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The characteristic finite volume element method for the nonlinear convection-dominated diffusion problem

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Abstract

In modern numerical simulation of prospecting and exploiting oil–gas resources and environmental science, it is important to consider a numerical method for nonlinear convection-dominated diffusion problems. Based on actual conditions, such as the three-dimensional characteristics of large-scale science-engineering computation, we present a kind of characteristic finite volume element method. Some techniques, such as calculus of variations, commutating operators, the theory of prior estimates and techniques, are adopted. Suboptimal order error estimate in L^2 norm and optimal order error estimate in H^1 norm are derived to determine the errors for the approximate solution. Numerical results are presented to verify the performance of the scheme.

Keywords: Nonlinear; Convection-dominated diffusion; Characteristic finite volume element; Error estimate; Numerical experiment

1. Introduction

In recent years, with the rapid development of energy resources and environmental science, it is very important to study the numerical computation of underground fluid flow and the history of its changes under heat. In actual numerical simulation, the nonlinear three-dimensional convection-dominated diffusion problems need to be considered.

We consider the mathematical model, the following nonlinear partial differential equations with initial-boundary value problems [1,2]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \vec{b}(x,u)\cdot\nabla u - \nabla\cdot\{a(x,u)\nabla u\} = f(x,t,u), & x=(x_1,x_2,x_3)^{\mathrm{T}}\in\Omega,\ t\in J=(0,T], \\ u(x,t)=0, & x\in\Gamma,\ t\in J, \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^{\mathrm{T}}$, $\vec{b}(x,u) = (b_1(x,u),\ b_2(x,u),\ b_3(x,u))^{\mathrm{T}}$, $\vec{b}(x,u)|_{\Gamma} = \vec{0} = (0,0,0)^{\mathrm{T}}$, $\Omega \subset R^3$ is a bounded region with boundary Γ,

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$$a(x,u) = \begin{pmatrix} a_1(x,u) & & \\ & a_2(x,u) & \\ & & a_3(x,u) \end{pmatrix}.$$

The finite volume element method (FVEM) is a discretization technique for partial differential equations, especially for those arising from physical conservation laws including mass, momentum and energy. This method has been introduced and analyzed by Li and his collaborators since 1980s [3]. The FVEM uses a volume integral formulation of the original problem and a finite element partition of the domain $\bar{\Omega} = \Omega \bigcup \Gamma$ to discretize the equations. The approximate solution is chosen from a finite element space [3–5]. The FVEM is widely used in computational fluid mechanics and heat transfer problems [4–7]. It possesses the important and crucial property of inheriting the physical conservation laws of the original problem locally. Thus it can be expected to capture shocks, to produce simple stencils, or to study other physical phenomena more effectively.

On the other hand, the convection-dominated diffusion problem has strongly hyperbolic characteristics, therefore constructing a numerical method to solve such a problem is very difficult in mathematics and mechanics. When the central difference method is used to solve the convection-dominated diffusion problem, although it has second-order accuracy, it produces numerical diffusion and oscillation near discontinuity. Douglas and Russell published an important paper on the characteristic finite element method and finite difference method [1] to overcome the difficulties. Tabata and his collaborators have been studying upwind schemes based on triangulation for the convection–diffusion problem since 1977 [8–12]. In modern numerical simulation of prospecting and exploiting oil–gas resources and environmental science, the problems are often three-dimensional large-scale ones. Yuan presented a characteristic finite element alternating direction method with moving meshes [2] and an upwind finite difference fractional step method [13] for simulating these problems.

Most of the papers in the literature concern FVEMs for one- and two-dimensional linear partial differential equations [3–6,14,15]. In recent years, Feistauer [16,17], by introducing a lumping operator, constructed a finite volume–finite element method for nonlinear convection–diffusion problems. On the other hand, because a finite element method (FEM) involves great expense in solving the multiple space problems, we usually adopt finite difference methods (FDMs) to approximate the problems [13]. In this paper, we present a characteristic finite volume element method (CFVEM) for multiple space nonlinear convection-dominated diffusion problems. We adopt some techniques, such as calculus of variations, commutating operator, the theory of prior estimates and techniques, and derive the suboptimal order error estimate in L^2 norm and the optimal order error estimate in H^1 norm to determine the errors for the approximate solution. Finally, numerical results show that the CFVEM is effective in avoiding numerical diffusion and nonphysical oscillations.

The paper is organized as follows. In Section 2, we describe the CFVEM for problem (1.1). In this section, we introduce notation, and construct mesh partition T_h of Ω and its dual partition. Some auxiliary lemmas and the corresponding proofs are shown in Section 3. The error estimates in L^2 norm and H^1 norm of the scheme are derived in Section 4. In Section 5, a numerical experiment shows that the method is effective in avoiding numerical diffusion and nonphysical oscillations.

Throughout this paper, we use C (without or with a subscript) to denote a generic constant independent of discretization parameters. We also adopt the standard notations of Sobolev spaces and norms and semi-norms as in [18,19].

2. The characteristic finite volume element method

We define the bounded set G on $\mathbf{R}$ as

$$G = \{u : |u| \le K_0\},$$

where K_0 is a positive constant.

