Maximally non-abelian vortices from self-dual Yang–Mills fields

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A B S T R A C T

A particular dimensional reduction of $SU(2N)$ Yang–Mills theory on $\Sigma \times S^2$, with Σ a Riemann surface, yields an $S(U(N) \times U(N))$ gauge theory on Σ, with a matrix Higgs field. The $SU(2N)$ self-dual Yang–Mills equations reduce to Bogomolny equations for vortices on Σ. These equations are formally integrable if Σ is the hyperbolic plane, and we present a subclass of solutions.

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1. Introduction

The generalization of abelian Higgs vortices to the non-abelian case has recently gained much attention [1–3]. There are many variants of non-abelian vortices, and in this Letter we shall investigate one of these, one that has not been explicitly investigated before, but which has a mathematically elegant and symmetric structure. All these types of vortices satisfy static, first order Bogomolny equations, defined in two-dimensional space. Vortices are most commonly studied on the plane $\mathbb{R}^2$, but the Bogomolny equations are not integrable there. The vortex equations on the hyperbolic plane $\mathbb{H}^2$ are, however, integrable [4–6]. The reason is that these vortex equations arise by dimensional reduction of the self-dual Yang–Mills equations on $\mathbb{H}^2 \times S^2$, where the curvatures on $\mathbb{H}^2$ and the 2-sphere S^2 are opposite; moreover there is a conformal equivalence $\mathbb{H}^2 \times S^2 \cong \mathbb{R}^4 - \mathbb{R}^1$, and self-dual Yang–Mills is both conformally invariant, and integrable on $\mathbb{R}^4$. The vortex equations on $\mathbb{R}^2$ also arise by dimensional reduction of self-dual Yang–Mills, this time on $\mathbb{R}^2 \times S^2$, but here there is no integrability. Solutions exist despite this, but they are transcendental, and their existence has to be established by methods of analysis, or numerics [7].

The dimensional reduction leading from self-dual Yang–Mills fields to vortices arises by imposing spherical symmetry (i.e. $SO(3)$ symmetry) on the gauge field over the S^2 factor of a Riemannian product 4-manifold $\Sigma \times S^2$, where Σ is a Riemann surface. The resulting vortex equations are on Σ. Since $SO(3)$ is non-abelian, the dimensional reduction is non-trivial, and there are various possible outcomes. Spherically symmetric $SU(2)$ gauge fields were first presented in the 1970s in the context of monopoles and instantons. A systematic understanding was achieved by Romanov et al. [8,9], and a more general overview of symmetric gauge fields was given in Ref. [10]. The mathematical basis for this can be traced back to the earlier theorem of Wang [11], but the later work incorporated dynamical aspects like the Yang–Mills action and field equations.

We will briefly review the general structure of $SO(3)$-symmetric pure Yang–Mills fields with gauge group $\mathcal{G}$ on $\Sigma \times S^2$, and show that the dimensionally reduced self-dual Yang–Mills equations are Bogomolny equations for vortices on Σ, with a gauge group G that is a subgroup of $\mathcal{G}$. We then focus on an example where G is a particularly large subgroup of $\mathcal{G}$. Here $\mathcal{G} = SU(2N)$ and $G = S(U(N) \times U(N))$. This is at the opposite extreme from another well-studied case, where G is particularly small, namely $G = U(1)^{2N-1}$ [12–14].

The Bogomolny equations on Σ involve a G-gauge potential and also Higgs fields. The latter arise from the components of the original $\mathcal{G}$-gauge potential tangent (more accurately, co-tangent) to S^2. In our example, the Higgs field is a complex $N \times N$ matrix, gauge transforming from the left and right by the two $U(N)$ factors of G. Our example is therefore closely related to the well-known non-abelian vortex equations with an $N_c \times N_f$ matrix of Higgs fields, where there is a "colour" $U(N_c)$ gauge group acting from the left, and a "flavour" $SU(N_f)$ global symmetry group acting from the right. These colour–flavour theories arise naturally in supersym-

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metric gauge theories with eight supercharges [15]. It is usually assumed that $N_f \geqslant N_c$, to have a vacuum solution of zero energy, where the colour and the flavour are locked together.

