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Nonsmooth multiobjective optimization using limiting subdifferentials

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Abstract

In this study, using the properties of limiting subdifferentials in nonsmooth analysis and regarding a separation theorem, some weak Pareto-optimality (necessary and sufficient) conditions for nonsmooth multiobjective optimization problems are proved.

Keywords: Limiting subdifferential; Lipschitz function; Invex function; Multiobjective optimization; Weak Pareto-optimality

1. Introduction

Recently, much attention has been paid to characterize the Pareto-optimality conditions for multiple-objective optimization problems, under various types of conditions. Under smooth conditions (say, convexity and generalized convexity as well as differentiability) optimality conditions for these problems have been studied by some scholars, see, e.g., Hanson and Mond [8], Giorgi and Guerraggio [5], Kaul et al. [10], Rueda and Hanson [18], Aghezzaf and Hachimi [1,2], Mishra et al. [11–13,15], Soleimani-damaneh [19] and references therein. The nondifferentiable case has been studied by some scholars, too. See, e.g., Hachimi and Aghezzaf [6], Mishra et al. [14,16], and Yang et al. [20], among others.

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In this paper we deal with the limiting subdifferentials for Lipschitz functions following [4,17] as well as the concept of invexity along the lines of [7,9] and characterize the weak Pareto-optimal solutions under locally Lipschitz and invexity conditions. Our idea is based on an algebraic construction for the set of improving directions and the set of feasible directions using limiting subdifferential properties in nonsmooth analysis and finally using a separation theorem in convex analysis to derive the optimality conditions. Section 2 contains some basic definitions and primary results. In Section 3 the main results of our study have been sketched and established through four theorems.

2. Preliminaries

For $x, y \in \mathbb{R}^n$, $x < y$ means $x_i < y_i$ for all $i = 1, \ldots, n$. $x \leqslant y$ means $x_i \leqslant y_i$ for all $i = 1, \ldots, n$, but $x \neq y$. $x \leqq y$ allows equality. In this paper we consider the multiobjective optimization problem

$$\min\bigl\{f(x) = \bigl(f_1(x), \ldots, f_m(x)\bigr)\colon\ x \in X,\ g_j(x) \leqq 0;\ j = 1, 2, \ldots, p\bigr\}, \tag{1}$$

where X is a nonempty open set in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$, and $g = (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$.

This section contains some basic definitions and results which are useful for the rest of the paper.

Consider the feasible set of (1) as follows:

$$S = \bigl\{x \in X\colon\ g(x) = \bigl(g_1(x), \ldots, g_p(x)\bigr) \leqq 0\bigr\}.$$

Definition 2.1. Let $\phi \neq S \subseteq \mathbb{R}^n$ and let $\bar{x} \in cl S$. The set of *feasible directions* of S at $\bar{x}$, denoted by $D^{\bar{x}}$, is given by

$$D^{\bar{x}} = \bigl\{d \in \mathbb{R}^n\colon\ d \neq 0 \text{ and } \exists \delta > 0 \text{ such that } \bar{x} + \lambda d \in S\ \forall \lambda \in (0, \delta)\bigr\}.$$

Note that if $\bar{x} \in \operatorname{int} S$, then $D^{\bar{x}} = \mathbb{R}^n$.

Definition 2.2. Let $\phi \neq S \subseteq \mathbb{R}^n$ and let $\bar{x} \in S$. The set of *descent directions* at $\bar{x}$, denoted by $F^{\bar{x}}$, is given by

$$F^{\bar{x}} = \bigl\{d \in \mathbb{R}^n\colon\ \exists \delta > 0 \text{ such that } f(\bar{x} + \lambda d) < f(\bar{x})\ \forall \lambda \in (0, \delta)\bigr\}.$$

Definition 2.3. We say that $\bar{x} \in S$ is a weak Pareto-optimal solution (WPOS) of problem (1) if there exists no $x \in S$ such that $f(x) < f(\bar{x})$.

