two measures are obtained for the perturbed system while unperturbed one dissatisfied which because of the effect of the perturbed terms.

Keywords: Perturbed impulsive integro-differential equations; Stability in terms of two measures; Variation of Lyapunov second method; Comparison principle

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1. Introduction

In the study of the nonlinear system, one of the most used techniques is the variation of parameters when the unperturbed terms are smooth enough especially when they are linear, the other is the Lyapunov second method. Combining these two techniques, a flexible mechanism-variation of Lyapunov second method is introduced, see [1].

Employing this introduction, a new comparison principle is presented, which connects the solutions of the perturbed system and unperturbed one through the solutions of the comparison system. This has been used by many authors, see [2–4]. For example, Devi [2] considered the following impulsive differential system:

$$\begin{cases} x' = F(t,x), \quad t \neq t_k, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), \\ x(t_0^+) = x_0, \quad t_0 \geqslant 0,\ k \in N, \end{cases}$$

where $0 \leqslant t_0 < t_1 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$, $F: [0, +\infty) \times R^n \to R^n$ is continuous on $(t_k, t_{k+1}] \times R^n$ and $I_k : R^n \to R^n$, $k = 1, 2, \ldots$. By using the variation of Lyapunov second method together with the comparison theorem, the uniformly asymptotical stabilities of such perturbed system are studied.

While many stability concepts are presented in the literature such as the Lyapunov stability, partial stability, conditional stability, relative stability and so on. In 1960, Movchan [5] introduced the concept of stability in terms of two measures which unified the forgoing stability concepts. Following his study, the theories of the stability in terms of two measures have been successfully developed and become important in the investigation of the quality analysis, see [5–9].

In this paper, we consider the perturbed impulsive integro-differential equations

$$\begin{cases} x' = F(t,x,L_1 x), \quad t \neq t_k, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), \\ x(t_0^+) = x_0, \quad t_0 \geqslant 0,\ k \in N, \end{cases}$$

where t_k, F, I_k are similar to the above system while L_1 is a kind of integral function. We extend the Lyapunov stability for impulsive differential equations in [2] to the stability in terms of two measures for this impulsive integro-differential equations through the variation of Lyapunov second method together with the comparison theorem. Obviously, the results obtained in this paper generalize the ones in [2].

Some preliminaries are presented in Section 2 including definitions and concepts. An new comparison theorem is also given in this section, which is important to complete the main results of this paper. In Section 3, sufficient conditions for stability in terms of two measures are given for perturbed impulsive integro-differential equations with fixed moments of impulsive effect while the unperturbed one may fail to satisfy which because of the effect of the perturbed terms. An example is also worked out at the end of the paper.

2. Preliminaries

Let $R_+ = [0, +\infty)$ and R^n denotes the n-dimensional Euclidean space with appropriate norm $\|\cdot\|$.

Consider the following perturbed impulsive integro-differential equations with fixed moments of impulsive effect:

$$\begin{cases} x' = F(t, x, L_1 x), & t \neq t_k, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), \\ x(t_0^+) = x_0, & t_0 \geqslant 0,\ k \in N, \end{cases} \tag{1}$$

together with the unperturbed ones

$$\begin{cases} y' = f(t, y, L_2 y), & t \neq t_k, \\ y(t_k^+) = y(t_k) + J_k(y(t_k)), \\ y(t_0^+) = x_0, & t_0 \geqslant 0,\ k \in N, \end{cases} \tag{2}$$

where

(1) $t_0 < t_1 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$;
(2) $F, f : R_+ \times R^n \times R^n \to R^n$ are continuous on $(t_{k-1}, t_k] \times R^n \times R^n$;
(3) $L_i x = \int_{t_0}^{t} K_i(t, s, x(s))\, ds$, $K_i : R_+ \times R_+ \times R^n \to R^n$ are continuous on $(t_{k-1}, t_k] \times (t_{k-1}, t_k] \times R^n$, $i = 1, 2$;
(4) $I_k, J_k : R^n \to R^n$.

Here we note that system (2), the unperturbed system is a system with f smooth enough or even the linear terms of F in system (1). And suppose that the following hypothesis (H) holds:

(H) The solution $y(t) = y(t, t_0, x_0)$ of (2) exists for all $t \geqslant t_0$, unique, continuous with respect to the initial values and $y(t_0) = x_0$, $y(t, t_0, x_0)$ is locally Lipschitzian in x_0.

