1-FACTORIZING REGULAR GRAPHS OF HIGH DEGREE - AN IMPROVED BOUND

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We showed earlier that a regular simple graph of even order satisfying $d(G) \geq \frac{5}{2} |V(G)|$ was the union of edge-disjoint 1-factors. Here we improve this to regular simple graphs of even order satisfying $d(G) \geq \frac{1}{2} (\sqrt{7} - 1) |V(G)|$.

1. Introduction

The graphs we shall consider will be simple, that is they will have no multiple edge or loops. An edge-colouring of a graph is a map $\Phi: E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours and $E(G)$ is the set of edges of $G$, such that no two incident edges receive the same colour. The chromatic index $\chi'(G)$ of $G$ is the least value of $|\mathcal{C}|$ for which an edge-colouring of $G$ exists. A well-known theorem of Vizing [7] states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where $\Delta(G)$ is the maximum degree of $G$. Graphs for which $\Delta(G) = \chi'(G)$ are said to be Class 1, and otherwise they are Class 2. A regular Class 1 graph is often called 1-factorizable, as it is the union of edge disjoint 1-factors.

For a regular graph $G$, let us denote the common degree of the vertices by $d(G)$. A well-known conjecture which may be due to G.A. Dirac (he told one of us that it was 'going around' in the early 1950s) is as follows.

**Conjecture 1.** A regular graph of even order satisfying

$$d(G) \geq \frac{1}{2} |V(G)|$$

is 1-factorizable.

The present authors took the first significant step towards solving this conjecture by proving it with the more restrictive bound $d(G) \geq \frac{5}{2} |V(G)|$ in [1]; they actually proved it with $d(G) \geq 0.849 |V(G)|$. Here we improve this to
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\[ d(G) \geq \frac{\delta}{6} |V(G)|; \] in fact again we prove a slightly stronger bound, namely
\[ d(G) \geq 0.823 |V(G)|. \]

In a regular graph \( G \) of even order on vertices \( v_1, \ldots, v_{2n} \), let \( \tilde{p}_{ij} = \tilde{p}_{ij}(G) \) be
the number of paths in \( \tilde{G} \), the complement of \( G \), of length 2 which join \( v_i \) and \( v_j \), and let \( \tilde{p} = \tilde{p}(G) = \max_{i,j} \tilde{p}_{ij} \). Clearly \( \tilde{p} \leq d(\tilde{G}) = |V(\tilde{G})| - d(G) - 1. \)

First we prove the following result.

**Theorem 1.** Let \( G \) be a regular graph of even order satisfying
\[ d(G) > \frac{\delta}{6} |V(G)| - \frac{1}{2} \tilde{p} - \frac{1}{6}. \]

Then \( G \) is 1-factorizable.

By proving an easy bound on \( \tilde{p} \) we obtain the following corollary. (Note that \( \frac{\delta}{6} = 0.833 \) and \( \frac{1}{2}(\sqrt{7} - 1) \approx 0.823 \).)

**Theorem 2.** Let \( G \) be a regular graph of even order satisfying
\[ d(G) > \frac{1}{2}(\sqrt{7} - 1) |V(G)|. \]

Then \( G \) is 1-factorizable.

For the case when \( \tilde{p} = |V(G)| - d(G) - 1 \), Theorem 1 reduces to Theorem 3.

**Theorem 3.** Let \( G \) be a regular graph of even order containing two vertices which, in \( \tilde{G} \), are joined by \( |V(G)| - d(G) - 1 \) paths of length 2. Furthermore, let
\[ d(G) > \frac{3}{4} |V(G)| + \frac{1}{4}. \]

Then \( G \) is 1-factorizable.

