Linear Equations over Commutative Rings

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ABSTRACT

A generalized rank (McCoy rank) of a matrix with entries in a commutative ring R with identity is discussed. Some necessary and sufficient conditions for the solvability of the linear equation $Ax = b$ are derived, where $x, b$ are vectors and $A$ is a matrix with entries in either a Noetherian full quotient ring or a zero dimensional ring.

1. INTRODUCTION

In this paper all rings are commutative with the identity. We shall consider the equation

$$Ax = b,$$  \hspace{1cm} (1)

where $A = (a_{ij})_{n \times m}$ is a matrix over $R$, and $x, b$ are column vectors in $R$. We shall be explicitly concerned about the solvability of Eq. (1). When $b = 0$, N. H. McCoy defined a generalized notion of rank (in this paper, it will be denoted by $\text{rank}_{MA}$), and proved the following [6, p. 159]:

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A paper by Camion, Levy and Mann [3] also discussed the solvability of Eq. (1). They considered special classes of commutative rings (including valuation rings and Prüfer domains). Among the neatest results they derived is a necessary and sufficient condition for the solvability of Eq. (1) over a Prüfer domain, which of course bears its familiar appearance when $R$ is a field.

The present paper was originally motivated by the study of discrete dynamical systems over commutative rings (i.e., those dynamical systems which allow the input and output data to take values in a commutative ring), where the solvability of certain matrix equations is essential [7,8]. We shall consider rings with zero divisors, and shall call a ring $R$ Noetherian full quotient if it is a commutative Noetherian ring with the property that each element in $R$ is either a unit or a zero divisor. We shall also consider rings with Krull dimension zero [4, p. 32]. In Sec. 2, we discuss the solvability of Eq. (1) and related problems over a general commutative ring. In Sec. 3, we consider the case when $R$ is a Noetherian full quotient ring. In Sec. 4, we consider the case when $R$ is a zero dimensional ring.

**Notation**

(i) In Eq. (1), we will assume $n < m$, since we can always add columns of zeros to make a matrix equation with $n < m$.

(ii) If every element of ring $R$ is either a unit or a zero divisor, we write

$$R = U(R) \cup Z(R),$$

where $U(R)$ is the set of all units in $R$, and $Z(R)$ is the set of all zero divisors in $R$.

(iii) If $L$ is an ideal in $R$, we denote by

$$L^* = \{ r \in R | rL = 0 \}$$

the annihilator ideal of $L$.

(iv) We shall denote by $N$ the nilradical of $R$.

2. **COMMUTATIVE RINGS WITH IDENTITY**

For a matrix over a commutative ring, the following generalized rank was first introduced by N. H. McCoy.
DEFINITION 1. Let $R$ be a non-trivial commutative ring with identity, and let $A = (a_{ij})_{n \times m}$ be a matrix over $R$. If all the entries $a_{ij}$ have a non-zero annihilator, then $\operatorname{rank}_M A$ is defined to be zero. If $A$ does not have rank zero, its rank is the greatest positive integer $r < n$ such that the set of all $r \times r$ subdeterminants of $A$ does not have a common non-zero annihilator. Denote this rank by $\operatorname{rank}_M A = r$.

THEOREM 1. Let $R$ be a commutative ring with identity. If Eq. (1) has a solution for a given $b$, then the solution is unique if and only if $\operatorname{rank}_M A = m$.

Proof. This is just another form of McCoy’s theorem [6, p. 159].

COROLLARY. If $n = m$ and Eq. (1) has a solution for a given $b$, then the solution is unique if and only if $\det A$ is not a divisor of zero in $R$.

Note that this Corollary was obtained in Bourbaki [1, Proposition 3, p. 524] by using arguments involving the exterior algebra.

The following is a local and global connection [2, p. 89] which is essential in understanding Theorem 2.