Let ρ be a real positive number and we give the following classes of functions for convenience:

$$K = \{a : [0, \rho) \to R_+ \text{ is continuous, strictly increasing and } a(0) = 0\};$$

$$PC = \{\sigma : R_+ \to R_+ \text{ is continuous on } (t_{k-1}, t_k] \text{ and } \sigma(t) \to \sigma(t_k^+) \text{ exists as } t \to t_k^+\};$$

$$PCK = \{\phi : R_+ \times [0, \rho) \to R_+,\ \phi(\cdot, u) \in PC \text{ for each } u \in [0, \rho),\ \phi(t, \cdot) \in K \text{ for each } t \in R_+\};$$

$$\Gamma = \Big\{h : R_+ \times R^n \to R_+,\ \inf_{x \in R^n} h(t, x) = 0,\ h(\cdot, x) \in PC \text{ for each } x \in R^n \text{ and } h(t, \cdot) \in C(R^n, R_+) \text{ for each } t \in R_+\Big\};$$

$$S(h, \rho) = \{(t, x) \in R_+ \times R^n:\ h(t, x) < \rho,\ h \in \Gamma\};$$

$$S(\rho) = \{x \in R^n:\ (t, x) \in S(h, \rho) \text{ for each } t \in R_+\}.$$

Definition 2.1. $V(t,x)$ belongs to V_0 if $V(\cdot,x) \in PC$ for each $x \in S(\rho)$, $V(t,x)$ is locally Lipschitzian with respect to x uniformly in t.

Definition 2.2. Let $V \in V_0$, then for any fixed $t > t_0$, we define for $(s,x) \in (t_{k-1}, t_k) \times S(\rho)$, $t_0 \leqslant s < t$,

$$\begin{aligned} &D^+V\big(s, y(t,s,x)\big) \\ &\quad = \limsup_{h\to 0^+} \frac{1}{h}\big[V\big(s+h, y\big(t, s+h, x+hF(s,x,L_1x)\big)\big) - V\big(s, y(t,s,x)\big)\big], \end{aligned}$$

where $y(t,s,x)$ is any solution of (2) such that $y(s,s,x) = x$.

Remark 2.1. Suppose $x(s) = x(s,t_0,x_0)$ is any solution of system (1) such that $x(s) \in S(\rho)$ for some certain $s \in R_+$. Then for some certain s such that $t_0 \leqslant s < t$, $s \neq t_k$ and $x = x(s)$, we have

$$\begin{aligned} D^+V\big(s, y(t,s,x)\big) &= V_s\big(s, y(t,s,x)\big) + V_y\big(s, y(t,s,x)\big) \\ &\quad \times \big[y_s(t,s,x) + y_x(t,s,x)F(s,x,L_1x)\big], \end{aligned}$$

where

$$\begin{aligned} V_s\big(s, y(t,s,x)\big) &= \limsup_{h\to 0^+} \frac{1}{h}\big[V\big(s+h, y(t,s,x)\big) - V\big(s, y(t,s,x)\big)\big], \\ V_y\big(s, y(t,s,x)\big) &= \limsup_{h\to 0^+} \frac{V(s, y(t,s+h,x+hF(s,x,L_1x))) - V(s, y(t,s,x))}{y(t,s+h,x+hF(s,x,L_1x)) - y(t,s,x)}, \\ y_s(t,s,x) &= \limsup_{h\to 0^+} \frac{1}{h}\big[y(t,s+h,x) - y(t,s,x)\big], \\ y_s(t,s,x) &= \limsup_{h\to 0^+} \frac{y(t,s,x+hF(s,x,L_1x)) - y(t,s,x)}{hF(s,x,L_1x)}. \end{aligned}$$

Further suppose that $F(t,x,L_1x) = f(t,x,L_2x) + R(t,x,Lx)$ and the solution of system (2) is differential with respect to the initial value. Then we have

$$\begin{cases} \frac{\partial y}{\partial x_0}(t,t_0,x_0) = \Phi(t,t_0,x_0), \\ \frac{\partial y}{\partial t_0}(t,t_0,x_0) = -\Phi(t,t_0,x_0)\cdot f(t_0,x_0,L_2x_0), \quad t \geqslant t_0, \end{cases}$$

where $\Phi(t,t_0,x_0)$ is the fundamental matrix solution of the corresponding variational equation. Set $V(s,y) = \|y\|^2$ and we have

$$D^+V\big(s, y(t,s,x)\big) = 2y^T(t,s,x)\cdot\Phi(t,s,x)\cdot R(s,x,Lx),$$

which shows how the perturbation terms affect the stability properties of the perturbed system.

