On Maximal Asymptotic Nonbases of Zero Density

S. TURJÁNYI

Mathematics Department University of Debrecen, Debrecen, Hungary

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INTRODUCTION

Nathanson introduced in [1] the notion of a maximal asymptotic nonbasis of order \( h \) as the dual of the notion of a minimal asymptotic basis of order \( h \). He calls a strictly increasing sequence \( A \) of nonnegative integers a maximal asymptotic nonbasis of order \( h \), if \( A \) has the following two properties:

(i) \( A \) is an asymptotic nonbasis of order \( h \);

(ii) if \( b \) is a nonnegative integer and \( b \notin A \), then \( A \cup b \) is an asymptotic basis of order \( h \).

In the same paper Nathanson shows that under certain conditions the union of suitable congruence classes satisfies conditions (i) and (ii), that is, it yields a maximal asymptotic nonbasis of positive density. He posed the problem, on the one hand, of whether there exists an asymptotic nonbasis of order \( h \), which cannot be represented as the union of congruence classes and, on the other hand, whether there exists a maximal asymptotic nonbasis of order \( h \geq 2 \) such that \( \lim \inf \frac{A(x)}{x} = 0 \) (here \( A(x) \) denotes as usual the number of those elements of \( A \) which are not greater than \( x \)). Erdős and Nathanson constructed in [2] an asymptotic nonbasis of order \( h \geq 2 \) which cannot be represented as the union of congruence classes and thus they gave an answer to the first question. The second question, however, remained open.

It is the aim of our paper to give a positive answer to this open question in the case \( h = 2 \). We give a procedure by which from every basis of order 2 which satisfies the condition \( \lim_{x \to \infty} \frac{A(x)}{x} = 0 \), a maximal asymptotic nonbasis with the desired property can be constructed. In the first step of our construction we form a set of sequences starting with \( A \); with the aid of this set of sequences we define then a sequence \( A^* \) which can easily be proved to have properties (i) and (ii) and \( \lim \inf \frac{A^*(x)}{x} = 0 \).

In [3, 4] one can find examples of basis sequences for which \( \lim_{x \to \infty} \frac{A(x)}{x} = 0 \).
THE CONSTRUCTION OF MAXIMAL ASYMPTOTIC NONBASES OF ORDER 2 WITH ZERO DENSITY

LEMMA 1. Let $A$ be a sequence for which \( \lim_{x \to \infty} \left[ \frac{A(x)}{x} \right] = 0. \) Then for any sufficiently small $\epsilon > 0$ and for any sufficiently large $M$ there exist $a_i \in A$ such that the following conditions are simultaneously satisfied:

1. $\frac{A(a_i)}{a_i} < \epsilon$;
2. $a_i - a_{i-1} > M$;
3. if $i > j$ then $a_i - a_{i-1} > a_j - a_{j-1}$.

Proof. It is easy to see that if there exist such $a_i$ corresponding to a fixed $\epsilon$ and $M$, then for this $a_i$ the same $\epsilon$ and every $m < M$ will do since $a_i - a_{i-1} > M > m$. We may assume that $A^*$ is greater than $D$, where 

\[
D \overset{\text{def}}{=} \max_{a_i \in A^*} (a_i - a_{i-1}).
\]

Here $A^*$ denotes the set of those numbers $a_i \in A$ for which $\frac{A(a_i)}{a_i} \geq \epsilon$. This set is obviously finite, since \( \lim_{x \to \infty} \left[ \frac{A(x)}{x} \right] = 0. \) The inequality

\[
a_i - a_{i-1} > M \quad (a_i, a_{i-1} \in A)
\]

has infinitely many solutions satisfying (1). For, if (4) had only a finite number of solutions (but at least one), then $\lim \inf[A(x)/x] = 1/\bar{a}_j$, where $\bar{a}_j$ denotes the greatest of these solutions. On the other hand, if (4) had no solution at all, then $\lim \inf[A(x)/x] \geq 1/M$, that is, both cases contradict our assumption that \( \lim_{x \to \infty} [A(x)/x] = 0. \) Choose among the common solutions of (1) and (4) the smallest and denote it by $a_{i_0}$. It is obvious that this can be done in a unique manner. We have still to show that

\[
a_{i_0} - a_{i_0-1} > a_j - a_{j-1}
\]

is true for $i_0 > j$.