If $A : R^m \rightarrow R^n$ is a homomorphism with $m \geq n$, then $A$ is onto if and only if for every maximal ideal $M$ of $R$, $\pi(A) : (R/M)^m \rightarrow (R/M)^n$ is onto, where $\pi : R \rightarrow R/M$ is the natural homomorphism.

Let $I_r(A)$ denote the ideal generated by the set of all $r \times r$ subdeterminants of $A$.

THEOREM 2. Let $R$ be a commutative ring with identity. Then for every $b$ Eq. (1) has a solution if and only if

$$I_n(A) = R.$$ 

Proof. Suppose for every $b$ Eq. (1) has a solution; then $A$ represents an onto homomorphism. This means $\operatorname{rank}(\pi_M A) = n$ for all $M$. Hence for each $M$, there is some $n \times n$ subdeterminant of $A$ which is not zero in $R/M$. Therefore $I_n(A) = R$.

If $I_n(A) = R$, then $I_n(\pi_M A) = R/M$, where $M$ is a maximal ideal in $R$ and $\pi_M : R \rightarrow R/M$ is the natural homomorphism. This means $\pi_M A$ is onto for every maximal ideal $M$. By the local and global connection mentioned earlier we conclude that $A$ is onto as a linear transformation $A : R^m \rightarrow R^n$. Therefore Eq. (1) is solvable for every $b$. □
The following two corollaries are two immediate consequences of the above theorem.

**Corollary 1.** If \( n = m \), then for every \( b \) Eq. (1) has a solution if and only if \( \det A \) is a unit in \( R \).

**Corollary 2.** Let \( R \) be a commutative ring with identity. If Eq. (1) has a solution for every \( b \), then \( \text{rank}_M A = n \).

**Remark.** \( \text{rank}_M A = n \) is certainly not sufficient in the above corollary. This can be seen by the example \( 2x = b \) over the ring of integers \( \mathbb{Z} \). However, over a large class of commutative rings, \( \text{rank}_M A = n \) is a sufficient condition. We shall see this in Sec. 3.

We shall denote by \((A, b)\) the augmented matrix of Eq. (1). With this notation it is then easy to see that the following theorem is a direct consequence of Definition 1.

**Theorem 3.** Let \( R \) be a commutative ring with identity. If for a given \( b \) Eq. (1) has a solution, then

\[
\text{rank}_M A = \text{rank}_M (A, b).
\]

3. **NOETHERIAN FULL QUOTIENT RINGS**

Recall that a ring is Noetherian full quotient if it is Noetherian with the property that every element is either a unit or a zero divisor.

**Example.**

(i) Every commutative finite ring is a Noetherian full quotient ring.

(ii) Let \( R \) be a commutative Noetherian ring. If we localize with respect to its non-zero divisors, we have a Noetherian full quotient ring.

**Theorem 4.** Let \( R \) be a Noetherian full quotient ring, and \( a_i \in R \) for \( i = 1, 2, \ldots, k \). Then the set \( \{a_i\}_{i=1}^k \) generates the unit ideal in \( R \) if and only if

\[
\bigcap_{i=1}^k (a_i)^* = (0).
\]

**Proof.** One way is trivial. The converse can be concluded from a special case of Theorem 82 [4, p. 56].
This theorem is an obvious generalization of the concept of elements being relatively prime in the ring of integers. It also implies some applications of the McCoy rank as described in Theorem 5 and Theorem 6. In Sec. 4 we shall prove the same theorem for zero dimensional rings.

**Theorem 5.** Let $R$ be a Noetherian full quotient ring. For every $b$ Eq. (1) has a solution if and only if

$$\text{rank}_M A = n.$$ 

**Proof.** Corollary 2 to Theorem 2 states that if for every $b$ Eq. (1) has a solution, then $\text{rank}_M A = n$. Now if $\text{rank}_M A = n$, by Theorem 4 we must have $I_n(A) = R$. Hence by Theorem 2 again, for every $b$ Eq. (1) has a solution.

