6. We prove, in this section, that the zeros of the random meromorphic function \( g(z, t) \), which belongs to the family \( \tilde{G} \), are contained in circular pits.

It will be shown, by theorem 2, that the function \( g(z, t) \) takes in each pit every value of a satisfying the inequality:

\[
|a| < \exp \left( -8(3R)^{3e} \right) \frac{m_b(R)}{m_e(R)},
\]

where \( R \) is the mean distance of the pit from the origin, as often as it has zeros in the same pit. The function \( g \) may have, in one pit, one or more zeros. Such a pit is reckoned with a multiplicity equal to the number of zeros it contains. Then there is just one circular pit to each zero, but circular pits may overlap and one circular pit may contain another, in which case, we count each pit separately. So the total number of pits of \( g(z, t) \), is about the same number of its zeros i.e. as it has been shown in section 5 case 1 [2].

It has been shown by lemma 2, that these pits are simply connected \( z \)-domains, of small diameter less or equal to \( 2 \exp (-R^4/8) \).

By the result of lemma 3, we find that these pits are separated from each other by level curves on which \( g(z, t) \) satisfies the inequality:

\[
\log^+ |g| > \log \left( \frac{m_b(R)}{m_e(R)} \right) - 8(3R)^{3e}.
\]

Lemma 4, gives us the following conclusion: If \( g(z, t) \) belongs to \( \tilde{G} \), then, on mapping the values of \( \log^+ |g| \) on the \( z \)-plane, we obtain a surface like a bowl, the height of which is of order \( m_b(R)/m_e(R) \). The upper part of this bowl is bounded by circles of diameter at most \( \exp (-R^4/10) \).

**Theorem 2.** Let \( r < R^{-e} \), \( |a| < \exp \left( -8(3R/2)\text{e} \right) \frac{m_b(R)}{m_e(R)} \) and \( v(z, r; a) = v(z, r; t, a) \) be as usual the number of \( a \)-values of \( g(z, t) \) in the circle of centre \( z \) and radius \( r \). Accordingly, \( v(z, r; 0) \) is the number of zeros of \( g(z, t) \) in the same circle. Let

\[
T(z, n, t) = \max (2s)^{-1} \int_{1-s}^{1+s} \int_0^w \left| \int (v(wz, n; a) - v(wz, n; 0)/n) \, dn \right|,
\]

where \( s = \exp \left( -(3R/2)\text{e} \right) \).
Then

\[ av \ T(z, n, t) = \int_0^1 T(z, n, t) \, dt \leq \exp(-R^2/3). \]

Proof: The symbol \( av \) denotes of course an average with respect to \( t \). Since the number of \( a \)-values of the random meromorphic function \( g(z, t) \) is the same as the number of zeros of the random integral function \( f(z, t; a) \) (of lemma 1).

Hence

\[ v(wz, n; a) = v(wz, n; 0, f). \]

Also, the number of zeros of \( g(z, t) \) is the same as the number of zeros of the random integral function \( b(z, t) \).

Write \( j = z + re \) then by Jensen’s theorem we have:

\[
\int_0^\infty (v(wz, n; a) - v(wz, n; 0))/n \, dn = (2\pi)^{-1} \int_0^{2\pi} \log |f(wj, t; a)/b(wj, t)| \, du - \log |f(wz, t; a)/b(wz, t)|.
\]

Then

\[
\max (2s)^{-1} \int_0^{1+s} dw |(v(wz, n; a) - v(wz, n; 0))/n| \, dn |
\]

\[ = \max (2s)^{-1} \int_0^{1+s} dw (2\pi)^{-1} \int_0^{2\pi} \log |f(wj, t; a)/b(wj, t)| \, du +
\]

\[ + \max (2s)^{-1} \int_0^{1+s} \log |f(wz, t; a)/f(wz, t)| \, dw.
\]

\[ = (2\pi)^{-1} \int_0^{2\pi} av I(j) \, du + av I(z),
\]

where

\[ I(j) = \max (2s)^{-1} \int_0^{1+s} \log |f(wj, t; a)/b(wj, t)| \, dw,
\]

and \( I(z) \) has the same definition with \( j \) replaced by \( z \); or alternatively we take \( r = 0 \).

We begin by finding the upper bound for \( I(j) \) for all \( f \) of \( F \) i.e. all \( f \) which belong to \( F - x \).