Definition 2.3. Let $h_0, h \in \Gamma$, then

(I) h_0 is finer than h if there exits a $\lambda^* > 0$ and a function $\phi \in PCK$ such that

$$h_0(t,x) < \lambda^* \quad \text{implies} \quad h(t,x) \leqslant \phi\big(t, h_0(t,x)\big);$$

(II) h_0 is uniformly finer than h if (I) holds with $\phi \in K$.

Definition 2.4. Let $V \in V_0$ and $h, h_0 \in \Gamma$, then $V(t,x)$ is said to be

(i) h-positive definite if there exists a $\lambda > 0$ and a function $b \in K$ such that

$$h(t,x) < \lambda \quad \text{implies} \quad b\big(h(t,x)\big) \leqslant V(t,x);$$

(ii) weakly h_0-decrescent if there exists a $\lambda_0 > 0$ and a function $a \in PCK$ such that

$$h_0(t,x) < \lambda_0 \quad \text{implies} \quad V(t,x) \leqslant a\big(t, h_0(t,x)\big);$$

(iii) h_0-decrescent if (ii) holds with $a \in K$.

Definition 2.5. Let $h_0, h \in \Gamma$ and $x(t) = x(t, t_0, x_0)$ be any solution of (1), then system (1) is said to be

(S_1) (h_0, h)-stable if for each $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, x_0) < \delta \quad \text{implies} \quad h\big(t, x(t)\big) < \varepsilon, \quad t \geqslant t_0;$$

(S_2) (h_0, h)-uniformly stable if (S_1) holds with δ independent of t_0;

(S_3) (h_0, h)-attractive if there exists a $\delta_0 = \delta_0(t_0) > 0$ and for each $\varepsilon > 0$, there exists $T = T(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, x_0) < \delta_0 \quad \text{implies} \quad h\big(t, x(t)\big) < \varepsilon, \quad t \geqslant t_0 + T;$$

(S_4) (h_0, h)-uniformly attractive if (S_3) holds with δ and T independent of t_0;

(S_5) (h_0, h)-asymptotically stable if it is (h_0, h)-stable and (h_0, h)-attractive;

(S_6) (h_0, h)-uniformly asymptotically stable if it is (h_0, h)-uniformly stable and (h_0, h)-uniformly attractive.

Remark 2.2. When we endow h_0, h with explicit form, the (h_0, h)-stability reduces to the other stability such as

(1) set $h_0(t,x) = h(t,x) = \|x\|$, then (h_0, h)-stability means the corresponding Lyapunov stability of the trivial solution;

(2) set $h_0(t,x) = h(t,x) = \|x - x^*\|$, then (h_0, h)-stability means the corresponding Lyapunov stability of solution x^*;

(3) set $h_0(t,x) = \|x\|$, $h(t,x) = \|x\|_s$, $1 \leqslant s < n$, then (h_0, h)-stability means the corresponding partial stability of the trivial solution;

(4) set $h_0(t,x) = h(t,x) = d(x, A)$, where $A \in R^n$, then (h_0, h)-stability means the corresponding stability of an invariant set A;

(5) set $h_0(t,x) = d(x, A)$, $h(t,x) = d(x, B)$, where $A \subset B \subset R^n$, then (h_0, h)-stability means the corresponding stability of a conditionally invariant set B with respect to A.

In the following we always suppose that $x(t) = x(t, t_0, x_0)$, $y(t) = y(t, t_0, x_0)$ are the solutions of (1) and (2) such that $x(t_0) = x_0$, $y(t_0) = y_0$, respectively.

Next, a comparison principle is presented which is necessary for completing our main results.