We assume that the coefficients of problem (1.1) satisfy the following conditions:

(A1) Generally, (1.1) is a positive definite problem, i.e.,

$$0 < a_* \le a_i(x,u) \le a^*, \quad \forall (x,t) \in \Omega \times (0,T), u \in G\ (i = 1, 2, 3),$$

and there exist constants $C_1,\ C_2$ which depend on K_0 such that

$$|f(x,t,u)| \le C_1|u| + C_2, \quad \forall (x,t) \in \Omega \times (0,T), u \in R,$$
$$\vec{b}(x,u) = (b_1(x,u), b_2(x,u), b_3(x,u))^{\mathrm{T}} \in W^1_\infty(G) \times W^1_\infty(G) \times W^1_\infty(G),$$
$$f(x,t,u) \in W^1_\infty(G \times \Omega \times (0,T]),\ u_0(x) \in C(\bar{\Omega}) \bigcap H^1_0(\Omega).$$

(A2) $a(x,u),\ \vec{b}(x,u)$ and $f(x,t,u)$ are Lipschitz continuous with respect to the last variable u

$$|a_i(x,u) - a_i(x,v)| \le L|u-v|, \quad \forall u,\ v \in G,$$
$$\|\vec{b}(x,u) - \vec{b}(x,v)\| \le L|u-v|, \quad \forall u,\ v \in G,$$
$$|f(x,t,u) - f(x,t,v)| \le L|u-v|, \quad \forall u,\ v \in G,$$

where L is a Lipschitz constant related to K_0 and $\|\vec{b}(x,u) - \vec{b}(x,v)\|$ is defined as

$$\|\vec{b}(x,u) - \vec{b}(x,v)\| = \left\{ \sum_{i=1}^{3} (|b_i(x,u) - b_i(x,v)|)^2 \right\}^{1/2}.$$

The weak form of problem (1.1) is

$$\left(\frac{\partial u}{\partial t}, v\right)_\Omega + (\vec{b}(x,u)\cdot\nabla u, v)_\Omega + (a(x,u)\nabla u, \nabla v)_\Omega = (f(x,u), v)_\Omega, \quad \forall v \in H^1_0(\Omega),\ t \in (0,T], \tag{2.1}$$

where Ω is the given three-dimensional region with coordinates $x = (x_1, x_2, x_3)^{\mathrm{T}} \in R^3$. For simplicity, we omit subscript Ω in $(\cdot,\cdot)_\Omega$ and define $a(w;u,v) = (a(x,w)\nabla u, \nabla v)$; further, we have

$$\left(\frac{\partial u}{\partial t}, v\right) + (\vec{b}(x,u)\cdot\nabla u, v) + a(u;u,v) = (f(x,u), v), \quad \forall v \in H^1_0(\Omega), t \in (0,T], \tag{2.2}$$

and we assume that the weak solution u of problem (1.1) satisfies the following regularity:

(A3) $u \in L^\infty(0,T;H^2(\Omega)) \bigcap L^\infty(0,T;W^{1,\infty}_0(\Omega))$, $|u(x,t)| \le K_0, \forall (x,t) \in \Omega \times (0,T]$, $u_t, u_{tt} \in L^2(0,T;H^2(\Omega))$.

We discuss the CFVEM approximation of nonlinear convection-dominated diffusion problem (1.1). For convenience, let Ω be a cuboid domain $\Omega = (0,X_L)\times(0,Y_L)\times(0,Z_L)$. First, we consider a family of regular cuboid partition T_h of the domain $\bar{\Omega}$ [3]. Let h be maximum diameter of cell of T_h. For a fixed cuboid partition $T_h = \{K\}$, we define a closed cuboid set $\{K_i\}_{i=1}^{N_K}$ and node set $\bar{\Omega}_h = \{P_i\}_{i=1}^{M_2}$, where $\Omega_0 = \{P_i\}_{i=1}^{M_1}$ is the inner node set of Ω and $\Gamma_h = \bar{\Omega} - \Omega_0 = \{P_i\}_{i=M_1+1}^{M_2}$ is the boundary node set on $\partial\Omega$. Let $E_h = \{e_i : 1 \le i \le M_E\}$ be a set of all edges.

Definition 2.1. Suppose that $T = \{T_h : 0 < h \le h_0\}$ is a set of cuboid partition of Ω; the set T is termed regular if there exists a positive constant σ_1 independent of h, such that

$$\max_{K\in T_h} \frac{h_K}{\rho_K} \le \sigma_1, \quad \forall h \in (0,h_0),$$

where h_K and ρ_K are the diameter of K and the maximum diameter of the circumscribing sphere of cuboid K, respectively.

Definition 2.2. The cuboid partition T_h is called a Delaunay mesh if K does not include the remainder of the nodes of Ω_h for each $K \in T_h$.

Definition 2.3. The two cuboid cells are called face-adjacent if they have one common face, but edge-adjacent if they have one common edge.

Definition 2.4. The two nodes are called adjacent if they form one edge which belongs to E_h. Denote $\bigwedge_i = \{j : P_j$ is adjacent to $P_i, P_i, P_j \in \Omega_h\}$.

For a given cuboid partition T_h with nodes $\{P_i\} \in \Omega_h$ and edges $\{e_i\} \in E_h$, we construct two kinds of dual partitions. First, we define an average center dual partition of T_h. $\forall P_i \in \Omega_h$; let $\Omega_h(P_i) = \{K : K \in T_h,\ P_i$ is a vertex of $K\}$. Let Q_j be a center of $K(\in \Omega_h(P_i))$. Connecting Q_j $(1 \le j \le 8)$ of the two face-adjacent cuboid cells which belong to $\Omega_h(P_i)$, then we can derive a cuboid $K^*_{P_i}$ which surrounds the node P_i. Q_j $(1 \le j \le 8)$ are vertexes of $K^*_{P_i}$, which is called an average center dual partition corresponding to node P_i. $T^*_h = \{K^*_{P_i} : P_i \in \Omega_h\}$ is the average center dual partition of T_h. Suppose P_{ij} is the midpoint of P_i and its adjacent node P_j.