Let \( G_\Delta \) be the subgraph of a graph \( G \) induced by the vertices of degree \( \Delta = \Delta(G) \). We call \( G_\Delta \) the core of \( G \). A very useful result, due to Fournier [5], is that if \( G_\Delta \) is a forest, then \( G \) is Class 1. As a preliminary to our proof of Theorem 1, we extend Fournier’s theorem. A general discussion of the possibilities for extending Fournier’s theorem was provided by Hoffman and Rodger in [6]; see also [2] and [3].

2. Preliminary results

For a vertex \( v \) in a graph \( G \), let \( d^*(v) \) denote the number of vertices of \( G \) of maximum degree to which \( v \) is adjacent. The following lemma was proved in [1].
Lemma 1. For a graph $G$, let $e \in E(G)$ be incident with $w \in V(G)$. Let $d^*(w) \leq 1$. Then

$$\Delta(G - e) = \Delta(G) \Rightarrow \chi'(G - e) = \chi'(G)$$
and
$$\Delta(G - w) = \Delta(G) \Rightarrow \chi'(G - w) = \chi'(G).$$

The next lemma is a well-known result of Dirac [4].

Lemma 2. Let $G$ be a graph whose minimum degree $\delta(G)$ satisfies

$$\delta(G) \geq \frac{1}{2} |V(G)|.$$ 

Then $G$ possesses a Hamiltonian circuit.

3. Extensions of Fournier's theorem

We first prove the following theorem. Define a proper tree to be a tree with at least one edge.

Theorem 4. Let the connected components of $G_\Delta$ be $G_\Delta(1), \ldots, G_\Delta(r)$. For each $i \in \{1, \ldots, r\}$ assume that $G_\Delta(i)$ consists of disjoint proper trees $T_{i1}, \ldots, T_{is(i)}$ which are rooted on a graph $H_i$, where, for each $j \in \{1, \ldots, s(i)\}$, $H_i \cap T_{ij}$ is a single vertex $v_{ij}$ (the root vertex), and such that $G_\Delta(i) \setminus V(T_{i1} \cup \cdots \cup T_{is(i)})$ contains no edges. Then $G$ is Class 1.

The type of graph permitted for a $G_\Delta(i)$ is illustrated in Fig. 1.

In the particular special case when each $T_{ij}$ is a single edge, Theorem 4 was used (without being explicitly stated) in [1].

Proof of Theorem 4. We first colour all the edges of $G \setminus E(G_\Delta)$ with $\Delta(G)$ colours. Since the only vertices of degree $\Delta(G)$ in this graph are non-adjacent, it follows from Fournier's theorem that this is possible. For each $i$ we colour all the edges of $H_i$ using Vizing's fan argument; we first colour the edges of the subgraph of $H_i$ induced by $v_{i1}, \ldots, v_{is(i)}$, using vertices in $\{v_{i1}, \ldots, v_{is(i)}\}$ as pivots, and then we colour the remaining edges of $H_i$, using the vertices of $V(H_i) \setminus \{v_{i1}, \ldots, v_{is(i)}\}$ as pivots. Finally we colour the edges of each $T_{ij}$ as follows. We may order the edges $e_1, \ldots, e_t$ of a tree $T_u$ so that $e_1$ is incident with $v_{ij}$, and, for $1 \leq k \leq t$, the edges $e_1, \ldots, e_k$ induce a subtree. We then colour $e_1, \ldots, e_t$ in that order using Vizing's fan argument, always choosing as pivot the vertex of $e_k$ which is non-adjacent to any of the vertices of $e_1, \ldots, e_{k-1}$. □
Next we show that Theorem 4 can be extended.

**Theorem 5.** Let $G_\Delta$ be the edge-disjoint union of two graphs $B$ and $R_\Delta$ having the following properties.

(i) If $R_\Delta(1), \ldots, R_\Delta(r)$ are the connected components of $R_\Delta$, then for each $i \in \{1, \ldots, r\}$, $R_\Delta(i)$ consists of disjoint proper trees $T_{i1}, \ldots, T_{is(i)}$ which are rooted on a graph $H_i$, where, for each $j \in \{1, \ldots, s(i)\}$, $H_i \cap T_{ij}$ is a single vertex $v_{ij}$ (the root vertex), and such that $R_\Delta(i) \setminus V(T_{i1} \cup \cdots \cup T_{is(i)})$ contains no edges.