Lemma 2.1. *Suppose that* (H) *holds and*

(i) $V \in V_0$ *satisfies the inequalities for* $(s, x) \in S(h, \rho)$, $t_0 \leqslant s < t$,

$$\begin{cases} D^+V(s, y(t, s, x)) \leqslant g(s, V(s, y(t, s, x))), \quad t \neq t_k, \\ V(t_k^+, y(t, t_k^+, x(t_k^+))) \leqslant \psi_k(V(t_k, y(t, t_k, x(t_k)))), \\ V(t_0^+, y(t, t_0^+, x_0)) \leqslant u_0, \end{cases}$$

where $g(\cdot, u) \in PC$ *for each* $u \in R_+$ *and* $\psi_k : R_+ \to R_+$ *are nondecreasing functions for all* $k \in N$;

(ii) $r(t) = r(t, t_0, u_0)$ *is the maximal solution of the following scalar impulsive differential equation*

$$\begin{cases} u' = g(t, u), \quad t \neq t_k, \\ u(t_k^+) = \psi_k(u(t_k)), \\ u(t_0^+) = u_0 \geqslant 0, \end{cases} \tag{3}$$

existing on $[t_0, +\infty)$.

Then we have

$$V\big(t, x(t, t_0, x_0)\big) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0.$$

Proof. Denote $x(t) = x(t, t_0, x_0)$ any solution of system (1) satisfying $(t_0, x_0) \in S(h, \rho)$. Set

$$m(s) = V\big(s, y\big(t, s, x(s)\big)\big), \quad \text{for } t_0 \leqslant s \leqslant t,$$

where $m(t) = \lim_{s \to t-0} m(s)$. Thus we have

$$D^+m(s) \leqslant g\big(s, m(s)\big), \quad t \neq t_k,$$
$$m\big(t_k^+\big) \leqslant \psi_k\big(m(t_k)\big),$$
$$m(t_0) \leqslant u_0, \quad k = 1, 2, \ldots.$$

It follows from [6] that $m(s) \leqslant r(s, t_0, u_0)$ for $t_0 \leqslant s \leqslant t$, which implies that

$$V\big(s, y\big(t, s, x(s)\big)\big) \leqslant r(s, t_0, u_0), \quad t_0 \leqslant s \leqslant t.$$

Notice that $y(t, t, x(t)) = x(t)$ and we have

$$V\big(t, x(t, t_0, x_0)\big) = V\big(t, y\big(t, t, x(t)\big)\big) \leqslant r(t, t_0, u_0).$$

So the proof is complete. □

Remark 2.3. u_i $(i = 1, 2)$ are two different initial values, then from Lemma 2.1, we have

$$r(t, t_0, u_1) \leqslant r(t, t_0, u_2), \quad \text{if } u_1 \leqslant u_2. \tag{4}$$

Also when $g(t, u)$ and $\psi_k(u)$ are special (see [2]), we can get some explicit comparison results which we omit here.

3. Stability criteria

Theorem 3.1. *Suppose that* (H) *holds and*

(A$_1$) $f(t, 0) = 0$, $g(t, 0) = 0$ *and* $J_k(0) = 0$, $\psi_k(0) = 0$ *for all* $k \in N$;
(A$_2$) $h_0, h \in \Gamma$, $h_0(t, 0) = 0$ *for* $t \in R_+$, h_0 *is finer than* h;
(A$_3$) $V \in V_0$, $V(t, x)$ *is* h*-positive definite and weakly* h_0*-decrescent for* $(t, x) \in S(h, \rho)$, *and*

$$D^+V\big(s, y(t, s, x)\big) \leqslant g\big(s, V\big(s, y(t, s, x)\big)\big),$$
$$\text{for } s \neq t_k,\ (s, x) \in S(h, \rho),\ t_0 \leqslant s < t;$$

(A$_4$) $V(t_k^+, y(t, t_k^+, x(t_k^+))) \leqslant \psi_k(V(t_k, y(t, t_k, x(t_k))))$;
(A$_5$) *there exists a* $\rho_0 \in (0, \rho]$ *such that*

$$h\big(t_k, x(t_k)\big) < \rho_0 \quad \text{implies} \quad h\big(t_k^+, x\big(t_k^+\big)\big) < \rho, \quad k = 1, 2, \ldots.$$

Then the stability of the trivial solution of system (2) *and the* (*asymptotical*) *stability of the trivial solution of* (3) *imply the* (h_0, h)-(*asymptotical*) *stability of system* (1).