The other dual partition is as follows. $\forall e_k \in E_h$, let $\Omega_h(e_k) = \{K : K \in T_h$ and e_k be an edge of $K\}$. Denote two vertexes of the edge e_k by P_{k_1} and P_{k_2}. Q_j $(1 \le j \le 4)$ are the centers of the cuboids $K \in \Omega_h(e_k)$. Let $K^*_{e_k}$ be a polyhedron whose vertexes are P_{k_1}, P_{k_2} and Q_j $(1 \le j \le 4)$. $K^*_{e_k}$ is called a dual cell for edge e_k. $\bar{T}^*_h = \{K^*_{e_k}\}_{k=1}^{M_E}$ is a dual partition to T_h.

Let Ω^*_h denote the node set of dual partition. For $Q \in \Omega^*_h$, let K_Q denote a cuboid cell which includes Q. Let $|K^*_P|$ and $|K_Q|$ be the volumes of the dual cell K^*_P and cuboid cell K_Q, respectively. As follows, we assume that the partition family T_h is regular, i.e., there exist positive constants C_3, C_4 independent of h, such that the following condition (A4) satisfies:

$$\begin{cases} C_3h^3 \le |K^*_P| \le C_4h^3, & P \in \bar{\Omega}_h, \\ C_3h^3 \le |K_Q| \le C_4h^3, & Q \in \Omega^*_h. \end{cases} \tag{2.3}$$

Let trial function space $U_h \subset H^1_0(\Omega)$, whose basis functions are $\{\varphi(P),\ P \in \bar{\Omega}_h\}$, be an isoparametric three linear space based on T_h [3] and let test function space $V_h \subset L^2(\Omega)$ be a piecewise constant space on dual partition T^*_h, whose basis functions are $\{\psi(P),\ P \in \Omega^*_h\}$, defined by

$$\psi(P) = \begin{cases} 1, & P \in K^*_P, \\ 0, & \text{otherwise}, \end{cases}$$

and $\psi(P) = 0,\ P \in \Gamma_h$.

Let Π^*_h be an interpolation operator from H^1_0 to V_h satisfying

$$\Pi^*_h u = \sum_{K^*_P \in T^*_h} u(P)\psi(P). \tag{2.4}$$

For using the characteristic procedure to treat the first-order part of problem (1.1), we rewrite the Eq. (1.1) in the form

$$\frac{\partial u}{\partial t} = -\vec{b}(x,u)\cdot\nabla u + \nabla\cdot(a(x,u)\nabla u) + f(x,t,u). \tag{2.5}$$

Let $\tau = \tau(x,t)$ be an unit vector in the direction $(b_1(x,u), b_2(x,u), b_3(x,u), 1)$ and

$$\Psi(x,u) = [1 + |\vec{b}(x,u)|^2]^{1/2}, \quad |\vec{b}(x,u)|^2 = \sum_{i=1}^{3} b_i^2(x,u). \tag{2.6}$$

Then we have

$$\frac{\partial}{\partial\tau(x,u)} = \frac{1}{\Psi(x,u)}\frac{\partial}{\partial t} + \frac{\vec{b}(x,u)}{\Psi(x,u)}\cdot\nabla. \tag{2.7}$$

Now we can write Eq. (2.5) in the form

$$\Psi(x,u)\frac{\partial u}{\partial\tau} - \nabla\cdot(a(x,u)\nabla u) = f(x,t,u). \tag{2.8}$$

The equivalent weak form of Eq. (2.5) is

$$\left(\Psi(x,u)\frac{\partial u}{\partial\tau}, v\right) + a(u;u,v) = (f(x,u),v), \quad \forall v \in H^1_0(\Omega),\ t \in (0,T], \tag{2.9}$$

where

$$a(w; u, v) = (a(x, w)\nabla u, \nabla v) = \sum_{P\in\bar{\Omega}_h}\left[\int_{K_P^*} a(x, w)\nabla u\cdot\nabla v\mathrm{d}x - \int_{\partial K_P^*} a(x, w)\nabla u\cdot\nu v\mathrm{d}s\right]. \tag{2.10}$$

For Eq. (2.9), we adopt the backward difference along the τ-characteristic tangent at (x, t^{n+1}) as the approximation of $\Psi(x, u)(\partial u/\partial\tau)$, and then we have

$$\left(\Psi(x, u)\frac{\partial u}{\partial\tau}\right)(x, t^{n+1}) \approx \Psi(x, u^{n+1})\frac{u(x, t^{n+1}) - u(\check{x}, t^n)}{\Delta t} = \frac{u(x, t^{n+1}) - u(\check{x}, t^n)}{\Delta t}, \tag{2.11}$$

where $\check{x} = x - \vec{b}(x, u^{n+1})\Delta t$. Let $u_h(x, t)$ be a finite element solution of (2.9), $u_h^{n+1} = u_h(x, t^{n+1})$, $\hat{u}_h^n = u_h(\hat{x}, t^n)$, $\hat{x} = x - \vec{b}(x, u_h^n)\Delta t$.

