Two low order characteristic finite element methods for a convection-dominated transport problem

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A B S T R A C T
Two low order rectangular finite elements for the convection–diffusion problem with a modified characteristic finite element scheme are studied in this paper. The \(O(h^{2})\) order error estimates in \(L^{2}\)-norm with respect to the space are obtained for one element with regular meshes and the other under anisotropic meshes. In the process, we use some distinct properties of the interpolation operator, the integral identity technique and the mean value technique, instead of the traditional elliptic projection which is an indispensable tool in the convergence analysis of the previous literature. Finally, some numerical results are provided to verify our theoretical analysis.

1. Introduction

We consider the convection–diffusion equation

\[
\begin{cases}
(a) & c_t + u(X, t) \cdot \nabla c - \nabla \cdot (a(X, t) \nabla c) = f(X, t), \quad \text{in } \Omega \times [0, T], \\
(b) & c(X, 0) = c_0(X), \quad \text{in } \Omega, \\
(c) & c(X, t) = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(1)

where \(\Omega \subset \mathbb{R}^{2}\) denotes an open bounded domain with the boundary \(\Gamma\) and a time interval \((0, T]\), \(X = (x, y)\), and the parameters appear in (1) satisfy the following assumptions [1].

(1) \(c(X, t)\) denotes, for example, the concentration of a possible substance;
(2) \(u(X, t)\) represents the velocity of the flow satisfying

\[
|u(X, t)| + |\nabla \cdot u(X, t)| \leq C, \quad \forall X \in \Omega,
\]

(2)

here \(C\) is a constant;
(3) \(a(X, t)\) is sufficiently smooth and there exist constants \(a_1\) and \(a_2\), such that

\[
0 < a_1 \leq a(X, t) \leq a_2 < +\infty, \quad \forall X \in \Omega;
\]

(3)

(4) \(f\) denotes a source term;
(5) \(\nabla\) and \(\nabla \cdot\) denote the gradient and the divergence operators, respectively.

In many diffusion processes arising in physical problems, convection essentially dominates diffusion, it is natural to seek numerical methods for such problems to reflect their almost hyperbolic nature. A lot of discretization schemes have been developed, such as the finite volume element methods [2,3], the streamline diffusion method [4], the least-squares mixed finite element method [5] and the modified method of characteristic Galerkin finite element procedure

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The interpolation operator $I_h$ is required to satisfy a regular assumption. Let $\Omega$ be a rectangular mesh of $\alpha_1, \alpha_2$, \(\alpha = (\alpha_1, \alpha_2)\) with $|\alpha| = 1$, we have

$$\|D^\alpha (\hat{\mathbf{u}} - I_h \mathbf{u})\|_{0,\hat{\Omega}} \leq C \|D^\alpha \hat{\mathbf{u}}\|_{1,\hat{\Omega}}.$$

For the convenience, let $\Omega \subset \mathbb{R}^2$ be a polygon with boundaries parallel to the axes, $T^h_i (i = 1, 2)$ be an axis-parallel rectangular mesh of $\Omega$, where $T^h_1$ does not need to satisfy the regularity assumption or quasi-uniform assumption, but $T^h_2$ is required to satisfy above regular assumption. Let $K = \{x_K - h_K, x_K + h_K\} \times \{y_K - h_K, y_K + h_K\}$, $h_K = \text{diam}(K)$, $S$ is the circle contained in $K$, $\rho_K = \max_{S \subset K} \text{diam}(S)$, $h = \max_{K \in T^h_2} h_K$, $l_k (k = 1, 2, 3, 4)$ be the edges of $K$.  

2. Construction of finite element schemes

Let $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference element on $\xi - \eta$ plane. The four vertices of $\hat{K}$ are $\hat{d}_1 = (-1, -1)$, $\hat{d}_2 = (1, -1)$, $\hat{d}_3 = (1, 1)$ and $\hat{d}_4 = (-1, 1)$, the four edges are $\hat{l}_1 = \hat{d}_1 \hat{d}_2$, $\hat{l}_2 = \hat{d}_2 \hat{d}_3$, $\hat{l}_3 = \hat{d}_3 \hat{d}_4$ and $\hat{l}_4 = \hat{d}_4 \hat{d}_1$.