Proof. Note that $x(t) = x(t, t_0, x_0)$, $y(t) = y(t, t_0, x_0)$, $u(t) = u(t, t_0, u_0)$ are any solutions of system (1), (2) and (3), respectively.

Since $V(t, x)$ is h-positive definite on $S(h, \rho)$, there exists a $b \in K$ such that

$$h(t, x) < \rho \quad \text{implies} \quad b\big(h(t, x)\big) \leqslant V(t, x). \tag{5}$$

Also $V(t, x)$ is weakly h_0-decrescent and h_0 is finer than h, so there exists a $\lambda_0 > 0$ and $a \in PCK$, $\phi \in PCK$ such that

$$h(t, x) \leqslant \phi\big(t, h_0(t, x)\big) \quad \text{and} \quad V(t, x) \leqslant a\big(t, h_0(t, x)\big), \quad \text{when } h_0(t, x) < \lambda_0, \tag{6}$$

where λ_0 is such that $\phi(t_0^+, \lambda_0) < \rho$.

Let $0 < \varepsilon < \rho_0$ and $t_0 \in R_+$ be given. Since the trivial solution of (3) is stable, for given $b(\varepsilon) > 0$, there exists a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that

$$0 < u_0 \leqslant \delta_1 \quad \text{implies} \quad u(t, t_0, u_0) < b(\varepsilon), \quad t \geqslant t_0. \tag{7}$$

While the trivial solution of (2) is also stable, so for this δ_1, there exists a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that

$$\|x_0\| < \delta_2 \quad \text{implies} \quad \big\|y(t)\big\| < a^{-1}(t_0, \delta_1),$$

while from condition (A_2), without loss of generality, we have

$$h_0(t_0^+, x_0) < \delta_2 \quad \text{implies} \quad h_0(t_0^+, y(t)) < a^{-1}(t_0, \delta_1). \tag{8}$$

Choosing $\delta = \delta(t_0, \varepsilon) > 0$ such that $\delta < \min\{\lambda_0, \delta_2\}$, then it follows from (5)–(8) that if $h_0(t_0^+, x_0) < \delta$,

$$b(h(t_0^+, x_0)) \leqslant V(t_0^+, x_0) \leqslant a(t_0^+, h_0(t_0^+, x_0)) < a(t_0^+, \delta_2) \leqslant \delta_1 \leqslant b(\varepsilon).$$

Which implies that $h(t_0^+, x_0) < \varepsilon$ when $h_0(t_0^+, x_0) < \delta$. We claim that

$$h(t, x(t)) < \varepsilon, \quad \text{whenever } h_0(t_0^+, x_0) < \delta. \tag{9}$$

In fact, if (9) is false, there exists $t^* > t_0$ such that $h(t^*, x(t^*)) \geqslant \varepsilon$. For $h \in \Gamma$, we have two cases:

Case I: $t_0 < t^* \leqslant t_1$. Without loss of generality we suppose that $t^* = \inf\{t\colon h(t, x(t)) \geqslant \varepsilon\}$ and so $h(t^*, x(t^*)) = \varepsilon$. From Lemma 2.1, (4) and (7) we have

$$\begin{aligned} V(t^*, x(t^*)) &\leqslant r(t^*, t_0, V(t_0^+, y(t^*, t_0, x_0))) \leqslant r(t^*, t_0, a(t_0, h_0(t_0^+, y(t^*, t_0, x_0)))) \\ &\leqslant r(t^*, t_0, \delta_1) < b(\varepsilon). \end{aligned}$$

On the other hand, from (5) we have

$$V(t^*, x(t^*)) \geqslant b(h(t^*, x(t^*))) = b(\varepsilon),$$

which is a contradiction.