So far, we can obtain the CFVEM: find $u_h \in U_h$, such that

$$\left(\frac{u_h^{n+1} - \hat{u}_h^n}{\Delta t}, v_h\right) + a(u_h^n; u_h^{n+1}, v_h) = (f(x, \hat{u}_h), v_h), \quad \forall v_h \in V_h. \tag{2.12}$$

Here V_h is a piecewise constant space, so

$$a(w_h; u_h, v_h) = -\sum_{P\in\bar{\Omega}_h} v_h(P)\int_{\partial K_P^*} a(x, w_h)\nabla u_h\cdot\nu\mathrm{d}s, \quad \forall w_h, u_h \in U_h. \tag{2.13}$$

Noting that $\vec{b}(x, u)|_\Gamma = (0, 0, 0)^T$, we can deduce that $\hat{x} = x - \vec{b}(x, u_h^n)\Delta t$ is a homeomorphism map, when Δt is suitably small. Generally, $\hat{u}_h^n$ are not node values, so they should be derived by interpolation formulas on u_h^n.

3. Auxiliary lemmas

Define the discrete norm and the discrete semi-norm:

$$|||u_h|||_0^2 = (u_h, \Pi_h^* u_h), \qquad |u_h|_{1,h}^2 = \sum_{K\in T_h} |u_h|_{1,h,K}^2,$$

$$|u_h|_{1,h,K}^2 = \sum_{i=1}^{4}[(u_h(P_{i+1}^b) - u_h^b(P_i))^2 + (u_h(P_{i+1}^u) - u_h(P_i^u))^2 + (u_h(P_i^u) - u_h(P_i^b))^2]h,$$

where $P_{i+1}^b = P_1^b$, $P_{i+1}^u = P_1^u$, as $i = 4$; $P_i^b(i = 1, 2, 3, 4)$ and $P_i^u(i = 1, 2, 3, 4)$ are the bottom vertexes and upper vertexes of cuboid cell K. Obviously, the discrete norm and discrete semi-norm are equivalent to the continuous norm and full-norm on U_h, respectively. i.e., there exist positive constants C_5, C_6, such that

$$C_5\|u_h\|_0 \le |||u_h|||_0 \le C_6\|u_h\|_0, \qquad C_5|u_h|_1 \le |u_h|_{1,h} \le C_6|u_h|_1.$$

We assume that the cuboid cells are parallel to the coordinate axes, and denote the isometric partition steps along the x_1-, x_2- and x_3-directions by h_{x_1}, h_{x_2} and h_{x_3}. As follows, we assume that the partition family T_h is regular again, i.e., there exist positive constants C_3, C_4 independent of h, such that the following condition (A5) satisfies:

$$C_3 \le \frac{h_{x_1}}{h_{x_2}}, \frac{h_{x_1}}{h_{x_2}}, \frac{h_{x_2}}{h_{x_3}} \le C_4. \tag{3.1}$$

Lemma 3.1. *For $\forall u_h, \bar{u_h} \in U_h$, there exists a positive constant C, such that*

$$(u_h, \Pi_h^*\bar{u_h}) = (\bar{u_h}, \Pi_h^* u_h), \tag{3.2}$$

$$(u_h, \Pi_h^*\bar{u_h}) \le C\|u_h\|_0\cdot\|\bar{u_h}\|_0. \tag{3.3}$$

Proof. From the properties of U_h, for each partition cell $K(\in T_h)$, whose eight vertexes are $P_i^b, P_i^u (i = 1, 2, 3, 4)$ respectively, we know that $u_h|_K$ has the following expression:

$$u_h(x,t)|_K = \sum_{i=1}^{4}[u_h(P_i^b)\varphi(P_i^b) + u_h(P_i^u)\varphi(P_i^u)], \tag{3.4}$$

where

$$\varphi(P_i^r) = \phi_{P_i^r}(x_1)\chi_{P_i^r}(x_2)\psi_{P_i^r}(x_3), \quad r = b,\ u;\ i = 1,2,3,4.$$

$$\phi_{P_i^r}(x_1) = \begin{cases} 1 - \dfrac{1}{h_{x_1}}|x_1 - x_{1,P_i^r}|, & x_{1,P_i^r} - h_{x_1} \le x_1 \le x_{1,P_i^r} + h_{x_1}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\chi_{P_i^r}(x_2) = \begin{cases} 1 - \dfrac{1}{h_{x_2}}|x_2 - x_{2,P_i^r}|, & x_{2,P_i^r} - h_{x_2} \le x_2 \le x_{2,P_i^r} + h_{x_2}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{P_i^r}(x_3) = \begin{cases} 1 - \dfrac{1}{h_{x_3}}|x_3 - x_{3,P_i^r}|, & x_{3,P_i^r} - h_{x_3} \le x_3 \le x_{3,P_i^r} + h_{x_3}; \\ 0, & \text{otherwise.} \end{cases}$$

Here x_{j,P_i^r} $(j = 1, 2, 3)$ are the x_j-direction coordinates of nodes P_i^r. Let $|K| = h_{x_1}h_{x_2}h_{x_3}$ be the volume of the cuboid cell K. Furthermore, we have

$$(u_h, \Pi_h^*\bar{u_h}) = \sum_{K\in T_h}\sum_{l=1}^{4}\left(\bar{u_h}(P_l^b)\int\int\int_{K^*_{P_l^b}\bigcap K} u_h\mathrm{d}x\mathrm{d}y\mathrm{d}z + \bar{u_h}(P_l^u)\int\int\int_{K^*_{P_l^u}\bigcap K} u_h\mathrm{d}x\mathrm{d}y\mathrm{d}z\right).$$