Let $D_r(A)$ denote the set of all $r \times r$ subdeterminants of $A$, and $D_r(A, b)$ denote the set of all $r \times r$ subdeterminants of the augmented matrix $(A, b)$. Let $D^*_r(A, b)$ be the ideal generated by all elements in $D_r(A, b)$ but not in $D_r(A)$. Some special cases are as follows: if $r = 1$, then $D^*_1(A, b) = \{0\}$ means $b = 0$. If $A$ is an $n \times m$ matrix with $n < m$, then $D^*_{n+1}(A, b) = \{0\}$.

The following theorem gives a sufficient condition for the solvability of Eq. (1) for a given $b$. The proof can be found in [3, Theorem 9] provided the rank is properly replaced by $\text{rank}_M$.

**Theorem 6.** Let $R$ be a Noetherian full quotient ring. Then for a given $b$ Eq. (1) has a solution if

$$D^*_\text{rank}_M A + 1(A, b) = \{0\}.$$ 

4. **ZERO DIMENSIONAL RINGS**

A non-trivial commutative ring with identity is a zero dimensional ring if its Krull dimension is zero. The Krull dimension of a ring $R$ is defined to be the sup of all lengths of chains of prime ideals [4, p. 32].

**Example.**

1. Every direct sum (or product) of fields.
2. Every Boolean ring.
3. Every $p$-ring, where $p$ is a prime integer in $\mathbb{Z}$ [6, p. 144].
4. $R = k[x]/(x^n)$, where $k$ is a field and $n$ is a positive integer.
A zero dimensional ring is a full quotient ring [4, p. 60], but need not be a
Noetherian ring as indicated by the infinite product of fields. Conversely, it
is easy to see that a Noetherian full quotient ring need not be a zero
dimensional ring. In this section we shall show that all the results in Sec. 3
hold true if the Noetherian full quotient rings are replaced by the zero
dimensional rings.

A ring $R$ is a von Neumann regular ring if for every $r \in R$, there exists
$r' \in R$ such that $rr' = r$. A von Neumann regular ring is a zero dimensional
ring since every prime ideal in a von Neumann regular ring is a maximal
ideal [5, p. 33]. Now Theorem 4 in Sec. 3 can be stated over von Neumann
regular rings.

**Theorem 7.** Let $R$ be a von Neumann regular ring and $a_i \in R$, $i = 1, 2, \ldots, k$. Then the set \( \{a_i\}_{i=1}^{k} \) generates the unit ideal if and only if \( \bigcap_{i=1}^{k} (a_i)^* = (0) \).

**Proof.** It is easy to see that if \( \{a_i\}_{i=1}^{k} \) generates the unit ideal, then (2) follows.

Now assume (2). Suppose \( \{a_i\}_{i=1}^{k} \) does not generate the unit ideal. Then

\[
\sum_{i=1}^{k} (a_i) = (\alpha) \subseteq \mathcal{Z}(R),
\]

where the equality comes from the fact that in a von Neumann regular ring
every finitely generated ideal is principal [5, p. 36]. The inclusion comes from
the fact that $R = \mathcal{Z}(R) \cup U(R)$ [5, p. 33]. But then

\[
(\alpha)^* = \bigcap_{i=1}^{k} (a_i)^* \neq (0)
\]

gives a contradiction. Therefore \( \{a_i\}_{i=1}^{k} \) generates $R$. \[ \square \]

Before we establish the same theorem over a zero dimensional ring, we
need a few lemmas.