Given $\hat{\mathbf{u}} \in H^1(\hat{K})$, we define the finite elements $(\hat{K}, \hat{P}, \Sigma^i), i = 1, 2$ on $\hat{K}$ as follows:

$$\Sigma^1 = \{\hat{v}^1, \hat{v}^2, \hat{v}^3, \hat{v}^4\}, \quad \hat{P} = \text{span}\{1, \xi, \eta, \xi \eta\},$$

where $\hat{v}^i = \hat{v}(\hat{d}_i)$, $i = 1, 2, 3, 4$.

$$\Sigma^2 = \{\hat{v}^1, \hat{v}^2, \hat{v}^3, \hat{v}^4\}, \quad \hat{P} = \text{span}\{1, \xi, \eta, \xi^2 - \eta^2\},$$

where $\hat{v}^i = \frac{1}{|\hat{d}_i|} \int_{\hat{d}_i} \hat{v} ds$, $i = 1, 2, 3, 4$.

It can be easily checked that interpolations defined above are well-posed and the interpolation function $\hat{P} \hat{v}$ ($i = 1, 2$) can be expressed as

$$\hat{I}^1 \hat{v} = \frac{1}{4} (1 - \xi)(1 - \eta) \hat{v}^1 + \frac{1}{4} (1 + \xi)(1 - \eta) \hat{v}^2 + \frac{1}{4} (1 + \xi)(1 + \eta) \hat{v}^3 + \frac{1}{4} (1 - \xi)(1 + \eta) \hat{v}^4,$$

$$\hat{I}^2 \hat{v} = \left(\frac{1}{4} - \frac{1}{2} \eta - \frac{3}{8} \xi^2 + \frac{3}{8} \eta^2\right) \hat{v}^1 + \left(\frac{1}{4} + \frac{1}{2} \xi + \frac{3}{8} \xi^2 - \frac{3}{8} \eta^2\right) \hat{v}^2 + \left(\frac{1}{4} + \frac{1}{2} \eta - \frac{3}{8} \xi^2 + \frac{3}{8} \eta^2\right) \hat{v}^3 + \left(\frac{1}{4} - \frac{1}{2} \xi + \frac{3}{8} \xi^2 - \frac{3}{8} \eta^2\right) \hat{v}^4.$$
Define the affine mapping \( F : \tilde{K} \rightarrow K \) as follows:
\[
\begin{align*}
x &= x_K + h_x, \\
y &= y_K + h_y.
\end{align*}
\]
(9)

Then the associated finite element space \( V_h^i \) (\( i = 1, 2 \)) is
\[
\begin{align*}
V_h^1 &= \{ v \big| v|_K = v|_K \circ F \in \hat{P}, \forall K \in T_h, \forall \mathbf{d} \subset \partial \Omega \}, \\
V_h^2 &= \{ v \big| v|_K = v|_K \circ F \in \hat{P}, \forall K \in T_h, \int [v]_l ds = 0, \forall l \subset \partial K \},
\end{align*}
\]
(10)
(11)

where \([v]_l \) stands for the jump of \( v \) across the edge \( l \) if \( l \) is an internal edge, and it is equal to \( v \) itself if \( l \) belongs to \( \partial \Omega \).

For any \( v \in H^1(\Omega) \), let \( \Pi^i \) (\( i = 1, 2 \)) be the associated interpolation operator on \( V_h^i \) (\( i = 1, 2 \)) satisfying \( \Pi^i|_K = \Pi^i_K \), \( \Pi^i_K = \hat{I} \circ F^{-1} \), then we have
\[
\begin{align*}
\Pi^i K (v|_l) &= v|_l, & i &= 1, 2, 3, 4, \\
\int_k (v - \Pi^i_k v) ds &= 0, & k &= 1, 2, 3, 4.
\end{align*}
\]
(12)
(13)

We denote by \( H^k(\Omega) \) the standard Sobolev space of \( k \)-differential functions and less than \( k \)-differential functions in \( L^2(\Omega) \) with the usual norm \( \| \cdot \|_k \) and semi-norm \( | \cdot |_k \) respectively. When \( k = 0 \), we let \( L^2(\Omega) \) denote the corresponding space defined on \( \Omega \) with norm \( \| \cdot \| \).