We will present our Bogomolny equations for both $\Sigma = \mathbb{R}^2$ and $\Sigma = \mathbb{H}^2$. One Bogomolny equation implies that in a certain sense the Higgs field is holomorphic. The free parameters of the holomorphic Higgs field are the moduli of the vortex solutions. The other Bogomolny equations then reduce to gauge-invariant "master equations", a generalization of Taubes' equation for abelian vortices [7]. It is expected that the master equations have unique solutions once the holomorphic Higgs field is fixed. In the hyperbolic case, $\Sigma = \mathbb{H}^2$, the master equations simplify, and are formally completely integrable. However, we have not found a general explicit solution satisfying the boundary conditions. We do show, however, that the explicitly known hyperbolic abelian vortices, found by Witten [4], can be embedded as solutions in the non-abelian system. These embedded abelian vortices are intrinsically non-abelian, in the same sense as the well-known non-abelian vortices in the Higgs phase [1–3].

More general explicit solutions could emerge from an application of the formulae of Leznov and Saveliev [5]. These rely on a good understanding of the structure of the gauge groups, but appear not to incorporate boundary conditions. The twistor approach of Popov could be useful, but so far has not yielded explicit solutions [6]. More promising, possibly, is the recent work of Manton and Rink, in which hyperbolic abelian vortices are constructed in a purely geometrical way, reproducing Witten's solutions and also giving novel solutions on surfaces Σ, other than $\mathbb{H}^2$, that have a hyperbolic metric [16]. Finding a non-abelian generalization of this approach would be useful and interesting.

2. Self-duality and Bogomolny equations

Bogomolny equations for vortices on a Riemann surface Σ arise naturally by dimensional reduction of the self-dual Yang–Mills equations on $\Sigma \times S^2$. Let z be a complex coordinate on Σ, and y the standard complex coordinate on S^2 obtained by stereographic projection (so that $y = \tan\frac{\theta}{2} e^{i\varphi}$ with θ, φ usual polar coordinates). The metric on $\Sigma \times S^2$ is taken to be

$$ds^2 = \sigma(z, \bar{z})\, dz\, d\bar{z} + \frac{8}{(1 + y\bar{y})^2}\, dy\, d\bar{y}. \tag{1}$$

σ is a generic conformal factor on Σ, and the second term describes a 2-sphere of fixed radius $\sqrt{2}$ and Gauss curvature $\frac{1}{2}$.

Let the gauge group be $\mathcal{G}$, a compact Lie group with Lie algebra $\mathfrak{g}$, whose complexification is $\mathfrak{g}^*$. The Yang–Mills gauge potential has components $\mathcal{A}_z$, $\mathcal{A}_{\bar{z}}$, $\mathcal{A}_y$, $\mathcal{A}_{\bar{y}}$ with values in $\mathfrak{g}^*$, but $\mathcal{A}_z + \mathcal{A}_{\bar{z}}$ and $i(\mathcal{A}_z - \mathcal{A}_{\bar{z}})$, being components in real directions, must be in $\mathfrak{g}$ itself,[1] and similarly for $\mathcal{A}_y$, $\mathcal{A}_{\bar{y}}$.