If $\frac{A(a_j)}{a_j} < \epsilon$ then this is clear because of the minimality of $a_{i_0}$. If $\frac{A(a_j)}{a_j} \geq \epsilon$ then the inequality $a_j - a_{j-1} \leq D < M < a_{i_0} - a_{i_0-1}$ shows that (3) is valid.

LEMMA 2. There exist a sequence of sequences $A_1, A_2, ..., A_k, ...$ and an
increasing sequence of natural numbers \( m_1, m_2, \ldots, m_k, \ldots \) such that the following conditions are satisfied:

(5) If \( x \in A_{k-1} \) and \( x \leq m_k \) then \( x \in A_k \).

(6) \( 2m_1 + 1, 2m_2 + 1, \ldots, 2m_k + 1 \notin 2A_k \) but for an arbitrary non-negative \( n \) which is distinct from these, the relation \( n \in A_k + A_k \) holds.

(7) If \( x \notin A_k \) and \( 0 \leq x < m_k \) then

\[
2m_k + 1 \in (A_k \cup \{x\}) + (A_k \cup \{x\}),
\]

(8) \( \lim_{k \to \infty} [A_k(m_k)/m_k] = 0 \).

Proof. We prove the lemma by induction on \( k \). Let \( A_0 \) be a basis of order 2, for which \( \lim_{x \to \infty} [A_0(x)/x] = 0 \) holds. Let \( m_1 = a_1 - 1 \), where \( a_1 \) denotes the smallest element of \( A_1 \), satisfying the conditions of Lemma 1 with \( \epsilon_1 = \frac{1}{2} \) and \( M_1 = 3 \). (Instead of \( \epsilon_1 \) and \( M_1 \), any other \( \epsilon \) and \( M \) could have been chosen provided that \( 1 > \epsilon > 0 \) and \( M > 2 \).) Let \( A_1 = A_0' \cup A_0'' \cup A_0''', \)

where

\[
A_0' = \{ x \mid x \leq m_k \text{ and } x \in A_0 \},
\]

\[
A_0'' = \{ x \mid x = 2m_1 + 1 - y, \text{ where } y \notin A_0 \text{ and } 0 \leq y \leq m_1 \},
\]

\[
A_0''' = \{ x \mid x > 2m_1 + 1 \text{ and } x \in A_0 + \{ z \mid z \leq 2m_1 + 1 \text{ and } z \in A_0 \} \}.
\]

We show first that every nonnegative integer with the exception of \( 2m_1 + 1 \) belongs to \( A_1 + A_1 \). Those integers \( z \) which are greater than \( 2m_1 + 1 \) belong to \( A_0''' + A_0'' \), since \( z \in A_0 + A_0 \). On the other hand, the integers which are smaller than \( m_1 \) belong to \( A_1 + A_1 \). It is therefore sufficient to show that the numbers of the form

\[
z = 2m_1 + 1 - a_i, \quad a_i \in A_0', \tag{9}
\]

are representable on the desired form.

We verify that the integers of the interval \( m_1 < z < 2m_1 + 1 \) are elements of \( A_0' + A_0'' \). \( A_0'' \) contains those integers which satisfy the inequality

\[
2m_1 + 1 - a_i < m_1 + 1 = a_i \leq u
\]

\[
\leq 2m_1 + 1 - (a_{i-1} + 1) = a_i - (a_{i-1} - 1) - 1, \tag{10}
\]

that is, it contains \( a_i = a_{i-1} - 1 \) consecutive integers.

Between the consecutive elements of \( A_0' \) there can be, on the other hand, an interval at most of length \( a_i - a_{i-1} - 1 \), since the \( a_i \) has been chosen in such a way that from \( i_0 > j \) the inequality

\[
a_{i_0} - a_{i_0-1} > a_j - a_{j-1}
\]
follows, which implies
\[ a_{i_0} - a_{i_0 - 1} - 1 \geq a_j - a_{j-1}. \]  
(11)

Because of (10) and (11) the numbers \( z \) with the property
\[ a_{i_0} + a_{j-1} \leq z \leq a_{i_0} + a_j - 1 \]  
(12)

can all be represented in the form \( a_j + u \), where \( u \) is an integer satisfying inequality (10). Finally, if \( j \) runs through 1, 2, ..., \( i_0 \) then from (12) it follows that the integers of the interval \((m_1, 2m_1 + 1)\) are indeed elements of \( A_{\theta'} + A_{\theta''} \). We remark that \( A_j(a_j)/a_j < \epsilon_1 = \frac{1}{2} \).