(ii) The graph $B$ is bipartite and has the property that the set of all proper trees $T_{ij}$ can be written in an order $T_{i1}, \ldots, T_{ip}$ such that the edges of $B$ join vertices of $V(T_k) \setminus \{v_k\}$ to vertices $v_1$ with $k < l$, where $v_k$ and $v_l$ denote the root vertices of $T_k$ and $T_l$ respectively.

Then $G$ is Class 1.

In the theorem above, the hypotheses on $R_\Delta$ are the same as the ones on $G_\Delta$ in Theorem 4. The graph $B$ and the trees $T_1, \ldots, T_p$ are illustrated in Fig. 2 in the
Fig. 2. The graphs $B$ together with trees $T_1, \ldots, T_p$ in the case when $T_1, \ldots, T_p$ each consists of one edge, and $p = 5$.

case where each $T_k$ consists of a single edge (this is, incidentally, the special case we shall use in the proof of Theorem 1).

**Proof of Theorem 5.** Adapting the proof of Theorem 4, first we colour the edges of $G \setminus E(G_a)$. Then we colour the edges of $H_1, \ldots, H_r$. Then we colour the edges of $B$, using Vizing’s fan argument with the vertices on the trees $T_1, \ldots, T_p$ as pivots. Finally we colour the edges of the trees as before, but colour them strictly in the order $T_1, \ldots, T_p$. □

4. The proof of Theorem 1

Let $G$ be a regular graph of order $2n$ and degree

$$d = d(G) \geq \frac{5}{6} |V(G)| - \frac{1}{3} \bar{d} - \frac{1}{6}.$$ 

Let $w, v^* \in V(G)$ be such that the number of paths of length two between $w$ and $v^*$ in $\tilde{G}$ is $\bar{d}$. Let $W$ be the set of vertices of maximum degree in $G$. Then $G - w$ has $2n - 1$ vertices, $|W| = 2n - d - 1$ of them having degree $d$, and the remaining $d$ of them having degree $d - 1$. The vertex $v^*$ is non-adjacent to $\bar{d}$ of the vertices of $W$. Thus $d^*(v^*) = |W| - \bar{d}$ or $|W| - \bar{d} - 1$.

Let $X$ be a set of $|W| - \bar{d} - 1$ vertices of $V(G - w)$ which are non-adjacent to $v^*$; as there are in $G - w$ at least $|W| - 1$ vertices non-adjacent to $v^*$, such a set $X$ does exist. Let $s = |X| = |W| - \bar{d} - 1$. Let $q = |(X \cup W) \setminus \{v^*\}| \leq (|W| - \bar{d} - 1) + |W| = 2|W| - \bar{d} - 1 = 4n - 2d - \bar{d} - 3$.

Now consider the subgraph $H$ of $G - w$ induced by $(X \cup W) \setminus \{v^*\}$. Let $M_0$ be a set of edges of $H$ forming a maximal matching, and let $m = |M_0|$. Let

$$(X \cup M) \setminus \{v^*\} = L \cup R,$$

where $L \cap R = \emptyset$, $|L| = m$, and where each edge of $M_0$ joins a vertex of $L$ to a
vertex of $R$. Let $L^* = L \cup \{v^*\}$ and let $H^*$ be the subgraph of $G - \{w\}$ induced by $L^* \cup R$. Let the elements of $L^*$ be denoted by $l_1, \ldots, l_{m+1}$, where $l_{m+1} = v^*$, and let the elements of $R$ be denoted by $r_t, r_{t+1}, \ldots, r_m$, where $t = 2m - q + 1$ (so $t$ may be negative). Suppose that

$$M_0 = \{l_1r_1, \ldots, l_mr_m\}.$$ 

Let $E^+$ consist of all edges of $H^*$, except those of the form $l_ir_j$, where $i \leq j$, and those with both endvertices in $R$.