We now suppose that the gauge potential is $SO(3)$-invariant over the 2-sphere, S^2. $SO(3)$ does not act freely on S^2. The isotropy group at each point of S^2 (the subgroup keeping that point fixed) is $SO(2)$. Let us focus on the particular point $y = 0$, and its $SO(2)$ isotropy group. For the gauge potential to be "invariant" at $y = 0$ and its infinitesimal neighbourhood, we mean that it is invariant under a combined $SO(2)$ rotation and gauge transformation. To define the gauge transformation, we must identify a subgroup $SO(2)_{\mathcal{G}}$ in $\mathcal{G}$ (which can be chosen to be constant over Σ). Let the generator of $SO(2)_{\mathcal{G}}$ be denoted by Λ, such that in the adjoint rep-

resentation of $\mathcal{G}$, $\exp(2\pi\Lambda)$ is the identity. The combined action of $SO(2)$ then consists of rotations by α combined with gauge transformations by $\exp(\alpha\Lambda)$, and the gauge potential must be invariant under this. Having chosen this lift of the $SO(2)$-action at $y = 0$, one can show that the notion of an $SO(3)$-invariant gauge potential over $\Sigma \times S^2$ is completely fixed, and in a convenient choice of gauge, the general invariant gauge potential on $\Sigma \times S^2$ is given by the formulae [9,17,6]

$$\mathcal{A}_z = \mathbf{A}_z(z, \bar{z}), \tag{2}$$

$$\mathcal{A}_{\bar{z}} = \mathbf{A}_{\bar{z}}(z, \bar{z}), \tag{3}$$

$$\mathcal{A}_y = \frac{1}{1 + y\bar{y}}\left(-\Phi(z, \bar{z}) - i\Lambda\bar{y}\right), \tag{4}$$

$$\mathcal{A}_{\bar{y}} = \frac{1}{1 + y\bar{y}}\left(\bar{\Phi}(z, \bar{z}) + i\Lambda y\right). \tag{5}$$

Here, the dependence on z and $\bar{z}$ is arbitrary, but the dependence on y and $\bar{y}$ is as shown. In addition, there are linear constraints, arising from the $SO(2)$ invariance at $y = 0$, namely

$$[\Lambda, \mathbf{A}_z] = [\Lambda, \mathbf{A}_{\bar{z}}] = 0, \tag{6}$$

$$[\Lambda, \Phi] = -i\Phi, \qquad [\Lambda, \bar{\Phi}] = i\bar{\Phi}. \tag{7}$$

The interpretation of these constraints is that $\mathbf{A}_z$, $\mathbf{A}_{\bar{z}}$ are components of a gauge potential on Σ for the gauge group G which is the centralizer of $SO(2)_{\mathcal{G}}$ in $\mathcal{G}$. Also, Φ, $\bar{\Phi}$ are scalar Higgs fields on Σ which must lie in the $\mp i$ eigenspaces of ad Λ in $\mathfrak{g}^*$. These eigenspaces are representation spaces for G, so Φ, $\bar{\Phi}$ are Higgs fields transforming under these representations of G.

The self-dual Yang–Mills equations on $\Sigma \times S^2$, with metric (1) and gauge group $\mathcal{G}$, are

$$\frac{8}{(1 + y\bar{y})^2}\mathcal{F}_{z\bar{z}} = \sigma\mathcal{F}_{y\bar{y}}, \tag{8}$$

$$\mathcal{F}_{z\bar{y}} = 0, \tag{9}$$

$$\mathcal{F}_{\bar{z}y} = 0, \tag{10}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$ for any coordinate indices μ, ν. Substituting the $SO(3)$-invariant fields (2)–(5) into this set of equations yields

$$\mathbf{F}_{z\bar{z}} = \frac{\sigma}{8}\left(2i\Lambda - [\Phi, \bar{\Phi}]\right), \tag{11}$$

$$D_z\bar{\Phi} = 0, \tag{12}$$

$$D_{\bar{z}}\Phi = 0, \tag{13}$$

where $\mathbf{F}_{z\bar{z}} = \partial_z\mathbf{A}_{\bar{z}} - \partial_{\bar{z}}\mathbf{A}_z + [\mathbf{A}_z, \mathbf{A}_{\bar{z}}]$, $D_z\bar{\Phi} = \partial_z\bar{\Phi} + [\mathbf{A}_z, \bar{\Phi}]$ and $D_{\bar{z}}\Phi = \partial_{\bar{z}}\Phi + [\mathbf{A}_{\bar{z}}, \Phi]$. It is consistent to interpret these as unconstrained Bogomolny equations with gauge group G, and this is seen explicitly if the linear constraints (6) and (7) are solved. For example, both left- and right-hand sides of (11) are in the zero eigenspace of ad Λ, which is the Lie algebra of G.