Defining

$$\alpha = (\bar{u_h}(P_1^b), \bar{u_h}(P_2^b), \bar{u_h}(P_3^b), \bar{u_h}(P_4^b), \bar{u_h}(P_1^u), \bar{u_h}(P_2^u), \bar{u_h}(P_3^u), \bar{u_h}(P_4^u))$$

and

$$\beta = (u_h(P_1^b), u_h(P_2^b), u_h(P_3^b), u_h(P_4^b), u_h(P_1^u), u_h(P_2^u), u_h(P_3^u), u_h(P_4^u))^{\mathrm{T}},$$

and noting equality (3.4), after complex integration computing we have that

$$(u_h, \Pi_h^*\bar{u_h}) = \sum_{K\in T_h}\frac{|K|}{512}\alpha\begin{pmatrix} 27 & 9 & 3 & 9 & 9 & 3 & 1 & 3 \\ 9 & 27 & 9 & 3 & 3 & 9 & 3 & 1 \\ 3 & 9 & 27 & 9 & 1 & 3 & 9 & 3 \\ 9 & 3 & 9 & 27 & 3 & 1 & 3 & 9 \\ 9 & 3 & 1 & 3 & 27 & 9 & 3 & 9 \\ 3 & 9 & 3 & 1 & 9 & 27 & 9 & 3 \\ 1 & 3 & 9 & 3 & 3 & 9 & 27 & 9 \\ 3 & 1 & 3 & 9 & 9 & 3 & 9 & 27 \end{pmatrix}\beta.$$

In the above computation, for simplicity we omit the variable t in function $u(x, y, z, t)$. Based on the above equality, we can complete the proof of Lemma 3.1 easily. □

Lemma 3.2. *Supposing that all cells K_Q of the partition T_h satisfy conditions* (A4) *and* (A5), *T_h^* is a circumcenter dual partition. For $\forall w_h, u_h, \bar{u_h} \in U_h$, there exist positive constants γ, C_7, C_8 independent of h, such that*

$$a(w_h; u_h, \Pi_h^* u_h) \ge \gamma\|u_h\|_1^2. \tag{3.5}$$

$$a(w_h; u_h, \Pi_h^*\bar{u}_h) \le C_7\|u_h\|_1\|\bar{u}_h\|_1. \tag{3.6}$$

$$|a(w_h; u_h, \Pi_h^*\bar{u_h}) - a(w_h; \bar{u_h}, \Pi_h^* u_h)| \le C_8 h\|u_h\|_1\|\bar{u_h}\|_1. \tag{3.7}$$

Using positive definite condition (A1)*, Lipschitz continuity of the functions* $a_i(x,u)$*, the definition of* $a(\cdot;\cdot,\cdot)$*, the property of* U_h *and the curve integration theorem, after complicated computation, we can complete proof of this lemma.*

Lemma 3.3. *Let* $|||u_h|||_0 = (\Pi_h^* u_h, \Pi_h^* u_h)^{\frac{1}{2}}$. $|||\cdot|||_0$ *is equivalent to* $\|\cdot\|_0$ *in* U_h.

The proof of Lemma 3.3 can be completed by computing the integral on cell K_Q *directly.*

Theorem 3.1 (*Trace Theorem [20]*). *Supposing that* Ω *has a Lipschitz boundary, and that* p *is a real number in the range* $1 \le p \le \infty$*, then there exists a constant* C*, such that*

$$\|v\|_{L^p(\partial\Omega)} \le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}, \quad \forall v \in W_p^1(\Omega).$$

Lemma 3.4. *Supposing that* P' *is a random point in dual partition cell* $K_{P_i}^*$, $\Gamma_{ij} = K_{P_i}^* \bigcap K_{P_j}^*$*, then*

$$\sum_{j\in\bigwedge_i} \int\int_{\Gamma_{ij}} |u(P') - u(x)|\mathrm{d}s \le Ch^2(\|u\|_{1,K_{P_i}^*} + \|u\|_{2,K_{P_i}^*}). \tag{3.8}$$

Proof. From the Hölder inequality, we get that

$$\sum_{j\in\bigwedge_i} \int\int_{\Gamma_{ij}} |u(P') - u(x)|\mathrm{d}s \le Ch \sum_{j\in\bigwedge_i} \left(\int\int_{\Gamma_{ij}} |u(P') - u(x)|^2\mathrm{d}s\right)^{\frac{1}{2}}.$$

Using Taylor expansion, the trace theorem in which we choose $p = 2$, and the Hölder inequality, the proof of Lemma 3.4 can be completed. □

Using the positive definite condition (A1), local continuity of the functions $a_i(x,u)$, the definition of $a(\cdot;\cdot,\cdot)$, the property of the function in U_h, and the curve integration theorem, after complicated computation, we can derive the following two lemmas.