We now observe that $E^+$ is contained in the union of $q$ edge-disjoint matchings of the complete graph on $V(H^*)$, $M_1^+, \ldots, M_q^+$, where $M_i^+$ is defined as follows:

- $M_i^+ = \{l_1r_{i-1}, l_2r_{i-2}, \ldots, l_mr_{m-i}, l_{m+1}r_{m+1-i}\}$, if $1 \leq i \leq 1 - t$;
- $M_i^+ = \{l_1l_{i+t-1}, l_2l_{i+t-2}, \ldots, l_{(i+t-1)/2}l_{(i+t-1)/2} + 1\}$
  $\cup \{l_{i+t}r_{i+t+1}r_{i+t+2}, \ldots, l_mr_{m-i}, l_{m+1}r_{m+1-i}\}$, if $2 - t \leq i \leq m + 1 - t$;
- $M_i^+ = \{l_{i-m+t-1}l_{i-m}, l_{i-m+1}l_{m+1}, \ldots, l_{(i+t-1)/2}l_{(i+t-1)/2} + 1\}$, if $m + 2 - t \leq i \leq q$.

Notice that $\bigcup_{i=1}^{q} M_i^+$ contains all the edges of the complete graph on the vertices of $V(H^*)$ except for the edges of $M_0$, the edges with both endvertices in $R$, and the edges which join $l_i \in L$ to $r_j \in R$ with $i < j$. Finally we notice that

$$|M_i^+| \leq \frac{1}{2}(q + 2 - i) \quad (1 \leq i \leq q).$$

The matchings $M_i^+$ ($1 \leq i \leq q$) are illustrated in Fig. 3.
Let $W_s$ be a set of $s$ elements of $W$ which are adjacent to $v^*$. (Recall that there are either $s$ or $s + 1$ such elements.) Let the vertices of $X$ be $x_1, \ldots, x_s$ and the vertices of $W_s$ be $w_1, \ldots, w_s$. If an edge $xw$ is in $M_0$ with $x \in X$ and $w \in W_s$, we may suppose that $x \in R$ and $w \in L$. We may moreover suppose that $l_1, \ldots, l_m, r_1, \ldots, r_m, x_1, \ldots, x_s, w_1, \ldots, w_s$ are labelled so that, for $1 \leq j \leq s - 1$, $x_j$ comes before $w_j$ in the list $(r_m, \ldots, r_1, l_1, \ldots, l_m)$. (In fact, except in the case when each edge of $M_0$ joins either two vertices of $X$ or two vertices of $W_s$, we could suppose that $x_i$ comes before $w_i$ also.)

We now construct matchings $M_i^*$ ($1 \leq i \leq q + 1$) by slightly modifying the $M_i^+$. If $x_i$ comes before $w_i$, define $M_i^* = M_i^+ (1 \leq i \leq q)$ and $M_{q+1}^* = \phi$. If $x_i$ comes after $w_i$, then we may suppose that $v^*w_i \in MG$ for some $i$. If $x_0$ is not incident with any edge in $M_i^+$, then define $M_i^* = M_i^+ (1 \leq i \leq q)$ and $M_{q+1}^* = \phi$. If there is an edge in $M_i^+$ incident with $x_0$, say $e_{i_0}$, then define $M_i^* = M_i^+ (i \in \{1, \ldots, q\} \setminus \{i_0\})$, $M_i^* = M_i^0 \setminus \{e_{i_0}\}$ and $M_{q+1}^* = \{e_{i_0}\}$.