Lemma 2.1. *If $\bar{x}$ is a WPOS of* (1) *then $F^{\bar{x}} \cap D^{\bar{x}} = \phi$.*

Proof. Straightforward. □

Let $\mathcal{A}$ be a nonempty subset of $\mathbb{R}^n$ and $y \in \mathbb{R}^n$. The *metric projection* of y on $\mathcal{A}$ is defined as

$$\operatorname{Proj}_{\mathcal{A}}(y) = \bigl\{\bar{x} \in \mathcal{A}\colon\ \|y - \bar{x}\| \leqq \|y - x\|\ \forall x \in \mathcal{A}\bigr\}.$$

The *proximal cone* to $\mathcal{A}$ at $\bar{x}$, denoted by $N^P_{\mathcal{A}}(\bar{x})$, is given by

$$N^P_{\mathcal{A}}(\bar{x}) = \bigl\{\zeta \in \mathbb{R}^n\colon\ \exists\bigl(\lambda \geqq 0, y \in \mathbb{R}^n\bigr) \text{ such that } \bar{x} \in \operatorname{Proj}_{\mathcal{A}}(y) \text{ and } \zeta = \lambda(y - \bar{x})\bigr\}.$$

Considering $h : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}$, the set of *proximal subdifferential vectors* of h at $\bar{x}$, denoted by $\partial_P h(\bar{x})$, is defined as

$$\partial_P h(\bar{x}) = \{\zeta \in \mathbb{R}^n: \ (\zeta, -1) \in N^P_{\operatorname{epi} h}(\bar{x}, h(\bar{x}))\}.$$

Definition 2.4. A vector $\eta \in \mathbb{R}^n$ is a *limiting subdifferential vector* of h at $\bar{x}$ if there exist two sequences $\{\zeta_i\}$ and $\{x_i\}$ in $\mathbb{R}^n$ such that $\zeta_i \in \partial_P h(x_i)$, $\zeta_i \to \eta$, $x_i \to \bar{x}$, and $h(x_i) \to h(\bar{x})$. The set of all limiting subdifferential vectors of h at $\bar{x}$ is denoted by $\partial_L h(\bar{x})$.

Definition 2.5. A function $h : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}$ is (V, α)*-invex* at $\bar{x} \in \mathcal{A}$ if there exist a positive real-valued function α and an n-dimensional vector-valued function $V : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^n$ such that

$$h(x) - h(\bar{x}) \geqq \alpha(x, \bar{x})\eta^T V(x, \bar{x}),$$

for every $x \in \mathcal{A}$ and every $\eta \in \partial_L h(\bar{x})$. h is said to be invex near $\bar{x}$ if it is invex at each point of a neighborhood of $\bar{x}$.

Recall that a function $h : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz on $\mathcal{A}$ if there exists a $k \in \mathbb{R}$ such that

$$|h(x) - h(y)| \leqq k\|x - y\| \quad \forall x, y \in \mathcal{A}.$$

h is said to be Lipschitz near x if it is Lipschitz on a neighborhood of x. Also, h is locally Lipschitz on $\mathcal{A}$ if it is Lipschitz near x for every $x \in \mathcal{A}$.

The following results are known in nonsmooth analysis (see [4,17]).

Lemma 2.2. *Let h be Lipschitz near x, then $\partial_L h(\bar{x}) \neq \phi$.*

Proof. See [17, Remark 2.1]. □

Lemma 2.3. *Let h be locally Lipschitz on $\bar{x} \in S$, then $\partial_L h(\bar{x})$ is a convex and closed set. In fact, if $x_i \to \bar{x}$, $\eta_i \in \partial_L h(x_i)$, $\eta_i \to \eta$, then $\eta \in \partial_L h(\bar{x})$.*

Theorem 2.1. *Let function h be locally Lipschitz on a neighborhood of line segment* $[x, y]$. *Then for every $\epsilon > 0$ there exists a point z in the ϵ-neighborhood of* $[x, y]$ *and $\zeta \in \partial_P h(z)$ such that* $h(x) - h(y) \leqq \zeta^T(y - x) + \epsilon$.

The following separation theorem is known in convex analysis (see [3]).