We have so far presented the most general type of $SO(3)$-invariant gauge field. There are two related reasons to restrict the choice of Λ. The first comes from requiring that the vortex solutions of the Bogomolny equations have finite energy. If Σ has infinite area, as $\mathbb{R}^2$ and $\mathbb{H}^2$ do, then approaching infinity (the boundary of Σ), the solution must approach the vacuum. This means that $\mathbf{F}_{z\bar{z}} = 0$ there, and hence

$$2i\Lambda - [\Phi, \bar{\Phi}] = 0. \tag{14}$$

If we denote the vacuum values of Φ, $\bar{\Phi}$ by Φ_0, $\bar{\Phi}_0$ respectively, then, combining (14) and the constraints (7), we have

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[1] More explicitly, if $\mathcal{G}$ is a group of unitary matrices, with a Lie algebra $\mathfrak{g}$ of anti-hermitian matrices, then $\mathcal{A}_z + \mathcal{A}_{\bar{z}} = -(\mathcal{A}_z + \mathcal{A}_{\bar{z}})^\dagger$ and $\mathcal{A}_z - \mathcal{A}_{\bar{z}} = (\mathcal{A}_z - \mathcal{A}_{\bar{z}})^\dagger$. So $\mathcal{A}_z$ and $\mathcal{A}_{\bar{z}}$ are not in general anti-hermitian, but by adding or subtracting these equations we see that $\mathcal{A}_{\bar{z}} = -\mathcal{A}_z^\dagger$.

$$[\Lambda, \Phi_0] = -i\Phi_0, \qquad [\Lambda, \bar{\Phi}_0] = i\bar{\Phi}_0, \tag{15}$$

$$[\Phi_0, \bar{\Phi}_0] = 2i\Lambda. \tag{16}$$

In other words, the elements Λ, Φ_0, $\bar{\Phi}_0$ generate an $SO(3)$ subgroup of $\mathcal{G}$, which we denote by $SO(3)_{\mathcal{G}}$. The $SO(2)_{\mathcal{G}}$ subgroup generated by Λ is therefore not arbitrary, but must extend to $SO(3)_{\mathcal{G}}$.

The related reason for restricting Λ applies in the case that $\Sigma = \mathbb{H}^2$. Consider the action of $SO(3)$ on $\mathbb{R}^4 = \mathbb{R}^1 \times \mathbb{R}^3$. It acts in the standard way on the $\mathbb{R}^3$ factor, with 2-spheres as generic orbits. The conformal equivalence $\mathbb{H}^2 \times S^2 \cong \mathbb{R}^4 - \mathbb{R}^1$ arises from the manipulation of the $\mathbb{R}^4$ metric,

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{17}$$

$$\cong \frac{2}{r^2}(d\tau^2 + dr^2) + 2(d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{18}$$

The first factor in (18) is the metric on $\mathbb{H}^2$ in the upper-half-plane model, with $r > 0$, and the Gauss curvature is $-\frac{1}{2}$. In terms of the complex coordinate[2] $z = \tau + ir$, the metric is $\frac{2}{(\mathrm{Im}\,z)^2} dz\,d\bar{z}$. Now notice that the τ-axis of $\mathbb{R}^4$, where $r = 0$, is excluded here. This is the excluded $\mathbb{R}^1$, and it is the boundary of $\mathbb{H}^2$. To have well-defined $SO(3)$-invariant, self-dual Yang–Mills fields on all of $\mathbb{R}^4$, the $SO(3)$ invariance must hold also on this line. But here the isotropy group jumps – it is all of $SO(3)$. So we need to be able to lift $SO(3)$ to a subgroup $SO(3)_{\mathcal{G}}$ in $\mathcal{G}$, and for consistency, Λ must be one generator of $SO(3)_{\mathcal{G}}$. In other words, in addition to Λ, there should be two elements Φ_0, $\bar{\Phi}_0$ of $\mathfrak{g}^*$, such that the algebra (15) and (16) holds. As we saw above, this implies that the fields on $\mathbb{H}^2$ can approach vacuum values on the boundary. The lift of these fields to $\mathbb{H}^2 \times S^2$ can then be extended to the τ-axis of $\mathbb{R}^4$, to give finite-action self-dual Yang–Mills fields on $\mathbb{R}^4$.