Note that

$$|M_k^*| \leq \frac{1}{2}(q + 3 - k) \quad (1 \leq k \leq q + 1).$$

A near 1-factor $F$ of $G - w$ is a set of $\frac{1}{2}((V(G - w)) - 1)$ independent edges of $G - w$. We say that the vertex which is not incident with any edge of $F$ is "missed" by $F$. We choose $q + 1$ edge-disjoint near 1-factors $F_1, \ldots, F_{q+1}$ of $G - w$ such that

$$E^+ \cap (M_1^* \cup \cdots \cup M_k^*) \subseteq F_1 \cup \cdots \cup F_k \quad (1 \leq k \leq q + 1),$$

$$M_0 \cap (F_1 \cup \cdots \cup F_{q+1}) = \emptyset$$

and furthermore,

if $v^*w_i \in M_k^*$ for some $i \in \{1, \ldots, s\}$, then $v^*w_i \in F_k$ and $F_k$ misses $x_i$,

and

if $v^*w_i \notin M_k$ for all $i \in \{1, \ldots, s\}$, then $F_k$ misses $v^*$.

To choose $F_k$ ($1 \leq k \leq q + 1$), suppose that $F_1, \ldots, F_{k-1}$ have been chosen already. Let

$$M_k = (E^+ \cap M_k^*) \setminus (F_1 \cup \cdots \cup F_{k-1} \cup M_0).$$

Then

$$|M_k^*| \leq \frac{1}{2}(q + 3 - k) \quad (1 \leq k \leq q + 1).$$

Consider

$$G_{k-1} = (G - w) \setminus (F_1 \cup \cdots \cup F_{k-1} \cup M_0).$$

We choose $F_k$ to be a near 1-factor of $G_{k-1}$ containing $M_k$ and missing $x_i$ if $v^*w_i \in M_k$ for some $i \in \{1, \ldots, s\}$, or missing $v^*$ if $v^*w_i \notin M_k$ for all $i \in \{1, \ldots, s\}$.

Let $V(M_k)$ denote the set of vertices of $G$ which are incident with the edges of $M_k$, and define $G_{k-1}^* = G_{k-1} \setminus V(M_k)$. To see that we can choose $F_k$ in the way described, we apply Lemma 2 (Dirac's theorem) to show that $G_{k-1}^*$ has a
Hamiltonian circuit. We have
\[ \delta(G^*_{k-1}) \geq (d - 1) - \{(k - 1) + 1\} - |V(M_k)| = d - k - 1 - |V(M_k)|. \]
Also
\begin{align*}
\frac{1}{2} |V(G^*_{k-1})| &= \frac{1}{2}(|V(G_{k-1})| - |V(M_k)|) \\
&= \frac{1}{2}(2n - 1 - |V(M_k)|) \\
&= n - \frac{1}{2} - \frac{1}{2} |V(M_k)|.
\end{align*}
Therefore
\begin{align*}
\delta(G^*_{k-1}) - \frac{1}{2} |V(G^*_{k-1})| &\geq d - k - 1 - |V(M_k)| - n + \frac{1}{2} + \frac{1}{2} |V(M_k)| \\
&= d - n - k - \frac{1}{2} - \frac{1}{2} |V(M_k)| \\
&\geq d - n - k - \frac{1}{2} - \frac{1}{2}(q + 3 - k) \\
&= d - n - \frac{1}{2}q - \frac{1}{2}k - 2 \\
&\geq d - n - q - \frac{3}{2} \\
&\geq d - n - (4n - 2d - \tilde{p} - 3) - \frac{3}{2} \\
&= 3d - 5n + \tilde{p} + \frac{1}{2} \\
&\geq 0,
\end{align*}
since \( d \geq \frac{5}{8}(2n) - \frac{1}{2}\tilde{p} - \frac{1}{6}. \) Therefore by Lemma 2, \( G^*_{k-1} \) does have a Hamiltonian circuit. It follows that \( G_{k-1} \) contains a near 1-factor \( F_k \) which contains \( M_k \) and misses \( x_i \) if \( v^*w_i \in M^*_k \) for some \( i \in \{1, \ldots, s\} \), or misses \( v^* \) if \( v^*w_i \notin M^*_k \) for all \( i \in \{1, \ldots, s\} \). It is easy now to check that \( F_k \) has all the various properties required of it.