Let \([a, b] \subset [0, T], Y \) be a Sobolev space, and \( f(X, t) \) be smooth function on \( \Omega \times [a, b] \), also we define \( L^p(a, b; Y) \) and \( \| f \|_{L^p(a, b; Y)} \) as follows:
\[
\begin{align*}
L^p(a, b; Y) &= \left\{ f \big| \int_a^b \| f(\cdot, t) \|_Y^p dt < \infty \right\}, \\
\| f \|_{L^p(a, b; Y)} &= \left( \int_a^b \| f(\cdot, t) \|_Y^p dt \right)^{\frac{1}{p}}.
\end{align*}
\]

where if \( p = \infty \), the integral is replaced by the essential supremum.

Under the above assumptions, we begin to discretize the Eq. (1). Let
\[
\psi(X, t) = (1 + |u|^2)^{\frac{1}{2}}
\]
(14)

and the characteristic direction associated with the operator \( c_t + u \cdot \nabla c \) be denoted by \( \tau = \tau (X, t) \), where
\[
\frac{\partial}{\partial \tau} = \frac{1}{\psi(X, t)} \frac{\partial}{\partial t} + \frac{u}{\psi(X, t)} \cdot \nabla.
\]
(15)

Then the Eq. (1) can be put in the form
\[
\begin{align*}
\left\{ \begin{array}{ll}
\psi(X, t) \frac{\partial c}{\partial \tau} - \nabla \cdot (a(X, t) \nabla c) = f(X, t), & (X, t) \in \Omega \times (0, T), \\
c(X, 0) = c_0(X), & \forall X \in \Omega.
\end{array} \right.
\end{align*}
\]
(16)

The weak form of (16) is as follows: to find \( c \in H^1_0(\Omega) \times (0, T) \), such that
\[
\begin{align*}
\left\{ \begin{array}{ll}
\left( \psi(X, t) \frac{\partial c}{\partial \tau}, v \right) + (a(X, t) \nabla c, \nabla v) = (f, v), & \forall v \in H^1_0(\Omega) \cap H^2(\Omega), \\
c(X, 0) = c_0(X), & \forall X \in \Omega.
\end{array} \right.
\end{align*}
\]
(17)

This form will be discretized in details below.

Let \( e_\rho = c_\rho - \Pi^i c, \rho = \Pi^i c - c \).

In the procedure, we consider a time step \( \Delta t > 0 \) and approximate the solution at times \( t^n = n \Delta t \), and the characteristic derivative will be approximated basically in the following manner.

Let
\[
\overline{X} = X - u(X, t^n) \Delta t,
\]
then we have the following approximation similar to [9]
\[
\psi(X, t^n) \frac{\partial c}{\partial \tau} \bigg|_{t^n} \approx \psi(X, t^n) \frac{c(X, t^n) - c(\overline{X}, t^{n-1})}{\sqrt{(X - \overline{X})^2 + (\Delta t)^2}} = \frac{c(X, t^n) - c(\overline{X}, t^{n-1})}{\Delta t}.
\]
Our finite element scheme approximating (17) is to find \( c_h : \{t^0, t^1, \ldots, t^n\} \to V_h^i \) \((i = 1, 2)\), such that

\[
\left\{ \begin{aligned}
\frac{c_h^n - c_h^{n-1}}{\Delta t}, \quad v_h \\
\quad \quad + (a(X, t^n) \nabla c_h^n, \nabla v_h)_h = (f^n, v_h), \quad \forall v_h \in V_h^i,
\end{aligned} \right.
\]