Lemma 3.5. *Supposing that all cells* K_Q *of the partition* T_h *satisfy conditions* (A4) *and* (A5)*, then* T_h^* *is a circumcenter dual partition. For* $\forall\, w_h, u_h, \bar{u_h} \in U_h$*, there exists a positive constant* C_9 *independent of* h*, such that*

$$|a(w_h; u - u_h, \Pi_h^*\bar{u_h})| \le C_9(h\|u\|_2 + \|u - u_h\|_1)\|\bar{u_h}\|_1. \tag{3.9}$$

Lemma 3.6. *Supposing that all cells* K_Q *of the partition* T_h *satisfy conditions* (A4) *and* (A5)*, then* T_h^* *is a circumcenter dual partition. For* $\forall\, w_h, \bar{u_h} \in U_h$*, there exists a positive constant* C_{10} *independent of* h*, such that*

$$|a(w; u, \Pi_h^*\bar{u_h}) - a(w_h; u, \Pi_h^*\bar{u_h})| \le C_{10}(h^2\|w\|_2 + \|w - w_h\|_0)\|u\|_{1,\infty}\|\bar{u_h}\|_1. \tag{3.10}$$

4. Convergence analysis

Now we consider the error estimates of the approximate solution. Let

$$u^{n+1} - u_h^{n+1} = (u^{n+1} - \Pi_h u^{n+1}) + (\Pi_h u^{n+1} - u_h^{n+1}) = \rho_h^{n+1} + e_h^{n+1}.$$

Choosing $t = t^{n+1}$ in (2.9), then we have

$$\left(\Psi(x, u^{n+1})\frac{\partial u}{\partial \tau}(t^{n+1}), v\right) + a(u^{n+1}; u^{n+1}, v) = (f(x, u^{n+1}), v), \quad \forall v \in V_h. \tag{4.1}$$

Subtracting (2.12) from (4.1), we obtain that

$$\left(\frac{e_h^{n+1}-e_h^n}{\Delta t}, v_h\right) + a(u_h^n; e_h^{n+1}, v_h) = \left(\frac{\partial u^{n+1}}{\partial t} + \vec{b}(x, u_h^n)\cdot\nabla u^{n+1} - \frac{u^{n+1}-\hat{u}_h^n}{\Delta t}, v_h\right)$$
$$+ \left(\frac{\hat{e}_h^n - e_h^n}{\Delta t}, v_h\right) + \left(\frac{\rho_h^{n+1}-\hat{\rho}_h^n}{\Delta t}, v_h\right) + \left(\frac{\check{u}_h^n - \hat{u}_h^n}{\Delta t}, v_h\right) + ([b(x, u^{n+1}) - b(x, u_h^n)]\cdot\nabla u^{n+1}, v_h)$$
$$+ a(u_h^n; \rho_h^{n+1}, v_h) + (a(u^{n+1}; u^{n+1}, v_h) - a(u_h^n; u^{n+1}, v_h)) - (f(x, u^{n+1}) - f(x, \hat{u}_h^n), v_h). \tag{4.2}$$

Choosing $v_h = \Pi_h^* e_h^{n+1}$ in (4.2), testing (4.2) against $2\Delta t$ and estimating the terms on the left-hand side of (4.2), from the definition of $\|\cdot\|_{0,h}$ we have that

$$2\left(\frac{e_h^{n+1}-e_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t = 2[\|e_h^{n+1}\|_{0,h}^2 - (e_h^n, \Pi_h^* e_h^{n+1})].$$

Noting the equivalence of $\|\cdot\|_{0,h}$ and $\|\cdot\|_0$, and using Lemma 3.1 and the Cauchy inequality with ϵ(suitably), we have

$$2\left(\frac{e_h^{n+1}-e_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t \ge C_\epsilon(\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2); \tag{4.3}$$

$$2a(u_h^n; e_h^{n+1}, \Pi_h^* e_h^{n+1})\Delta t \ge 2\gamma\|e_h^{n+1}\|_1^2\Delta t. \tag{4.4}$$

For each term of the right-hand side of (4.2), noting that

$$\left\|\frac{\partial u^{n+1}}{\partial t} + \vec{b}(x, u_h^n)\cdot\nabla u^{n+1} - \frac{u^{n+1}-\hat{u}_h^n}{\Delta t}\right\|_0^2$$
$$\le \int_\Omega \left(\frac{1}{\Delta t}\right)^2 (\Psi\Delta t)^3 \left|\int_{(\hat{x},t^n)}^{x,t^{n+1}} \frac{\partial^2 u}{\partial t^2}\mathrm{d}t\right| \mathrm{d}x \le \Delta t\|\Psi^3\|_{0,\infty}\int_\Omega\int_{(\hat{x},t^n)}^{(x,t^{n+1})}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 \mathrm{d}t\mathrm{d}x$$
$$\le C\Delta t\int_\Omega\int_{t^n}^{t^{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2\mathrm{d}t\mathrm{d}x,$$

then, using Lemma 3.1 and the Cauchy inequality, we have

$$\left|2\left(\frac{\partial u^{n+1}}{\partial t} + \vec{b}(x, u_h^n)\cdot\nabla u^{n+1} - \frac{u^{n+1}-\hat{u}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t\right|$$
$$\le C\left\{\Delta t\int_\Omega\int_{t^n}^{t^{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2\mathrm{d}t\mathrm{d}x + \|e_h^{n+1}\|_0^2\right\}\Delta t, \tag{4.5}$$

$$\left|2\left(\frac{\hat{e}_h^n - e_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t\right| \le C\|e_h^n\|_0^2\Delta t + \epsilon\|\nabla e_h^{n+1}\|_0^2\Delta t \le C\|e_h^n\|_0^2\Delta t + \bar{\epsilon}\|e_h^{n+1}\|_1^2\Delta t. \tag{4.6}$$