From now on, we shall suppose that Λ is one generator of an $SO(3)_{\mathcal{G}}$ subgroup of $\mathcal{G}$.

3. A maximally non-abelian example

Let us now choose $\mathcal{G} = SU(2N)$, whose Lie algebra consists of $2N \times 2N$, anti-hermitian traceless matrices. Λ can always be conjugated into the Cartan subalgebra of diagonal matrices

$$\Lambda = i\begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_{2N} \end{pmatrix}, \tag{19}$$

with Λ_α real and $\sum \Lambda_\alpha = 0$. To obtain a large non-abelian centralizer of Λ and hence $SO(2)_{\mathcal{G}}$, we want as many as possible of the Λ_α to be equal. The constraint $[\Lambda, \Phi] = -i\Phi$ is satisfied by the $2N \times 2N$ matrices Φ, where the matrix element $\Phi_{\alpha\beta}$ can be non-zero only if $\Lambda_\beta - \Lambda_\alpha = 1$. To obtain a large non-zero part of Φ, we want as many as possible of the differences $\Lambda_\beta - \Lambda_\alpha$ to be 1. Combining these requirements, the optimal choice is

$$\Lambda = \frac{i}{2}\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix}, \tag{20}$$

where $\mathbf{1}_N$ is the unit $N \times N$ matrix. This gives a maximally large gauge group and Higgs field after dimensional reduction.

The constraints (6) and (7) are satisfied by fields of the form

$$\mathbf{A}_z = \begin{pmatrix} A_z & 0 \\ 0 & \tilde{A}_z \end{pmatrix}, \qquad \mathbf{A}_{\bar{z}} = \begin{pmatrix} A_{\bar{z}} & 0 \\ 0 & \tilde{A}_{\bar{z}} \end{pmatrix}, \tag{21}$$

$$\Phi = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \qquad \bar{\Phi} = \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix}, \tag{22}$$

where the non-zero parts are $N \times N$ blocks. The reduced gauge group G is $S(U(N) \times \widetilde{U(N)})$, i.e. $U(N) \times \widetilde{U(N)}$ with overall determinant 1. The Lie algebra is that of $SU(N) \times \widetilde{SU(N)} \times U(1)$. The notation $\tilde{}$ conveniently distinguishes the factors of the gauge group and the corresponding gauge potentials A and $\tilde{A}$.

There is an $SO(3)_{\mathcal{G}}$ algebra here, satisfying (15) and (16), with Λ as above and

$$\Phi_0 = \begin{pmatrix} 0 & 0 \\ \mathbf{1}_N & 0 \end{pmatrix}, \qquad \bar{\Phi}_0 = \begin{pmatrix} 0 & \mathbf{1}_N \\ 0 & 0 \end{pmatrix}. \tag{23}$$

Hence there is a zero-energy vacuum, with $H = \mathbf{1}_N$, where the $SU(N)$ and $\widetilde{SU(N)}$ gauge groups are locked, instead of the colour–flavour locking mentioned in the introduction.