where \( c_h^0 = c_0(t^0) \), \( \bar{X} = X - u_h(X, t^{n-1}) \Delta t \), \( c_h^{n-1} = c_h^{n-1}(\bar{X}, t^{n-1}) = c_h^{n-1}(X - u_h(X, t^{n-1}) \Delta t, t^{n-1}) \), \( f^n = f(X, t^n) \), \( \Pi^i c_0 \) is the finite element interpolation of \( c_0, i = 1, 2 \), \( (u, v)_h = \sum_K \int_K uv \text{d}x \text{d}y \). For conforming bilinear finite element, it means \((u, v)_h = \int_{\Omega} uv \text{d}x \text{d}y \).

\[\text{3. The existence and uniqueness of the solution of discrete problem}\]

In this section we prove the existence and uniqueness of the solution of the discrete problem (18).

**Theorem 1.** Under the assumption of (3), there exists a unique solution \( c_h \in V_h^i \) \((i = 1, 2)\) to the finite element scheme (18).

**Proof.** The linear system generated by (18) is square, so the existence of the solution is implied by its uniqueness. Let \( c_h^{n-1} \) and \( f \) be zero, thus \( \frac{c_h^n - c_h^{n-1}}{\Delta t} \) is zero too, thus we have

\[
\left( \frac{c_h^n}{\Delta t}, v_h \right) + (a(X, t^n) \nabla c_h^n, \nabla v_h)_h = 0, \quad \forall v_h \in V_h^i.
\]

Choosing \( v_h = c_h^n \) in (19) gives

\[
\frac{1}{\Delta t} \| c_h^n \|^2 + (a(X, t^n) \nabla c_h^n, \nabla c_h^n)_h = 0,
\]

then assumption (3) follows \( c_h^n = 0 \), the proof is completed.

Let \( \| \cdot \| = \left( \sum_K | \cdot |_{1, K}^2 \right)^{\frac{1}{2}} \), then it is a norm over \( V_h^i \) \((i = 1, 2)\).

To get error estimates, we state the following important lemmas.

**Lemma 3.1.** For any \( v \in H^2(\Omega) \), we have

\[
\| v - \Pi^i v \| + h \| v - \Pi^i v \|_h \leq C h^2 \| v \|_2, \quad i = 1, 2.
\]

Here and later, the positive \( C \) is independent of \( h_K \) and \( \frac{h_K}{\rho_K} \), which may be different in different places.

**Proof.** The desired result comes from the interpolation theorem [11,17].

**Lemma 3.2** ([18]). For all \( c \in H^1 \Omega \) and all \( v \in V_h^i \), we have

\[
(\nabla (c - \Pi^i c), \nabla v)_h \leq C h^2 \| c \|_3 \| v \|_1, \quad \forall v \in V_h^i.
\]

**Proof.** We only to prove \( \int_{\Omega^i} w_x v_x \leq C h^2 \| c \|_3 \| v \|_1, \forall v \in V_h^i \).

We expand \( v \in V_h^i \) in a Taylor series about the point \((x_K, y_K)\) as follows:

\[
v(x, y) = v(x_K, y_K) + (y - y_K) v_y.
\]

Let \( w = c - \Pi^1 c, F(y) = \frac{1}{2}[(y - y_K)^2 - h_y^2] \), then

\[
\int_K w_x v_x = \int_K w_x v(x_K, y_K) + \int_K w_y (y - y_K) v_y.
\]

Note that \( w(x_K \pm h, y_K \pm h) = 0, F(y_K \pm h) = 0 \), thus

\[
\int_K w_x = \int_K \frac{\partial^2 F}{\partial y^2} w_x \\
= \frac{\partial F(y_K + h_y)}{\partial y} [w(x_K + h, y_K + h_y) - w(x_K - h, y_K + h_y)] \\
- \frac{\partial F(y_K - h_y)}{\partial y} [w(x_K + h, y_K - h_y) - w(x_K - h, y_K - h_y)]
\]
On square meshes, for all \( c \in H^1(\Omega) \) and all \( v_h \in V^h \), we have
\[
\mathbf{Lemma 3.3.} \quad (\nabla (c - \Pi^2 c), \nabla v_h)_h = 0, \quad \forall v_h \in V^h.
\]