From

$$\left(\frac{\rho_h^{n+1}-\hat{\rho}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right) = \left(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right) + \left(\frac{\rho_h^n-\hat{\rho}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right),$$
$$\left|\left(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\right| = \left|\left(\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\frac{\partial\rho_h}{\partial t}\mathrm{d}t, \Pi_h^* e_h^{n+1}\right)\right| \le C\left\{\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\left\|\frac{\partial\rho_h}{\partial t}\right\|_0^2\mathrm{d}t + \|e_h^{n+1}\|_0^2\right\},$$
$$\left|\left(\frac{\rho_h^n-\hat{\rho}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\right| \le C\|\rho_h^n\|_0^2 + \epsilon\|\nabla e_h^{n+1}\|_0^2 \le C\|\rho_h^n\|_0^2 + \bar{\epsilon}\|e_h^{n+1}\|_1^2,$$

then we have

$$\left|2\left(\frac{\rho_h^{n+1}-\hat{\rho}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t\right| \le C\left\{\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\left\|\frac{\partial \rho_h}{\partial t}\right\|_0^2 \mathrm{d}t + \|e_h^{n+1}\|_0^2 + \|\rho_h^n\|_0^2\right\}\Delta t + \bar{\epsilon}\|e_h^{n+1}\|_1^2\Delta t. \tag{4.7}$$

From the continuity of the vector function $\vec{b}(x,u)$ and the triangle inequality, we know that

$$\begin{aligned}\frac{\check{u}_h^n-\hat{u}_h^n}{\Delta t} \le C(u)\frac{|\check{x}-\hat{x}|}{\Delta t} &= C(u)|\vec{b}(x,u^{n+1})-\vec{b}(x,u_h^n)| \\ &\le C|u^{n+1}-u_h^n| = C|u^{n+1}-u^n+\rho_h^n+e_h^n| \le C(|u^{n+1}-u^n|+|\rho_h^n|+|e_h^n|).\end{aligned}$$

With Lemma 3.1, we obtain

$$\left|2\left(\frac{\check{u}_h^n-\hat{u}_h^n}{\Delta t}, \Pi_h^* e_h^{n+1}\right)\Delta t\right| \le C\left\{\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_0^2 \mathrm{d}t + \|e_h^{n+1}\|_0^2 + \|\rho_h^n\|_0^2 + \|e_h^n\|_0^2\right\}\Delta t. \tag{4.8}$$

Analogously, we have

$$\left|2([\vec{b}(x,u^{n+1})-\vec{b}(x,u_h^n)]\cdot\nabla u^{n+1}, \Pi_h^* e_h^{n+1})\Delta t\right| \le L\left\{\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_0^2 \mathrm{d}t + \|e_h^{n+1}\|_0^2 + \|\rho_h^n\|_0^2 + \|e_h^n\|_0^2\right\}\Delta t. \tag{4.9}$$

From the Lipschitz continuous property of $f(x,u)$ in condition (A_2), making use of triangle inequality, an important inequality and Lemma 3.1, we have

$$\begin{aligned}&|2(f(x,u^{n+1})-f(x,\hat{u}_h^n), \Pi_h^* e_h^{n+1})\Delta t| \\ &\quad\le CL\|u^{n+1}-\hat{u}_h^n\|_0\cdot\|e_h^{n+1}\|_0 \\ &\quad\le C\left\{\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_0^2 \mathrm{d}t + \|\rho_h^n\|_0^2 + \|e_h^n\|_0^2 + \|e_h^{n+1}\|_0^2 + \|u_h^n\|_0^2\Delta t\right\}\Delta t + \epsilon\|\nabla e_h^{n+1}\|_0^2(\Delta t)^2 \\ &\quad\le C\left\{\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_0^2 \mathrm{d}t + \|\rho_h^n\|_0^2 + \|e_h^n\|_0^2 + \|e_h^{n+1}\|_0^2 + \|u_h^n\|_0^2\Delta t\right\}\Delta t + \bar{\epsilon}\|e_h^{n+1}\|_1^2(\Delta t)^2,\end{aligned} \tag{4.10}$$

where L is a Lipschitz constant.

From Lemmas 3.5 and 3.6, an important inequality, we can obtain the following important estimates:

$$|2a(u_h^n;\rho_h^n,\Pi_h^* e_h^{n+1})\Delta t| \le C(h\|u^n\|_2+\|\rho_h^n\|_1)\|e_h^{n+1}\|_1\Delta t \le C\left\{h^2\|u^n\|_2^2+\|\rho_h^n\|_1^2+\|e_h^{n+1}\|_1^2\right\}\Delta t, \tag{4.11}$$

$$\begin{aligned}&|2[a(u^{n+1};u^{n+1},\Pi_h^* e_h^{n+1})-a(u_h^n;u^{n+1},\Pi_h^* e_h^{n+1})]\Delta t| \\ &\quad\le C(h^2\|u^{n+1}\|_2+\|u^{n+1}-u_h^n\|_1)\|e_h^{n+1}\|_1\Delta t \\ &\quad\le C\left\{h^4\|u^{n+1}\|_2^2+\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_1^2 \mathrm{d}t + \|\rho_h^n\|_1^2+\|e_h^{n+1}\|_1^2\right\}\Delta t.\end{aligned} \tag{4.12}$$

For error equation (4.2), by (4.3)–(4.12) and applying the Sobolev space embedding theorem, the interpolation theorem and the inverse estimate, we can obtain

$$\begin{aligned}&(\|e_h^{n+1}\|_0^2-\|e_h^n\|_0^2)+\left(2\frac{\gamma}{C_\epsilon}-3\bar{\epsilon}\right)\|e_h^{n+1}\|_1^2\Delta t \\ &\quad\le C\left\{\Delta t\int_\Omega\int_{t^n}^{t^{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 \mathrm{d}t\mathrm{d}x+\int_{t^n}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_1^2 \mathrm{d}t\Delta t+\|e_h^{n+1}\|_0^2+\|e_h^n\|_0^2+h^2(\|u^n\|_2^2+h^2\|u^{n+1}\|_2^2)\right\}\Delta t.\end{aligned}$$