Let \( E_1 \) be the set of values of \( w \) in the interval \((1-s; 1+s)\) for which \(|f(wj, t; a)| > |b(wj, t)|\), and the set \( E_2 \) be \( C(E_1) \) with respect to the interval \((E_1, 2) \) vary with \( a, j \) and \( t \). Then we have:

\[ av I(j) = av \max (2s)^{-1} \int_{E_1} \log |f(wj, t; a)/b(wj, t)| \, dw.
\]

We find, now, the value of

\[ \int_{E_1} \log |f(wj, t; a)/b(wj, t)| \, dw.
\]

\[ 1 \leq |f(wj, t; a)/b(wj, t)| = |b(wj, t) - a c(wj, t)/b(wj, t)|
\]

\[ \leq 1 + (|a|/|c(wj, t)|/|b(wj, t)|). \]
But

\[
\begin{align*}
|c(wj, t)| &\leq |wj|^{\gamma \epsilon} m_\alpha(|wj|) \\
&\leq \left( (1 + s) (R + R^{-\eta}) \right)^{\gamma \epsilon} m_\alpha((1 + s) (R + R^{-\eta})) \\
&\leq A_1 e^{R^\s} m_\alpha(R) \\
&\quad \text{for } R \geq R_0(\epsilon)
\end{align*}
\]

(6.3)

and by lemma 10 [1], we have

\[
|b(wj, t)| \geq A_2 e^{-R^\s} m_\beta(R).
\]

(6.4)

Hence

\[
1 \leq |f(wj, t; a)|/b(wj, t)| \leq 1 + A_3 |a| e^{R^\s} (m_\epsilon(R)/m_\beta(R))
\]

\[
\leq 1 + A_3 \exp \left( -8 (3R/2)^{3\s} + 2R^\s \right)
\]

\[
\leq 1 + \exp \left( -6 (3R/2)^{3\s} \right), \text{ by a rechoise of } R_0(\epsilon),
\]

i.e.

\[
\log |f(wj, t; a)|/b(wj, t)| \leq \exp \left( -6 (3R/2)^{3\s} \right).
\]

Since \( |E_1| \leq 2s \), then

\[
\int \log |f(wj, t; a)|/b(wj, t)| \, dw \leq 2s \exp \left( -6 (3R/2)^{3\s} \right)
\]

(6.5)

We find, now, the value of

\[
\int \log |f(wj, t; a)|/b(wj, t)| \, dw.
\]

in this case, we have

\[
|b(wj, t)| > |f(wj, t; a)|.
\]

Let \( x = x(\epsilon, j) \), be the exceptional set of lemma 1; so we have

\[
|x| = \exp (-|j|^{\s}).
\]

Then, for all \( f \) belonging to \( F - x \), by lemma 1, we get:

\[
|f(wj, t; a)| \geq A \exp \left( -8 (3R/2)^{3\s} \right) m(|j|, |a|) \min ((w - y_i)^{q_1}; (y_{i+1} - w)^{q_1}),
\]

for \( y_i < w < y_{i+1} \), and some set of at most \( q_1 + 2 \) points: \( y_0, y_1, \ldots \) situated in \((1 - s; 1 + s);\), where \( s = \exp \left( -3R/2)^{3\s} \right) \), and \( q_1 < 4(3R/2)^{\s} \) (\( q_1 \) is that used in theorem 1).

Let \( E_3 \) be the sub-set of \( E_2 \), which denotes the set of intervals each of length \( s' = \exp \left( -3/2)(3R/2)^{3\s} \right) \) with their centres at the points \( y_i \). Then for \( w \) outside \( E_2 \) and \( R > R_0(\epsilon) \), we have:

\[
|w - y|^n = \exp (q_1 \log |w - y|) > \exp (-6(3R/2)^{3\s}),
\]

where \( y = y(w) \) denotes the number of the set \( y_i \) nearest to \( w \).