Theorem 2.2. *Let $\mathcal{A}$ be a nonempty closed convex set in $\mathbb{R}^n$ and $y \notin \mathcal{A}$. Then, there exists a nonzero $p \in \mathbb{R}^n$ and a scalar α such that $p^T y > \alpha$ and $p^T x \leqq \alpha$ for each $x \in \mathcal{A}$.*

3. Main results

The first two theorems of this section give an algebraic representation of one of the subsets of $F^{\bar{x}}$ and $D^{\bar{x}}$, respectively.

Theorem 3.1. *Consider $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ such that f_i for each i, $1 \leqq i \leqq m$, is Lipschitz near $\bar{x}$, and there exists a $d \in \mathbb{R}^n$ such that $\eta^T d < 0$ for all $\eta \in \bigcup_{1 \leqq i \leqq m} \partial_L f_i(\bar{x})$. Then $d \in F^{\bar{x}}$.*

Proof. By the assumption of the theorem for each $1 \leqq i \leqq m$ there exists a $\gamma_i > 0$ such that f_i is Lipschitz on $B_{\gamma_i}(\bar{x}) = \{x \in \mathbb{R}^n\colon \|x - \bar{x}\| < \gamma_i\}$. Considering $\lambda \in (0, \frac{\gamma_i}{2\|d\|})$, f_i is locally Lipschitz on a neighborhood of the line segment $[\bar{x}, \bar{x} + \lambda d]$. Using Theorem 2.1 for each $\epsilon > 0$, there exist z_ϵ in the ϵ-neighborhood of $[\bar{x}, \bar{x} + \lambda d]$ and $\zeta_\epsilon \in \partial_P f_i(z_\epsilon)$ such that

$$\frac{f_i(\bar{x} + \lambda d) - f_i(\bar{x})}{\lambda} \leqq d^T \zeta_\epsilon + \frac{\epsilon}{\lambda}.$$

If $\epsilon \to 0$, then the sequence $\{\zeta_\epsilon\}$ has a subsequence, say $\{\bar{\zeta}_\epsilon\}$, and there exists a corresponding subsequence of $\{z_\epsilon\}$, say $\{\bar{z}_\epsilon\}$, such that $\bar{\zeta}_\epsilon \in \partial_P f_i(\bar{z}_\epsilon)$, $\bar{z}_\epsilon \to \hat{x}$, and $\bar{\zeta}_\epsilon \to \hat{\zeta}$, where $\hat{x} \in [\bar{x}, \bar{x} + \lambda d]$ and hence $\hat{\zeta} \in \partial_L f_i(\hat{x})$ by Lemma 2.3. Now $\lambda \to 0$ implies $\hat{x} \to \bar{x}$ and $\hat{\zeta} \to \eta \in \partial_L f_i(\bar{x})$ by Lemma 2.3. Therefore

$$\lim_{\lambda \to 0^+} \lim_{\epsilon \to 0^+} d^T \zeta_\epsilon + \frac{\epsilon}{\lambda} = \eta^T d$$

and thus

$$\lim_{\lambda \to 0^+} \lim_{\epsilon \to 0^+} \frac{f_i(\bar{x} + \lambda d) - f_i(\bar{x})}{\lambda} < 0.$$

This implies that there exists a $\delta_i > 0$ such that $f_i(\bar{x} + \lambda d) < f_i(\bar{x})$ for each $\lambda \in (0, \delta_i)$. Therefore $f(\bar{x} + \lambda d) < f(\bar{x})$ for each $\lambda \in (0, \delta)$, where $\delta = \min_{1 \leqq i \leqq m}\{\delta_i\}$. Thus $d \in F^{\bar{x}}$ and the proof is complete. □

Theorem 3.2. *Let $\bar{x} \in S$ be a feasible solution for* (1) *and $I(\bar{x}) = \{j\colon g_j(\bar{x}) = 0\}$. Suppose that $g_j(x)$ for $j \in I(\bar{x})$ is Lipschitz near $\bar{x}$ and $g_j(x)$ for $j \notin I(\bar{x})$ is continuous at $\bar{x}$. Also suppose that there exists a $d \in \mathbb{R}^n$ such that $\eta^T d < 0$ for all $\eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x})$. Then $d \in D^{\bar{x}}$.*

Proof. By the assumption of the theorem and similar to the proof of Theorem 3.1, it can be shown that for each $j \in I(\bar{x})$ there exists a $\delta_j > 0$ such that $g_j(\bar{x} + \lambda d) < g_j(\bar{x}) = 0$ for each $\lambda \in (0, \delta_j)$.