Let \( J = \{i: F_i \text{ misses } v^*\} \). The graph \( ((G - w) \setminus (F_1 \cup \cdots \cup F_{q+1})) \setminus \{v^*\} \) has core of the form of Theorem 5, with each tree \( T_j \) just consisting of a single edge. Therefore \( ((G - w) \setminus (F_1 \cup \cdots \cup F_{q+1})) \setminus \{v^*\} \) is Class 1. The graph
\begin{align*}
(((G - w) \setminus (F_1 \cup \cdots \cup F_{q+1})) \setminus \{v^*\}) \cup \{F_i: i \in J\} \\
= ((G - w) \setminus \{F_i: i \in \{1, \ldots, q + 1\} \setminus J\}) \setminus \{v^*\}
\end{align*}
is therefore Class 1. By Lemma 1, it now follows that
\( (G - w) \setminus \{F_i: i \in \{1, \ldots, q + 1\} \setminus J\} \) is Class 1. It therefore follows that \( G - w \) is Class 1. In any edge-colouring of \( G - w \) with \( d(G) \) colours, it is easy to see by counting that each colour is missing from exactly one vertex. Therefore an edge-colouring of \( G - w \) can be extended to an edgicolouring of \( G \). Thus \( G \) is Class 1.

This completes the proof of Theorem 1. \( \square \)

5. The proof of Theorem 2

The argument in the last section improved our result from \( \frac{9}{8} |V(G)| \) to \( \frac{5}{8} |V(G)| \), an improvement of \( \frac{9}{8} - \frac{5}{8} \approx 0.024 \). In this section we use a counting argument to improve our bound further by about 0.01.
First we give a bound on $\bar{p}$.

**Lemma 3.**

$$\bar{p} \geq \frac{(2n - d - 1)(2n - d - 2)}{2n - 1}.$$ 

**Proof.** Each vertex in $\tilde{G}$ is the centre of $\tilde{G}$ of \(\binom{2n - d - 1}{2}\) paths of length 2. There are therefore altogether $2n \binom{2n - d - 1}{2}$ paths of length two in $\tilde{G}$. Therefore the average number of paths of length two joining an arbitrary pair of vertices is

$$2n \binom{2n - d - 1}{2} \geq \frac{(2n - d - 1)(2n - d - 2)}{2n - 1}.$$ 

Clearly $\bar{p}$ is greater than or equal to this average number. This proves Lemma 3. □

**Proof of Theorem 2.** From Theorem 1 and Lemma 3, it follows that if

$$d(G) \geq \left(\frac{1}{2}\right)(2n) - \left(\frac{1}{3}\right) \frac{(2n - d - 1)(2n - d - 2)}{2n - 1} - \frac{1}{6},$$

then $G$ is Class 1. After multiplying out and simplifying, the inequality becomes

$$d^2 + 2nd - (6n^2 - \frac{3}{2}) \geq 0,$$

so that

$$d \geq -n + \sqrt{7n^2 - \frac{3}{2}}.$$ 

suffices. Therefore $G$ is 1-factorizable if

$$d \geq \frac{1}{2}(\sqrt{7} - 1)|V(G)|.$$ 

This proves Theorem 2. □

**6. The proof of Theorem 3**

If we substitute $\bar{p} = |V(G)| - d(G) - 1$ into the inequality $d(G) \geq \frac{5}{6} |V(G)| - \frac{1}{2}\bar{p} - \frac{1}{6}$, we obtain the inequality $d(G) \geq \frac{3}{4} |V(G)| + \frac{1}{4}$. □

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