Consequently, for \( w \) of \( E_2 - E_2 E_3 \), we get from (6.6), the following:

\[
|f(wj, t; a)| > \exp (-7(3R/2)^{3\s}) m(|j|, |a|) \quad \text{for } R > R_0(\epsilon).
\]

(6.7)
Hence, combining the results (6.3) and (6.7), we have:

\[ 1 < |b(w_j, t)|/|f(w_j, t; a)| < 1 + |a c(w_j, t)|/|f(w_j, t; a)| < 1 + (\exp (-8(3R/2)\epsilon + R^\ast) m_\ast(R)m_\ast(R)/|\exp (-7(3R/2)\epsilon) m(|j|, a)m_\ast(R)| \]

but,

\[
m(|j|, a) > m_\ast(|j|) \quad \text{by (3) [2]},
\]

\[
\geq m_\ast(R - R^{-\epsilon}) > A m_\ast(R).
\]

Then,

\[
1 < |b(w_j, t)|/|f(w_j, t; a)| < 1 + \exp (-\frac{1}{2}(3R/2)\epsilon) \quad \text{for } R > R_0(\epsilon).
\]

Since \(|E_2| < 2s\), then

\[
(6.8) \quad \int_{E_\ast - E_\ast} \log |b(w_j, t)|/|f(w_j, t; a)| \, dw \leq 2s \exp (-\frac{1}{2}(3R/2)\epsilon),
\]

by a rechoice of \(R_0(\epsilon)\), and for all \(j\) of \(\hat{F}\).

Again, if \(w\) belongs to \(E_2 E_3\), and \(f\) still belongs to \(\hat{F}\), we have:

\[
(6.9) \quad \int_{E_\ast E_\ast} \log |b(w_j, t)| \, dw \leq |E_2 E_3| (R^\ast + \log m_\ast (|w_j|)) \quad \text{by (6.3)}
\]

\[
\leq |E_2 E_3| (R^\ast + \log m_\ast (R)).
\]

Also, in view of (6.6), we get:

\[
\int_{E_\ast E_\ast} \log |f(w_j, t; a)| \, dw \geq |E_2 E_3| (-R^{2\epsilon} + \log m (|j|, a)) + \int_{E_\ast E_\ast} \log (w-y)^{\alpha} \, dw.
\]

Since all \(|w-y|, |w-y|\) are less than 1, then

\[
\int_{E_\ast E_\ast} \log |w-y|^\alpha \, dw \geq q_1 \int_{E_\ast} \log |w-y| \, dw
\]

\[
\geq q_1 \sum_{i=0}^{r+1} (t+i) \int_{E_\ast E_\ast} \log |w-y| \, dw
\]

\[
\geq -R^{2\epsilon} \exp \left(-\frac{3}{2}(3R/2)\epsilon\right).
\]

Therefore

\[
(6.10) \quad \int_{E_\ast E_\ast} \log |f(w_j, t; a)| \, dw \geq |E_2 E_3| (\log m_\ast (R) - A_4 R^{2\epsilon}).
\]

But we know that:

\[
(6.11) \quad \left\{ \begin{array}{l} |E_2 E_3| < |E_3| < 2(q_1 + 2)\epsilon < AR^{\ast} \exp \left(-\frac{3}{2}(3R/2)\epsilon\right) \\ \leq \exp \left(-\frac{4}{3}(3R/2)\epsilon\right). \end{array} \right.
\]

Then in view of (6.9), (6.10) and (6.11), we have:

\[
(6.12) \quad \left\{ \begin{array}{l} \int_{E_\ast E_\ast} \log |b(w_j, t)|/|f(w_j, t; a)| \, dw \leq |E_2 E_3| A_5 R^{2\epsilon} \\ \leq A_6 R^{2\epsilon} \exp \left(-\frac{4}{3}(3R/2)\epsilon\right) \leq \exp \left(-\frac{5}{4}(3R/2)\epsilon\right), \end{array} \right. \quad \text{(by a rechoice of } R_0)\
\]

for all \(f\) of \(\hat{F}\).
Therefore, in view of (6.8) and (6.12), we have:

\[(6.13) \int_{E_{a}} \log |b(wj, t)/f(wj, t; a)| \, dw \leq 2s \exp \left(-\frac{1}{2} R^2\right) + \exp \left(-\frac{5}{4} R^2/2\right),\]

for all \( f \) of \( \hat{F} \).