For the proof of \( \lim_{x \to \infty} [A_j(x)/x] = 0 \), it suffices to remember that we obtained \( A_1 \) from \( A_0 \) by changing a finite number of elements of \( A_0 \) and by adding a finite set to \( A_0 \), and by these changes the density of the sequence is not influenced. This completes the proof in the case \( k = 1 \).

Let now \( k \geq 2 \) and let \( m_k = \kappa a_{i_0} - 1 \), where \( \kappa a_{i_0} \) is an element of \( A_{k-1} \) which satisfies conditions (1)-(3) of Lemma 1 with \( \epsilon_k = \epsilon_1^k \) and \( \mu_k = m_{k-1}^k \). Let us define the sequence \( A_k \) by \( A_k = A_{k-1} + A_{k-1}' \cup A_{k-1}'' \), where
\[ A_{k-1}' = \{ x \mid x \leq m_k \text{ and } x \in A_{k-1} \}, \]
\[ A_{k-1}'' = \{ x \mid x = 2m_k + 1 - y \text{ with } y \notin A_{k-1} \text{ and } 0 \leq y \leq m_k \} \]
and
\[ A_{k-1}''' = \{ x \mid x > 2m_k + 1 \text{ and } x \in A_{k-1} + \{ z \mid z \leq 2m_k + 1 \text{ and } z \in A_{k-1} \} \}. \]

Since we have changed only finitely many elements of \( A_{k-1} \) and added only a finite set to \( A_{k-1} \) the obtained set \( A_k \) likewise satisfies the condition
\[ \lim_{x \to \infty} [A_k(x)/x] = 0. \]

We have still to show that \( A_k + A_k \) contains every nonnegative integer with the exception of the numbers \( 2m_i + 1 \) \((i = 1, 2, ..., k)\). Here it is sufficient to prove that the numbers \( z \) with the property \( m_k \leq z < 2m_k + 1 \) are contained in \( A_k + A_k \), the other cases being trivial by the construction.

As for the representability of the numbers \( z \) with the property \( m_k \leq z < 2m_k + 1 \), we may, however, repeat what has been said above for \( k = 1 \), writing only \( \kappa a_{i_0} \) instead of \( a_{i_0} \). Furthermore, because of the construction of \( A_k \), the relation \( 2m_i + 1 \notin 2A_k \), \( i = 1, 2, ..., k \) can be valid only if \( 2m_k + 1 \in A_{k-1}' + A_{k-1}'' \). This is, however, impossible because of the construction of \( A_{k-1}' \) and \( A_{k-1}'' \).

The sequence \( A_1, A_1, ..., A_k, ... \) satisfies (5), (6), and (7) by the construction of the sequences \( A_0, A_1, ..., A_k, ... \). Furthermore, we have \( [A_k(m_k)/m_k] < \epsilon_1^k \).
because of the choice of the $m_k$, and (8) also holds, since $0 < \epsilon_1 < 1$. This proves the assertion of Lemma 2.

Now we form the sequence $A^*$ in the following way: Let the elements of $A^*$ be those nonnegative integers $N$, for which there exists a $k$, such that $N < m$ and $N \in A_k$. $A^* + A^*$ contains every natural number with the exception of $2m_1 + 1, 2m_2 + 1, \ldots, 2m_k + 1, \ldots$. Because of (8) we obviously have $\lim \inf [A^*(x)/x] = 0$. If $x_0 > 0$ is an arbitrary integer and $x_0 \notin A^*$, then $(A^* \cup \{x_0\}) + (A^* \cup \{x_0\})$ contains all those numbers $2m_i + 1$ which are greater than $x_0$. For, if $x_0 \leq m_i$ then $x_0 \notin A_i$ and thus $2m_i + 1 - x_0 \in A_i$ because of the construction of $A_i$. But $2m_i + 1 - x_0 < m_{i+1}$ and therefore $2m_i + 1 - x_0 \in A_{i+1}$ and thus $2m_i + 1 - x_0 \in A^*$.

Hence $(2m_i + 1 - x_0) + x_0 \in 2(A^* \cup \{x_0\})$ which yields that $\{A^* \cup \{x_0\}\}$ is an asymptotic basis of order 2.

References