For each $j \notin I(\bar{x})$, since $g_j(x)$ is continuous at $\bar{x}$ and $g_j(\bar{x}) < 0$, there exists a $\delta'_j > 0$ such that $g_j(\bar{x} + \lambda d) < 0$ for each $\lambda \in (0, \delta'_j)$.

Since X is open, there exists a $\delta'' > 0$ such that $\bar{x} + \lambda d \in X$ for each $\lambda \in (0, \delta'')$.

Now by setting

$$\delta = \min\Big\{\delta'', \min_{j \in I(\bar{x})}\{\delta_j\}, \min_{j \notin I(\bar{x})}\big\{\delta'_j\big\}\Big\}$$

we get $\bar{x} + \lambda d \in S$ for each $\lambda \in (0, \delta)$. Hence $d \in D^{\bar{x}}$ and the proof is complete. □

The rest of this section contains two theorems which characterize the WPOSs of problem (1).

Theorem 3.3 *(Necessary condition). Let $\bar{x} \in S$ be a feasible solution for* (1) *and $I(\bar{x}) = \{j\colon g_j(\bar{x}) = 0\}$. Suppose that $f_i(x)$ for $i = 1, 2, \ldots, m$ and $g_j(x)$ for $j \in I(\bar{x})$ are Lipschitz near $\bar{x}$ and $g_j(x)$ for $j \notin I(\bar{x})$ is continuous at $\bar{x}$. If $\bar{x}$ is a WPOS of* (1)*, then there exists a $u = (v_1, \ldots, v_m, u_1, \ldots, u_p) \geqslant 0$ such that*

$$0 \in \sum_{i=1}^{m} v_i \partial_L f_i(\bar{x}) + \sum_{j=1}^{p} u_j \partial_L g_j(\bar{x})$$

and

$$u_j g_j(\bar{x}) = 0, \quad j = 1, 2, \dots, p.$$

Proof. To start the proof we define

$$F_0^{\bar{x}} = \Big\{ d \in \mathbb{R}^n \colon \eta^T d < 0\ \forall \eta \in \bigcup_{1 \leqq i \leqq m} \partial_L f_i(\bar{x}) \Big\}, \tag{2}$$

$$D_0^{\bar{x}} = \Big\{ d \in \mathbb{R}^n \colon \eta^T d < 0\ \forall \eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x}) \Big\}. \tag{3}$$

Since $\bar{x}$ is a WPOS, then $F^{\bar{x}} \cap D^{\bar{x}} = \phi$ by Lemma 2.1 and hence $F_0^{\bar{x}} \cap D_0^{\bar{x}} = \phi$ regarding Theorems 3.1 and 3.2. Furthermore we form the set

$$\mathcal{B} = \Big\{ b \in \mathbb{R}^n \colon \exists u = \big(v_1, \dots, v_m, u_j;\ j \in I(\bar{x})\big) \geqslant 0 \text{ and } \exists \big(d_1, \dots, d_m, d'_j;\ j \in I(\bar{x})\big) \text{ such that } d_i \in \partial_L f_i(\bar{x}),\ d'_j \in \partial_L g_j(\bar{x}) \text{ and } b = \sum_{i=1}^m v_i d_i + \sum_{j \in I(\bar{x})} u_j d'_j \Big\}.$$

This set is convex and closed. If $0 \in \mathcal{B}$, then the claim of the theorem is proved by setting $u_j = 0$ for $j \notin I(\bar{x})$. By contradiction suppose that $0 \notin \mathcal{B}$, then using Theorem 2.2 there exist a nonzero $p \in \mathbb{R}^n$ and a scalar α such that $p^T(0) > \alpha$ and $p^T b \leqq \alpha$ for each $b \in \mathcal{B}$. Hence $p^T b < 0$ for each $b \in \mathcal{B}$.