Hence in view of (6.5) and (6.13), for all \( f \) of \( \hat{F} \) i.e. for all \( w \) in \( (1-s; 1+s) \), we have:

\[(6.14) \begin{cases} (2s)^{-1} \int_{1-s}^{1+s} \log |f(wj, t; a)/b(wj, t)| \, dw \\ \leq \exp \left(-6 (3R/2)R^2\right) + \exp \left(-\frac{1}{2} (3R/2)R^2\right) + \exp \left(-\frac{1}{2} (3R/2)R^2\right) \\ \leq A_7 \exp \left(-\frac{1}{2} R^2\right), \quad \text{(by a rechoice of } R_0(\varepsilon)) \end{cases}\]

By function theory, we have, for all \( f \) of \( F \) including all \( f \) belonging to \( x \) and for all \( b \).

\[(6.15) (2s)^{-1} \int_{1-s}^{1+s} \log \left| f(wj, t; a)/b(wj, t) \right| \, dw \leq B R^e + 2s.\]

So for \( f \) belonging to \( x \), we have:

\[(6.16) \begin{cases} (2s)^{-1} \int_{1-s}^{1+s} \log |f(wj, t; a)/b(wj, t)| \, dw \leq |x| B_1 R^e + 2s \\ \leq B_2 \exp \left(-\frac{1}{2} R^2\right), \quad \text{(by rechoice of } R_0(\varepsilon)), \end{cases}\]

(because \( \varepsilon < 1 \) and \( |x| \leq \exp (-j^s) \leq \exp (-R - \varepsilon) \leq \exp (-R^e + 1) \)).

Combining the results (6.14) and (6.15), we get:

\[av I(j) < A_7 \exp \left(-\frac{1}{2} R^e\right) + \exp \left(-\frac{1}{2} R^e\right) \leq A_8 \exp \left(-\frac{1}{2} R^e/3\right), \quad \text{(by a rechoice of } R_0)\]

for all \( f \) of \( F \).

In particular

\[av I(z) \leq A_9 \exp \left(-\frac{1}{2} R^e/3\right).\]

Further

\[(2\pi)^{-1} \int_{0}^{2\pi} av I(j) \, du + av I(z) \leq (2\pi)^{-1} \int_{0}^{2\pi} \exp \left(-R^e/3\right) \, du + \exp \left(-R^e/3\right) \leq A \exp \left(-R^e/3\right), \quad \text{(by a rechoice of } R_0),\]

for all \( f \) of \( F \) (i.e. for all \( t \)).

Hence we get the following result:

\[av T(z, n, t) \leq \exp \left(-\frac{1}{2} R^e/3\right),\]

for all \( g \) of \( G \) and \( R > R_0(\varepsilon) \).

It is clear that the value of \( \exp \left(-\frac{1}{2} R^e/3\right) \) is less than 1, so we get from the result of this theorem the outstanding fact that the number of the zeros of the random meromorphic function \( g \) (which belongs to \( G \), in the same pit, is equal to the number of its \( a \)-values, where \( a \) satisfies the inequality:

\[|a| \leq \exp \left(-8 (3R)^{2e} \right) m_{b} (R)/m_{c} (R).\]
Lemma 2. If the random meromorphic function \( g(z, t) \) belongs to the set \( G - x_1 \), where \( x_1 \) is an exceptional set of measure at most \( \varepsilon \), and if at a point \( z \) (where \( |z| = R > R_0(\varepsilon) \)), \( g(z, t) \) has an \( \alpha \)-value satisfying:

\[
| \alpha | < \exp \left( -8(3R)^{3\varepsilon} \frac{m_b(R)}{m_c(R)} \right),
\]

then \( g(z, t) \) has a zero within the \( j \)-circle where

\[
|j - z| < 2 \exp \left( -R^8/8 \right).
\]

(The set \( x_1 \) depends on \( \varepsilon, |b_n|'s \) and \( |c_n|'s \) only i.e. it is different from the set \( x \) of theorem 2.)

Proof: The proof of this lemma is on the same lines as the proof of lemma 15 [1].

Lemma 3. If \( g(z, t) \) belongs to \( G - x_1 \), then associated with each zero \( z_1 \) of \( g(z, t) \) with \( |z_1| = R_1 > R_0(\varepsilon) \), there is a real positive number \( r_1 \) satisfying:

\[
5 \exp \left( -R^8/8 \right) < r_1 < 5[R_1^{3+1}] \exp \left( -R^8/8 \right),
\]

such that on the circumference of the \( z \)-circle \( C(z_1, r_1) \) of centre \( z_1 \) and radius \( r_1 \)

\[
|g(z, t)| > \exp \left( -8(3R)^{3\varepsilon} \frac{m_b(R)}{m_c(R)} \right).
\]

This relation also holds for \( |z| = R > R_0(\varepsilon) \) outside the family of circles \( C(z_1, r_1) \).