Substituting the expressions (21) and (22) into the generic Bogomolny equations (11)–(13), we find the Bogomolny equations for the unconstrained fields

$$F_{z\bar{z}} = \frac{\sigma}{8}\left(-\mathbf{1}_N + H^\dagger H\right), \tag{24}$$

$$\tilde{F}_{z\bar{z}} = \frac{\sigma}{8}\left(\mathbf{1}_N - HH^\dagger\right), \tag{25}$$

$$D_z H^\dagger = 0, \tag{26}$$

$$D_{\bar{z}} H = 0, \tag{27}$$

where F, $\tilde{F}$ are the field tensors of A, $\tilde{A}$, respectively, and $D_z H^\dagger = \partial_z H^\dagger + A_z H^\dagger - H^\dagger \tilde{A}_z$, $D_{\bar{z}} H = \partial_{\bar{z}} H + \tilde{A}_{\bar{z}} H - H A_{\bar{z}}$. These equations are gauge invariant under G, with $U(N)$ acting on H from the right, and $\widetilde{U(N)}$ acting from the left. So H is a Higgs field in the bifundamental representation of G.

Note that if the sizes N and N' of the two blocks of matrices in Eqs. (20) and (21) were unequal, the Higgs fields coming from the off-diagonal elements in Eq. (22) would not be square matrices. By taking a trace, we can easily see that the corresponding Bogomolny equations (24) and (25) would then not allow the vacuum solution with vanishing field strengths $F_{z\bar{z}} = \tilde{F}_{z\bar{z}} = 0$. This is another reason why we should choose the symmetric situation which necessitates the even size $2N$ of the starting unitary gauge group $SU(2N)$.

4. Moduli matrix and master equations

Let us split the $U(N)$ and $\widetilde{U(N)}$ gauge potentials A and $\tilde{A}$ into their traceless $SU(N)$ and $\widetilde{SU(N)}$ parts $A^{(0)}$ and $\tilde{A}^{(0)}$, and a common $U(1)$ part a. The Bogomolny equations now take the form

$$F_{z\bar{z}}^{(0)} + \frac{i}{2}\mathbf{1}_N f_{z\bar{z}} = \frac{\sigma}{8}\left(-\mathbf{1}_N + H^\dagger H\right), \tag{28}$$

$$\tilde{F}_{z\bar{z}}^{(0)} - \frac{i}{2}\mathbf{1}_N f_{z\bar{z}} = \frac{\sigma}{8}\left(\mathbf{1}_N - HH^\dagger\right), \tag{29}$$

$$D_{\bar{z}} H = 0, \tag{30}$$

where $f_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$ and $D_{\bar{z}} H = \partial_{\bar{z}} H + \tilde{A}_{\bar{z}}^{(0)} H - H A_{\bar{z}}^{(0)} - i a_{\bar{z}} H$. We suppress Eq. (26), as this is just the hermitian conjugate of (27). By taking the traceless and trace parts of Eqs. (28) and (29), we could decompose the Bogomolny equations into a set of coupled equations for the $SU(N)$, $\widetilde{SU(N)}$ and $U(1)$ parts. For the rest of this section we drop the superscript $^{(0)}$, remembering that capital A, F, etc., refer to $SU(N)$.

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[2] We use $z = \tau + ir$ here, exchanging the role of z and $\bar{z}$ compared to $z = r + i\tau$ in Ref. [4].

Let us define a real gauge parameter function $\psi(z, \bar{z})$ and $SL(N, \mathbb{C})$ gauge parameter matrix functions $S(z, \bar{z})$ and $\tilde{S}(z, \bar{z})$ by

$$a_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}\psi, \qquad A_{\bar{z}} = S^{-1}\partial_{\bar{z}}S, \qquad \tilde{A}_{\bar{z}} = \tilde{S}^{-1}\partial_{\bar{z}}\tilde{S}. \qquad (31)$$

Using these, the Bogomolny equation (30) for H can be solved in terms of a holomorphic moduli matrix $H_0(z)$, as [3,18,19]

$$H(z, \bar{z}) = e^{\frac{1}{2}\psi(z,\bar{z})}\tilde{S}^{-1}(z, \bar{z})H_0(z)S(z, \bar{z}). \qquad (32)$$