For each $\eta \in \bigcup_{1 \leqq i \leqq m} \partial_L f_i(\bar{x})$ as well as for each $\eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x})$, we have $\eta \in \mathcal{B}$. Therefore $p^T \eta < 0$ for each $\eta \in \bigcup_{1 \leqq i \leqq m} \partial_L f_i(\bar{x})$ as well as for each $\eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x})$. Hence $p \in F_0^{\bar{x}} \cap D_0^{\bar{x}}$ which contradicts the result obtained in the first part of the proof and completes the proof. □

Theorem 3.4 *(Sufficient condition). Let $\bar{x} \in S$ be a feasible solution for* (1) *and $I(\bar{x}) = \{j \colon g_j(\bar{x}) = 0\}$. Suppose that $f_i(x)$ for $i = 1, 2, \dots, m$ and $g_j(x)$ for $j \in I(\bar{x})$ are (α_i, V)-invex and (β_j, V)-invex near $\bar{x}$, respectively, for some real-valued functions α_i, β_j, defined on $X \times X$ and the same vector-valued function $V : X \times X \to \mathbb{R}^n$. $F_0^{\bar{x}}$ is as defined in* (2). *If $F_0^{\bar{x}} \cap \{d \in \mathbb{R}^n \colon \eta^T d \leqq 0\ \forall \eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x})\} = \phi$, then $\bar{x}$ is a local WPOS of* (1).

Proof. By the assumption of the theorem, for each $1 \leqq i \leqq m$ there exists an $\epsilon_i > 0$ such that f_i is Lipschitz on $B_{\epsilon_i}(\bar{x})$ and

$$f_i(x) - f_i(\bar{x}) \geqq \alpha_i(x, \bar{x}) \eta_i^T V(x, \bar{x}), \tag{4}$$

for each $x \in B_{\epsilon_i}(\bar{x})$ and for every $\eta_i \in \partial_L f_i(\bar{x})$. Also for each $j \in I(\bar{x})$ there exists a $\delta_j > 0$ such that g_j is Lipschitz on $B_{\delta_j}(\bar{x})$ and

$$\beta_j(x, \bar{x}) \eta_j^T V(x, \bar{x}) \leqq g_j(x) - g_j(\bar{x}) = g_j(x) \leqq 0, \quad j \in I(\bar{x}), \tag{5}$$

for every $x \in B_{\delta_j}(\bar{x})$ and for every $\eta_j \in \partial_L g_j(\bar{x})$.

Setting $\epsilon = \min\{\min_{1 \leqq i \leqq m}\{\epsilon_i\}, \min_{j \in I(\bar{x})}\{\delta_j\}\}$, relations (4) and (5) are valid for each $x \in B_\epsilon(\bar{x})$.

If there exists an $\hat{x} \in B_\epsilon(\bar{x})$ such that $f(\hat{x}) < f(\bar{x})$, then by (4) and (5) and regarding the fact that α_js and β_js are positive, we get

$$\eta_i^T V(\hat{x},\bar{x}) < 0, \quad \forall \eta_i \in \partial_L f_i(\bar{x}),\ i = 1,2,\ldots,m,$$
$$\eta_j^T V(\hat{x},\bar{x}) \leqq 0, \quad \forall \eta_j \in \partial_L g_j(\bar{x}),\ j \in I(\bar{x}).$$

These imply that

$$V(\hat{x},\bar{x}) \in F_0^{\bar{x}} \cap \left\{ d \in \mathbb{R}^n\colon\ \eta^T d \leqq 0\ \forall \eta \in \bigcup_{j \in I(\bar{x})} \partial_L g_j(\bar{x}) \right\}$$

which contradicts the assumption. Therefore there exists no $x \in B_\epsilon(\bar{x})$ such that $f(x) < f(\bar{x})$, which completes the proof. □

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