\textbf{Proof.} By Green’s formula and the definition of \( \Pi^2 \), we get
\[
(\nabla (c - \Pi^2 c), \nabla v_h)_h = \sum_{K \in T_h} \int_K (c - \Pi^2 c) \nabla v_h \mathbf{d}x \mathbf{d}y
= \sum_{K \in T_h} \int_{\partial K} (c - \Pi^2 c) \frac{\partial v_h}{\partial n} \mathbf{d}s - \sum_{K \in T_h} \int_K (c - \Pi^2 c) \Delta v_h \mathbf{d}x \mathbf{d}y.
\]

Note that for all \( v_h \in V^h \), \( \frac{\partial v_h}{\partial n} |_{\partial K} \) is constant and \( \Delta v_h |_K = 0 \) on square meshes, thus
\[
(\nabla (c - \Pi^2 c), \nabla v_h)_h = 0.
\]

Here and later, \( n = (n_1, n_2) \) denotes the unit outer norm on \( \partial K \). The proof is completed. \( \square \)

\textbf{Lemma 3.4 ([16])}. For the rotated \( Q_1 \) element, all \( u \in H^3(\Omega) \cap H^3_0(\Omega) \), we have
\[
\left| \sum_{K \in T_h} \int_{\partial K} \frac{\partial u}{\partial n} v_h \mathbf{d}s \right| \leq Ch^2 |u|_3 \| v_h \|_h, \quad \forall v_h \in V^h.
\]

\textbf{Lemma 3.5 ([1])}. Let \( \eta \in L^2(\Omega) \), and \( \tilde{\eta} = \eta(X - g(X) \Delta t) \), where function \( g \) and its gradient \( \nabla g \) are bounded, then
\[
\| \eta - \tilde{\eta} \|_{-1} \leq C \| \eta \| \| \Delta t \|.
\]

\textbf{4. Error estimate}

Now we start to derive the main result of this paper, i.e., the optimal order estimate of \( (c_h - c) \) in \( L^2 \)-norm.
Theorem 2. Let \( c_h, c \) be the solutions of (18) and (1) respectively, for sufficiently small \( \Delta t > 0 \), we have
\[
\max_{0 \leq n \leq N} \| c_h^n - c(t^n) \| \leq m_0 \Delta t + h^2 m_1,
\]
where
\[
m_0 = C \left\| \frac{\partial^2 c}{\partial \tau^2} \right\|_{L^2(0,T;L^2)}, \quad m_1 = C (|c|_{L^2(0,T;H^2)} + |c|_{L^\infty(0,T;H^2)} + |c|_{L^\infty(0,T;H^3)}).
\]

Proof. By (17) and (18), we get the error equation as follows:
For the bilinear finite element,
\[
\left( \frac{e^n_h - \tilde{e}^{n-1}_h}{\Delta t}, v_h \right) + (a(X, t^n) \nabla e^n_h, \nabla v_h)_h + (a(X, t^n) \nabla (\Pi c^n - c^n), \nabla v_h)_h
\]
\[
= \left( \psi^n \frac{\partial c^n}{\partial \tau} - \frac{c^n - \tilde{c}^{n-1}}{\Delta t}, v_h \right) - \left( \frac{\rho^n - \tilde{\rho}^{n-1}}{\Delta t}, v_h \right), \quad \forall v_h \in V^h.
\]
(31)

For the rotated Q1 finite element,
\[
\left( \frac{e^n_h - \tilde{e}^{n-1}_h}{\Delta t}, v_h \right) + (a(X, t^n) \nabla e^n_h, \nabla v_h)_h + (a(X, t^n) \nabla (\Pi c^n - c^n), \nabla v_h)_h + \sum_{K \in T_h} \int_{\partial K} a(X, t^n) \frac{\partial c^n}{\partial n} v_h ds
\]
\[
= \left( \psi^n \frac{\partial c^n}{\partial \tau} - \frac{c^n - \tilde{c}^{n-1}}{\Delta t}, v_h \right) - \left( \frac{\rho^n - \tilde{\rho}^{n-1}}{\Delta t}, v_h \right), \quad \forall v_h \in V^h.
\]
(32)

Then we choose \( v_h = e^n_h \) in (31) and (32), respectively.