By defining the gauge invariant quantities $\Omega \equiv SS^{\dagger}$ and $\tilde{\Omega} \equiv \tilde{S}\tilde{S}^{\dagger}$, the matrix Bogomolny equations (28) and (29) can now be reexpressed as

$$\partial_z\partial_{\bar{z}}\psi = \frac{\sigma}{4}\left(-1 + \frac{1}{N}e^{\psi}\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})\right), \qquad (33)$$

$$\partial_z(\Omega^{-1}\partial_{\bar{z}}\Omega)$$
$$= \frac{\sigma}{8}e^{\psi}\left(H_0^{\dagger}\tilde{\Omega}^{-1}H_0\Omega - \frac{1}{N}\mathbf{1}_N\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})\right), \qquad (34)$$

$$\partial_z(\tilde{\Omega}^{-1}\partial_{\bar{z}}\tilde{\Omega})$$
$$= -\frac{\sigma}{8}e^{\psi}\left(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger} - \frac{1}{N}\mathbf{1}_N\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})\right). \qquad (35)$$

We call Eqs. (33)–(35) the master equations for the $U(1)$, $SU(N)$ and $\widetilde{SU(N)}$ gauge groups, respectively. It has been shown that the solution of the $U(1)$ master equation (33) exists and is unique for the given source $\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})$ [20]. Similarly, we conjecture that the solution ψ, Ω, $\tilde{\Omega}$ of the coupled $U(1)$ and $SU(N)$ master equations (33)–(35) exists and is unique for a given moduli matrix $H_0(z)$.

Note that the moduli matrix is defined up to holomorphic gauge equivalence by $SL(N, \mathbb{C})$ transformations from the left and right,

$$H_0(z) \to \tilde{V}(z)H_0(z)V(z), \qquad S \to V^{-1}(z)S, \qquad \tilde{S} \to \tilde{V}(z)\tilde{S}, \qquad (36)$$

with $V(z)$, $\tilde{V}(z)$ holomorphic in z, and of unit determinant. This moduli matrix formalism is very similar to the case of the $U(N)$ gauge theory with N flavours of Higgs fields in the fundamental representation [3,18,19], except that here we have two gauge groups $SU(N)$, $\widetilde{SU(N)}$ besides a $U(1)$ gauge group.

Transposing the $SU(N)$ master equation (34), we observe that the $\widetilde{SU(N)}$ master equation (35) can be obtained by the transformation

$$H_0 \longleftrightarrow H_0^T, \qquad \tilde{\Omega}^{-1} \longleftrightarrow \Omega^T. \qquad (37)$$

The same transformation also gives (34) from (35). This implies that for a symmetric moduli matrix $H_0 = H_0^T$, the solution has the symmetry $\tilde{\Omega}^{-1} = \Omega^T$.

On $\mathbb{R}^2$, where $\sigma = 1$, we cannot expect the master equations to be integrable. However on the hyperbolic plane $\mathbb{H}^2$, where $\sigma = \frac{2}{(\operatorname{Im} z)^2}$, the equations are formally integrable [5,6]. Possibly this also applies to the multi-flavour $U(N)$ gauge theory on $\mathbb{H}^2$, but this has not been established. It is interesting to observe that in the hyperbolic case, the explicit factor of σ can be eliminated from the Bogomolny equations and the master equations [5]. This is because σ satisfies the Liouville equation $\partial_z\partial_{\bar{z}}(\log \sigma) = \frac{1}{4}\sigma$, and if we make the transformation $\psi \to \psi' = \psi + \log \sigma$, the master equations become

$$\partial_z\partial_{\bar{z}}\psi' = \frac{1}{4N}e^{\psi'}\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger}), \qquad (38)$$

$$\partial_z(\Omega^{-1}\partial_{\bar{z}}\Omega)$$
$$= \frac{1}{8}e^{\psi'}\left(H_0^{\dagger}\tilde{\Omega}^{-1}H_0\Omega - \frac{1}{N}\mathbf{1}_N\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})\right), \qquad (39)$$

$$\partial_z(\tilde{\Omega}^{-1}\partial_{\bar{z}}\tilde{\Omega})$$
$$= -\frac{1}{8}e^{\psi'}\left(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger} - \frac{1}{N}\mathbf{1}_N\operatorname{Tr}(\tilde{\Omega}^{-1}H_0\Omega H_0^{\dagger})\right). \qquad (40)$$