Case II: $t_k < t^* \leqslant t_{k+1}$ *for some* $k \in N$. In this case, noticing the impulse effect, we have

$$h(t^*, x(t^*)) \geqslant \varepsilon \quad \text{and} \quad h(t, x(t)) < \varepsilon, \quad t \in [t_0, t_k].$$

Since $0 < \varepsilon < \rho_0$, it follows from condition (A_5) that

$$h(t_k^+, x(t_k^+)) = h(t_k^+, x(t_k) + I_k(x)) < \rho,$$

and so there exists $\tilde{t} \in (t_k, t^*]$ such that

$$\varepsilon \leqslant h(\tilde{t}, x(\tilde{t})) < \rho \quad \text{and} \quad h(t, x(t)) < \rho, \quad t \in [t_0, \tilde{t}). \tag{10}$$

By using Lemma 2.1 and (7), we have

$$\begin{aligned} V(\tilde{t}, x(\tilde{t})) &\leqslant r(\tilde{t}, t_0, V(t_0^+, y(\tilde{t}, t_0, x_0))) \leqslant r(\tilde{t}, t_0, a(t_0, h_0(t_0^+, y(\tilde{t}, t_0, x_0)))) \\ &\leqslant r(\tilde{t}, t_0, \delta_1) < b(\varepsilon). \end{aligned}$$

On the contrary, from (5) and (10) we have $V(\tilde{t}, x(\tilde{t})) \geqslant b(h(\tilde{t}, x(\tilde{t}))) \geqslant b(\varepsilon)$, which is also a contradiction. Thus the claim is true for proving the (h_0, h)-stability of system (1).

Next suppose further that the trivial solution of (3) is asymptotically stable. From above we have the (h_0, h)-stability of system (1). Consequently from (9), taking $\varepsilon = \rho_0$, there exists a $\delta^* = \delta^*(t_0, \rho_0) > 0$ such that

$$h_0(t_0^+, x_0) < \delta^* \quad \text{implies} \quad h(t, x(t)) < \rho_0 < \rho, \quad t \geqslant t_0.$$

To prove the (h_0, h)-attractive of system (1), let $t_0 \in R_+$. The trivial solution of (3) is attractive, so for $t_0 \in R_+$ there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that

$$u_0 \leqslant \delta_0^* \quad \text{implies} \quad \lim_{t\to\infty} u(t, t_0, u_0) = 0.$$

For this δ_0^*, there exists a $\delta_1^* = \delta_1^*(t_0, \delta_0^*) > 0$ such that

$$h_0\big(t_0^+, x_0\big) < \delta_1^* \quad \text{implies} \quad h_0\big(t_0^+, y(t, t_0, x_0)\big) < a^{-1}\big(t_0, \delta_0^*\big).$$

Choosing $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$, and it is obviously that $\delta_0 = \delta_0(t_0)$ independent of ε, then by similar argument to the above, we can get that when $h_0(t_0^+, x_0) < \delta_0$ and as $t \to \infty$

$$b\big(h\big(t, x(t)\big)\big) \leqslant V\big(t, x(t)\big) \leqslant r\big(t, t_0, V\big(t_0^+, y(t, t_0, x_0)\big)\big) \leqslant r\big(t, t_0, \delta_0^*\big) \to 0,$$

which implies that $\lim_{t\to\infty} h(t, x(t)) = 0$ when $h_0(t_0^+, x_0) < \delta_0$, that is, system (1) is (h_0, h)-attractive. Hence it follows that the system (1) is (h_0, h)-asymptotically stable. □

Remark 3.1. Set $h_0(t, x) \equiv h(t, x) \equiv \|x\|$, then we can get the (asymptotical) stability of the trivial solution of system (1), if further set $L_1 x \equiv L_2 x \equiv 0$, we can get the results in [2].

Strengthen certain assumptions of Theorem 3.1 and we can obtain the uniform stability criteria of the perturbed system (1).

Theorem 3.2. *Assume that the conditions in Theorem* 3.1 *hold except that*

(A_6) *just replacing h_0 is finer than h with h_0 is uniformly finer than h in* (A_2);
(A_7) *just replacing V is weakly h_0-decrescent with V is h_0-decrescent in* (A_3).

Then the uniform stability of the trivial solution of system (2) *and the uniformly* (*asymptotical*) *stability of the trivial solution of* (3) *imply the* (h_0, h)*-uniformly* (*asymptotical*) *stability of system* (1).