Proof: This lemma can be proved by following the same steps of the proof of lemma 16 [1].

Lemma 4. Associated with any positive number \( \varepsilon \), there is an \( R_0(\varepsilon) \) depending only on \( \varepsilon \), the \( |b_n|'s \) and the \( |c_n|'s \), and an exceptional set \( x_1 \) of \( G \) of measure less than \( \varepsilon \). If \( g(z, t) \) belongs to \( G - x_1 \) and \( R > R_0(\varepsilon) \), then the following relations hold

(i) \( \log^+ |g(z, t)| > \log (m_b(R)/m_c(R)) - 8(3R)^{3\varepsilon} \) except for the \( z \)'s of the pits of \( g(z, t) \).

(ii) If a pit of \( g(z, t) \) contains a \( z \) with \( R > R_0 \), then

(a) The pit is contained in the circle with \( z \) as centre and diameter \( \exp \left( -R^8/10 \right) \), on the circumference of which \( g(z, t) \) satisfies the inequality of (i).

(b) Every value of \( \alpha \) satisfying:

\[
| \alpha | < \exp \left( -8(3R)^{3\varepsilon} \frac{m_b(R)}{m_c(R)} \right).
\]

is taken in the pit as many times as there are zeros of \( g(z, t) \) in the same pit.

Proof: By this lemma, we complete the investigation of 'pits behaviour' of the random meromorphic function \( g(z, t) \).

It is shown by employing results of lemmas 2 and 3, that every zero of \( g(z, t) \) is, for sufficiently large \( R \), included in a pit.
By lemma 3, it has been proved that if $z_1$ is a zero of $g(z, t)$, then there is a circle of centre $z_1$, and radius $r_1$ such that:

$$|g(z, t)| > \exp (-8(3R)^{38} m_b(R)/m_e(R)).$$

Since, furthermore, $r_1 > 5 \exp (-R^e/8)$, this implies that each zero is contained in a circular pit.

Also, we can say that, at points $z|z| = R$, outside the family of circles $C(z_1, r_1)$,

$$|g(z, t)| > \exp (-8(3R)^{38} m_b(R)/m_e(R)) > 2 \exp (-8(3R_1)^{38} m_b(R_1)/m_e(R_1)).$$

If we consider the open set of points within this circle for which

$$|g(z, t)| < 2 \exp (-8(3R_1)^{38} m_b(R_1)/m_e(R_1)),$$

then this open set must contain a simply connected domain of which $z_1$ is an interior point. Therefore $z_1$ is contained in a pit.

Since each pit is contained in a circular pit of radius at most

$$5[R_i^{1+1}] \exp (-R^e_i/8) < \exp (-R^e_i/10),$$

it follows that the diameter of each pit is, for $R_1 > R_0(\varepsilon)$, at most $\exp (-R^e_i/10)$.

It follows, also, that each pit is bounded by a level curve on which

$$|g(z, t)| = 2 \exp (-8(3R_1)^{38} m_b(R_1)/m_e(R_1),$$

where $R_1$ is the modulus of the zero $z_1$ of $g(z, t)$, and that the value of $g(z, t)$ within the pit does not exceed the value:

$$2 \exp (-8(3R_1)^{38} m_b(R_1)/m_e(R_1)).$$

Since

$$2 \exp (-8(3R_1)^{38} m_b(R_1)/m_e(R_1)) > \exp (-8(3R)^{38} m_b(R)/m_e(R),$$

then $g(z, t)$ takes within the pit values of the order $(-8(3R)^{38} m_b(R)/m_e(R)$.

By lemma 2, it has been proved that each value of a satisfying

$$|a| < \exp (-8(3R)^{38} m_b(R)/m_e(R),$$

taken by $g(z, t)$ at a point $z$, where $|z| = R > R_0(\varepsilon)$, is associated with a zero of $g(z, t)$ within a distance $2 \exp (-R^e/8)$ of $z$, and as

$$2 \exp (-R^e/8) < \frac{1}{2} \exp (-R^e/10),$$

then we conclude that $g(z, t)$ takes every value not exceeding

$$\exp (-8(3R)^{38} m_b(R)/m_e(R),$$

within the pit, as often as it has zeros within the same pit.

To complete the proof of this lemma, it must be shown that the measure of the exceptional set $x_1$ is less than $\varepsilon$. This has been proved in lemma 2.
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