If further, by analogy with Eq. (31), we define $a'_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}\psi'$, then

$$a'_{\bar{z}} = a_{\bar{z}} - \frac{i}{2}\partial_{\bar{z}}(\log \sigma), \qquad (41)$$

and $\psi \to \psi'$, $a_{\bar{z}} \to a'_{\bar{z}}$ amounts to a complexified $U(1)$ gauge transformation.

5. Vacuum and non-abelian vortices

We revert here to the notation of Section 3, where the $U(N)$ gauge fields are not split up.

The vacuum of our model is given by the constant solution of the Bogomolny equations

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad A = 0, \qquad \tilde{A} = 0. \qquad (42)$$

This vacuum is invariant under the diagonal gauge group $SU(N)_{\mathrm{d}}$, which is therefore the unbroken local gauge invariance. This contrasts with the multi-flavour $U(N)$ model, which is in a Higgs phase, as the gauge group is fully broken in the vacuum.

Exact vortex solutions are obtained using the ansatz

$$H = \begin{pmatrix} h^{(1)} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad A_{\bar{z}} = \begin{pmatrix} ia_{\bar{z}}^{(1)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad (43)$$

with $\tilde{A}_{\bar{z}} = -A_{\bar{z}}$ so that one has an $S(U(N) \times \widetilde{U(N)})$ gauge potential. The Bogomolny equations (24) and (27) in this case reduce to

$$if_{z\bar{z}}^{(1)} = \frac{\sigma}{8}(-1 + |h^{(1)}|^2), \qquad (44)$$

$$\partial_{\bar{z}}h^{(1)} - 2ia_{\bar{z}}^{(1)}h^{(1)} = 0, \qquad (45)$$

where $f_{z\bar{z}}^{(1)} = \partial_z a_{\bar{z}}^{(1)} - \partial_{\bar{z}}a_z^{(1)}$, and Eqs. (25) and (26) give nothing further.

Setting $h^{(1)} = e^{\frac{1}{2}k+i\chi}$ with k and χ real, and eliminating $a_z^{(1)} = (a_{\bar{z}}^{(1)})^*$ using Eq. (45), one finds that Eq. (44) simplifies to

$$\partial_z\partial_{\bar{z}}k = -\frac{\sigma}{4}(1 - e^k). \qquad (46)$$

This is the standard gauge invariant Taubes equation for abelian vortices on a general surface. On the hyperbolic plane, where σ satisfies Liouville's equation, Eq. (46) itself reduces to Liouville's equation, as first shown by Witten [4], and its solutions have been completely worked out in terms of Blaschke product functions. The solutions are hyperbolic vortices and multi-vortices, that also arise from spherically symmetric self-dual Yang–Mills fields (i.e. instantons) in $SU(2)$ gauge theory on $\mathbb{R}^4$.

Note that these abelian vortices embedded in $S(U(N) \times \widetilde{U(N)})$ gauge theory do not have full unit winding in the $U(1)$ subgroup

of the gauge group, and they have $SU(N)$ parts. So they are truly non-abelian. This situation is quite analogous to the non-abelian vortices in $U(N)$ gauge theories [1–3].

It is clear that our construction can be extended to an arbitrary choice of embedding of the Witten solutions into diagonal elements of the $U(N)$ group, and this leads to all possible non-abelian vortex solutions which are restricted to lie in the diagonal $U(1)^N$ subgroup.

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