Proof. Since $V(t, x)$ is h_0-decrescent and h_0 is uniformly finer than h, there exists a $\lambda_0 > 0$ and $a \in K$, $\phi \in K$ such that

$$h(t, x) \leqslant \phi\big(h_0(t, x)\big) \quad \text{and} \quad V(t, x) \leqslant a\big(h_0(t, x)\big), \quad \text{when } h_0(t, x) < \lambda_0, \tag{11}$$

where λ_0 is such that $\phi(\lambda_0) < \rho$. Let $0 < \varepsilon < \rho_0$ and $t_0 \in R_+$ be given. The trivial solution of (3) is uniformly stable, then for given $b(\varepsilon) > 0$, there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ independent of t_0 such that

$$0 < u_0 < \delta_1 \quad \text{implies} \quad u(t, t_0, u_0) < b(\varepsilon), \quad t \geqslant t_0, \tag{12}$$

where b is the same as above. The trivial solution of (12) is also uniformly stable, then for this δ_1, there exists a $\delta_2 > 0$ independent of t_0 such that

$$h_0\big(t_0^+, x_0\big) < \delta_2 \quad \text{implies} \quad h_0\big(t_0^+, y(t)\big) < a^{-1}(\delta_1). \tag{13}$$

Choosing δ such that $0 < \delta = \delta(\varepsilon) < \min\{\lambda_0, \delta_2\}$. Then with a similar argument to Theorem 3.1, we can conclude that

$$h\big(t_0^+, x_0\big) < \delta \quad \text{implies} \quad h\big(t, x(t)\big) < \varepsilon, \quad t \geqslant t_0,$$

where δ is independent of t_0, so the system (1) is (h_0, h)-uniformly stable.

If further suppose that the trivial solution of (3) is uniformly asymptotically stable, there exists a $\delta_0^* > 0$ independent of t_0 and for any given $\varepsilon \in (0, \rho_0)$ there exists a $T = T(\varepsilon)$ such that for any $t_0 \in R_+$,

$$0 < u_0 < \delta_0^* \quad \text{implies} \quad u(t, t_0, u_0) < b(\varepsilon), \quad t \geqslant t_0 + T(\varepsilon). \tag{14}$$

Noticing that (2) is uniformly stable, so for this δ_0^*, there exists a $\delta_1^* > 0$ independent of t_0 such that

$$h_0\big(t_0^+, x_0\big) < \delta_1^* \quad \text{implies} \quad h_0\big(t_0^+, y(t, t_0, x_0)\big) < a^{-1}\big(\delta_0^*\big).$$

Uniformly asymptotically stability of system (3) implies its asymptotically stability. So system (1) is (h_0, h)-uniformly stable. For $\varepsilon = \rho_0$, there exists a $\delta^* = \delta^*(\rho_0)$ such that

$$h_0\big(t_0^+, x_0\big) < \delta^* \quad \text{implies} \quad h\big(t, x(t)\big) < \rho_0 < \rho, \quad t \geqslant t_0. \tag{15}$$

Choosing δ such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$, with a similar argument to Theorem 3.1, we can get that when $h_0(t_0^+, x_0) < \delta_0$,

$$h\big(t, x(t)\big) < \varepsilon, \quad t \geqslant t_0 + T,$$

where δ_0 and T are independent of t_0, that is, system (1) is uniformly attractive.

So system (1) is (h_0, h)-uniformly asymptotically stable. □

4. Example

In this section, we present a simple but an illustrative example. Consider the perturbed impulsive integro-differential equations

$$\begin{cases} x_1' = e^{-t}x_1^3 + \frac{1}{2}x_1x_2^2\int_{t_0}^t F_1(t, u, x_1(u))\,du + \frac{1}{2}x_1^3, & t \neq t_k; \\ x_2' = e^{-t}x_2^3 + \frac{1}{2}x_1^2x_2\int_{t_0}^t F_2(t, u, x_2(u))\,du + \frac{1}{2}x_2^3, & t \neq t_k; \\ x_1(t_k^+) = d_1x_1(t_k), \quad x_1(t_0) = x_{10} \geqslant 0; \\ x_2(t_k^+) = d_2x_2(t_k), \quad x_2(t_0) = x_{20} \geqslant 0, \quad k = 1, 2, \ldots, \end{cases} \tag{16}$$

where $\int_{t_0}^s F_i(t, u, x_i(u))\,du \leqslant 0$, for any $t_0 \leqslant s < t$, $i = 1, 2$, and $|d_1| \leqslant 1$, $|d_2| \leqslant 1$.