Supposing that the spatial and temporal discretization satisfies the relation $\Delta t = O(h)$, then we have

$$(\|e_h^{n+1}\|_0^2 - \|e_h^n\|_0^2) + \left(2\frac{\gamma}{C_\epsilon} - 3\bar{\epsilon}\right)\|e_h^{n+1}\|_1^2 \Delta t \\ \le C(\Delta t)^2 \int_{t^n}^{t^{n+1}} \left(\left\| \frac{\partial^2 u}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \right) \mathrm{d}t + C\{\|e_h^{n+1}\|_0^2 + \|e_h^n\|_0^2 + h^2(\|u^n\|_2^2 + h^2\|u^{n+1}\|_2^2)\}\Delta t. \quad (4.13)$$

Summing from 0 to $N-1$ with respect to n in the above inequality, we can obtain that

$$(\|e_h^N\|_0^2 - \|e_h^0\|_0^2) + \eta \Delta t \sum_{n=0}^{N-1} \|e_h^{n+1}\|_1^2 \\ \le C(\Delta t)^2 \int_0^{\mathrm{T}} \left(\left\| \frac{\partial^2 u}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \right) \mathrm{d}t + C \left\{ \sum_{n=0}^{N-1} (\|e_h^{n+1}\|_0^2 + \|e_h^n\|_0^2) + h^2 \sum_{n=0}^{N-1} (\|u^n\|_2^2 + h^2\|u^{n+1}\|_2^2) \right\} \Delta t \\ \le C(\Delta t)^2 \int_0^{\mathrm{T}} \left(\left\| \frac{\partial^2 u}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \right) \mathrm{d}t + C \left\{ \sum_{n=0}^{N} \|e_h^n\|_0^2 + h^2 \sum_{n=0}^{N-1} (\|u^n\|_2^2 + h^2\|u^{n+1}\|_2^2) \right\} \Delta t, \quad (4.14)$$

where $\eta = 2\frac{\gamma}{C_\epsilon} - 3\bar{\epsilon}$. Supposing that $e_h^0 = 0$, using the discrete Gronwall lemma, we know that

$$\|e_h^N\|_0^2 + \eta \Delta t \sum_{n=0}^{N} \|e_h^n\|_1^2 \\ \le C(\Delta t)^2 \int_0^{\mathrm{T}} \left(\left\| \frac{\partial^2 u}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \right) \mathrm{d}t + Ch^2 \sum_{n=0}^{N-1} (\|u^n\|_2^2 + h^2\|u^{n+1}\|_2^2)\Delta t. \quad (4.15)$$

Noting that $N\Delta t \le T$, combining with the interpolation theorem and the regularity condition (A3), we obtain the final error estimate to the approximate solution as

$$\|u - u_h\|_{\bar{L}_\infty((0,T],L^2(\Omega))} + \|u - u_h\|_{\bar{L}^2((0,T],H^1(\Omega))} = O(h + \Delta t), \quad (4.16)$$

where $\|v\|_{\bar{L}_\infty((0,T],X)} = \sup_{n\Delta t \le T} \|v^n\|_X$, $\|v\|_{\bar{L}^2((0,T],X)} = \sup_{N\Delta t \le T}\{\eta \sum_{n=0}^{N} \|v^n\|_X \Delta t\}^{1/2}$.

Therefore we have the following theorem.

Theorem 4.1. *Supposing that the solution of the problem (1.1) is sufficiently smooth. when h and Δt are sufficiently small, $\Delta t = O(h)$ and the initial value u_h^0 is chosen as an interpolation of u_0, then Eq. (4.16) holds.*

5. Numerical experiment

We test a one-dimensional problem on the cell $\Omega = [0, 1]$ with different ν:

$$\frac{\partial u}{\partial t} + b(x, u)\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = f(x, u).$$

Choosing $b(x, u) = u \sin t$, then $f(x, u) = \pi \cos(\pi(x - t))(u \sin t - 1) + \nu\pi^2 \sin(\pi(x - t))$, so that the exact solution is $u(x, t) = \sin(\pi(x - t))$. The initial condition and the boundary values are obtained directly. The maximum absolute and relative errors which will be defined between the exact solution u and the approximate solution u_h at $t = 0.5$ are listed in Table 1. We choose a spatial step $h = 0.01$ and a temporal step $\Delta t = 0.001$; $AE_{\max}$ and $RE_{\max}$ are defined as

$$AE_{\max} = \max_{1 \le i \le \frac{1}{h}} |u_i - u_{hi}|.$$

$$RE_{\max} = \max_{1 \le i \le \frac{1}{h}} \frac{|u_i - u_{hi}|}{|u_i|}.$$

Table 1
The error table

	$AE_{\max}$	$RE_{\max}$
$\nu = 10^{-3}$	2.6229e–003	4.0579e–002
$\nu = 10^{-2}$	1.4933e–003	3.9239e–002
$\nu = 10^{-1}$	1.1085e–003	2.7681e–002
$\nu = 1$	1.5783e–004	6.0205e–004

From Table 1, we can see that our scheme is effective for avoiding numerical diffusion and nonphysical oscillations, and that it is consistent with the theoretical analysis results.

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