**Lemma 1.** Let $R$ be a commutative ring and $a_i \in R$, $i = 1, 2, \ldots, k$. If
\( \bigcap_{i=1}^{k} (a_i)^* = (0) \), then \( \bigcap_{i=1}^{k} (a_i^t)^* = (0) \) for all $t \geq 1$. 
Proof. First we shall show that if \( r \in \cap_{i=1}^{k}(a_i^t)^* \), then

\[
ra_1^{t-i}a_2^{t-i_2} \cdots a_k^{t-i_k}a_k^{t-1} = 0
\]

for all \( 0 < i_1, i_2, \ldots, i_k < t \). Let \( i_1 + i_2 + \cdots + i_k = l \), and apply induction on \( l \). Clearly for \( l = 0 \) the statement is trivially true. Assume it is true for \( i_1 + i_2 + \cdots + i_{k-1} = l - 1 \). Now consider

\[ ra_1^{t-i_1}a_2^{t-i_2} \cdots a_k^{t-i_k}a_k^{t-1}, \]

with \( i_1 + i_2 + \cdots + i_{k-1} = l \). By the induction hypothesis, each of the following is true:

\[
ra_1^{t-(i_1-1)}a_2^{t-i_2} \cdots a_k^{t-i_k}a_k^{t-1} = 0,
\]

\[
ra_1^{t-i_1}a_2^{t-(i_2-1)} \cdots a_k^{t-i_k}a_k^{t-1} = 0,
\]

and so on.

But then

\[
a_i(ra_1^{t-i_1} \cdots a_k^{t-i_k}a_k^{t-1}) = 0,
\]

for all \( i = 1, 2, \ldots, k \). Therefore

\[
ra_1^{t-i_1} \cdots a_k^{t-i_k}a_k^{t-1} \in \cap_{i=1}^{k}(a_i^t)^* = (0).
\]

Hence (3) is true and the induction is completed. In particular we have

\[
ra_k^{t-1} = 0.
\]

By a similar induction procedure, we can show that

\[
ra_i^{t-1} = 0.
\]
for $i = 1, 2, \ldots, k - 1$. But then
\[
  r \in \bigcap_{i=1}^{k} (a_i^{t-1})^*,
\]
and the Lemma can be completed by induction on $t$.

**Lemma 2.** Let $R$ be a commutative ring, and $a_i \in R$, $i = 1, 2, \ldots, k$. If $N$ is the nilradical of $R$ and $ra_i \in N$, $i = 1, 2, \ldots, k$, then $\cap_{i=1}^{k} (a_i)^* = (0)$ implies $r \in N$.

**Proof.** Suppose $ra_i \in N$ for $i = 1, 2, \ldots, k$; then $(ra_i)^t = 0$ for some $t_i \geq 1$ and $i = 1, 2, \ldots, k$. Now let $t = \max_i \{t_i\}$; then $(ra_i)^t = r'a_i^t = 0$ for $i = 1, 2, \ldots, k$. Therefore
\[
  r^t \in \bigcap_{i=1}^{k} (a_i^t)^*.
\]
By Lemma 1, $r^t = 0$. This means $r \in N$.

**Lemma 3.** Let $\bar{R} = R/N$. Then $\cap_{i=1}^{k} (a_i)^* = (0)$ implies $\cap_{i=1}^{k} (\bar{a}_i)^* = (0)$, where $\bar{a}_i \in \bar{R}$, $i = 1, 2, \ldots, k$.

**Proof.** Let $\bar{r} \in \cap_{i=1}^{k} (\bar{a}_i)^*$, then $\bar{r} \bar{a}_i = 0$ for all $i = 1, 2, \ldots, k$. But then $ra_i \in N$ for $i = 1, 2, \ldots, k$. By Lemma 2, $r \in N$, i.e., $\bar{r} = 0$. Therefore $\cap_{i=1}^{k} (\bar{a}_i)^* = (0)$.

**Lemma 4.** Let $R$ be a commutative ring, $\bar{R} = R/N$ and $a_i \in R$, $i = 1, 2, \ldots, k$. Then $(a_1) + (a_2) + \cdots + (a_k) = R$ if and only if $(\bar{a}_1) + (\bar{a}_2) + \cdots + (\bar{a}_k) = \bar{R}$.