Here we consider the unperturbed system without impulse

$$\begin{cases} y_1' = e^{-t}y_1^3, \quad y_1(t_0) = x_{10}; \\ y_2' = e^{-t}y_2^3, \quad y_1(t_0) = x_{20}. \end{cases} \tag{17}$$

By direct calculation, we have the solution of (17) given by

$$y(t, t_0, x_0) = \begin{pmatrix} y_1(t, t_0, x_{10}) \\ y_2(t, t_0, x_{20}) \end{pmatrix} = \begin{pmatrix} \frac{x_{10}}{[1+2x_{10}^2(e^{-t}-e^{-t_0})]^{1/2}} \\ \frac{x_{20}}{[1+2x_{20}^2(e^{-t}-e^{-t_0})]^{1/2}} \end{pmatrix},$$

which exists for all $t \geqslant t_0$ such that $\|x_0\| < \sqrt{e^{t_0}/2}$ $(x_0 = (x_{10}, x_{20})^T)$ and the fundamental matrix solution of the corresponding variational equations is

$$\Phi(t, t_0, x_0) = \begin{pmatrix} \frac{1}{[1+2x_{10}^2(e^{-t}-e^{-t_0})]^{3/2}} & 0 \\ 0 & \frac{1}{[1+2x_{20}^2(e^{-t}-e^{-t_0})]^{3/2}} \end{pmatrix}.$$

Set $V(t,x)=\|x\|^2=x_1^2+x_2^2$ and $h_0(t,x)=h(t,x)=\|x\|=(x_1^2+x_2^2)^{1/2}$. It is obvious that V is differentiable so we have

$$\begin{aligned}
&D^+V\bigl(s,y(t,s,x)\bigr)\\
&\quad=2y^T(t,s,x)\Phi(t,s,x)R(s,x,Lx)\\
&\quad=\frac{x_1^2(s)}{[1+2x_1^2(s)(e^{-t}-e^{-s})]^2}\left(x_2^2\int_{t_0}^{s}F_1\bigl(t,u,x_1(u)\bigr)\,du+x_1^2(s)\right)\\
&\qquad+\frac{x_2^2(s)}{[1+2x_2^2(s)(e^{-t}-e^{-s})]^2}\left(x_1^2\int_{t_0}^{s}F_2\bigl(t,u,x_2(u)\bigr)\,du+x_2^2(s)\right)\\
&\quad\leqslant\frac{x_1^4(s)}{[1+2x_1^2(s)(e^{-t}-e^{-s})]^2}+\frac{x_2^4(s)}{[1+2x_2^2(s)(e^{-t}-e^{-s})]^2}\\
&\quad\leqslant V\bigl(s,y(t,s,x)\bigr)^2;\\
&V\bigl(t_k^+,y\bigl(t,t_k^+,x\bigl(t_k^+\bigr)\bigr)\bigr)\\
&\quad=\frac{d_1^2x_1^2(t_k)}{1+2d_1^2x_1^2(t_k)(e^{-t}-e^{-t_k})}+\frac{d_2^2x_2^2(t_k)}{1+2d_2^2x_2^2(t_k)(e^{-t}-e^{-t_k})}\\
&\quad\leqslant d^2V\bigl(t_k,y\bigl(t,t_k,x(t_k)\bigr)\bigr),
\end{aligned}$$

where $d=\max\{|d_1|,|d_2|\}$.

Then the comparison equation is given as follows:

$$\begin{cases}u'=u^2, & t\neq t_k,\\ u(t_k^+)=d^2u(t_k),\\ u(t_0^+)=u_0, & t_0\geqslant 0,\ k\in N.\end{cases}\tag{18}$$

It is easy to get that Eq. (18) is stable. So from Theorem 3.1, we can conclude that if $\|y(t,t_0,x_0)\|\leqslant u_0$, the impulsive integro-differential system (16) is stable.

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