**Proof.** That $(a_1) + (a_2) + \cdots + (a_k) = R$ implies $(\bar{a}_1) + (\bar{a}_2) + \cdots + (\bar{a}_k) = \bar{R}$ is trivial. Now let $(\bar{a}_1) + (\bar{a}_2) + \cdots + (\bar{a}_k) = \bar{R}$; then there exist $\bar{x}_i \in R$ such that
\[
  \bar{a}_1 \bar{x}_1 + \bar{a}_2 \bar{x}_2 + \cdots + \bar{a}_k \bar{x}_k = 1.
\]
But then
\[
  a_1 x_1 + a_2 x_2 + \cdots + a_k x_k - 1 \in N.
\]
Hence
\[(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k - 1)^s = 0\]
for some \(s \geq 1\). If we expand this expression, it is not difficult to see that there exist \(a_i \in R\) such that
\[a_1 a_1 + a_2 a_2 + \cdots + a_k a_k = 1.\]
Therefore
\[(a_1) + (a_2) + \cdots + (a_k) = R. \quad \blacksquare\]

**Theorem 8.** Let \(R\) be a zero dimensional ring and \(a_i \in R, i = 1, 2, \ldots, k\). Then the set \(\{a_i\}_{i=1}^k\) generates the unit ideal if and only if
\[\bigcap_{i=1}^k (a_i)^* = (0). \quad (4)\]

*Proof.* If the set \(\{a_i\}_{i=1}^k\) generates the unit ideal, then (4) is obviously true.

To prove the converse we observe that by virtue of Lemmas 3, 4 it is enough to prove the statement over \(\overline{R} = R/N\), which is a reduced [i.e., nilradical \(= (0)\)] zero dimensional ring, where \(N\) is the nilradical of \(R\). But a reduced zero dimensional ring is a von Neumann regular ring [4,p. 64]. Hence the theorem follows from Theorem 7. \(\blacksquare\)

With Theorem 8 available, we can state the following theorem.

**Theorem 9.** Theorems 5, 6 in Sec. 3 hold true if the Noetherian full quotient ring is replaced by a zero dimensional ring.

*Proof.* It is enough to observe that in the proofs of Theorems 5, 6 the essential thing we needed was Theorem 4, which in the present case is Theorem 8. \(\blacksquare\)

**Theorem 10.** Let \(R\) be a zero dimensional ring and \(N\) be the nilradical of \(R\). If \(A\) is a \(n \times m\) matrix over \(R\) and \(\overline{A}\) is the corresponding matrix over \(\overline{R} = R/N\), then
\[\text{rank}_M A = \text{rank}_M \overline{A}.\]
In other words, the McCoy rank is invariant under factoring out the nilradical in a zero dimensional ring.

Proof.

(i) If \( \operatorname{rank}_M A = \gamma \), then \( \cap \{ (d_x)^* | d_x \in D_\gamma(A) \} = (0) \). By Lemma 3, \( \cap \{ (\tilde{d}_x)^* | \tilde{d}_x \in D_\gamma(A) \} = (0) \). Hence \( \operatorname{rank}_M A \leq \operatorname{rank}_{\tilde{M}} \tilde{A} \).

(ii) If \( \operatorname{rank}_M A = k \), then \( \cap \{ (\tilde{d}_k)^* | \tilde{d}_k \in D_k(A) \} = (0) \). By Theorem 9, the set \( \{ \tilde{d}_k | \tilde{d}_k \in D_k(A) \} \) generates \( \tilde{R} \). But then Lemma 4 implies that the set \( \{ d_k | d_k \in D_k(A) \} \) generates \( R \). Hence \( \cap \{ (d_k)^* | d_k \in D_k(A) \} = (0) \). Therefore \( \operatorname{rank}_M A \geq \operatorname{rank}_{\tilde{M}} \tilde{A} \).

(i) and (ii) together imply \( \operatorname{rank}_M A = \operatorname{rank}_{\tilde{M}} \tilde{A} \).

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