Then using the argument similar to [9], the following result is obtained
\[
\left| \left( \psi^n \frac{\partial c^n}{\partial \tau} - \frac{c^n - \tilde{c}^{n-1}}{\Delta t}, e^n_h \right) \right| \leq C \left\| \frac{\partial^2 c}{\partial \tau^2} \right\|_{L^2(0,T;L^2)} + \| e^n_h \|^2.
\]
(33)

Due to
\[
\rho^n - \tilde{\rho}^{n-1} = (\rho^n - \rho^{n-1}) + (\rho^{n-1} - \tilde{\rho}^{n-1}),
\]
we have
\[
\left| \left( \frac{\rho^n - \rho^{n-1}}{\Delta t}, e^n_h \right) \right| \leq C \frac{\| \rho^n \|^2}{\| \rho^{n-1} \|^2} + C \| e^n_h \|^2.
\]
(34)

By Lemma 3.5
\[
\left| \left( \frac{\rho^{n-1} - \tilde{\rho}^{n-1}}{\Delta t}, e^n_h \right) \right| \leq C \left( \| e^n_h \| \frac{\rho^{n-1} - \tilde{\rho}^{n-1}}{\Delta t} \right)^{1/2} \| e^n_h \|^2
\]
\[
\leq C \| e^n_h \|^2 + C \| \rho^{n-1} \|^2.
\]
(35)

Next we estimate the left hand of (31) and (32).

Firstly, the first two terms on the left hand of (31) and (32) can be estimated as
\[
\left( \frac{e^n_h - \tilde{e}^{n-1}_h}{\Delta t}, e^n_h \right) + (a(X, t^n) \nabla e^n_h, \nabla e^n_h) \geq \frac{1}{2\Delta t} \left[ (e^n_h, e^n_h) - (\tilde{e}^{n-1}_h, \tilde{e}^{n-1}_h) \right] + (a(X, t^n) \nabla e^n_h, \nabla e^n_h)
\]
\[
\geq \frac{1}{2\Delta t} \left[ (e^n_h, e^n_h) - (1 + C\Delta t)(e^{n-1}_h, e^{n-1}_h) \right] + a_1 \| e^n_h \|^2,
\]
(36)

where the inequality \( \| e^n_h \|^2 \leq (1 + C\Delta t) \| e^n_h \|^2 \) (cf. [19]) is used in the last step.

Secondly, by Lemma 3.2 we can estimate the third term on the left hand of (31) as
\[
|(a(X, t^n) \nabla (\Pi c^n - c^n), \nabla e^n_h)| = |(a(X, t^n) - \tilde{a}) \nabla (\Pi c^n - c^n), \nabla e^n_h| + |(\tilde{a} \nabla (\Pi c^n - c^n), \nabla e^n_h)|
\]
\[
\leq C \| \Pi c^n - c^n \|_h \| e^n_h \| + C \| e^n_h \|_3 \| e^n_h \|_h
\]
\[
\leq C \| e^n_h \|_h \| c^n \|_3,
\]
(37)
By Lemma 3.3 we can estimate the third term on the right hand of (32) as
\[
| (a(X, t^n) \nabla (\Pi c^n - c^n), \nabla e^n_n) | = | (a(X, t^n) - \tilde{a}) \nabla (\Pi c^n - c^n), \nabla e^n_n) | + | (\tilde{a} \nabla (\Pi c^n - c^n), \nabla e^n_n) | \\
\leq C h \| \Pi c^n - c^n \| \| e^n_n \| + 0 \\
\leq C h^2 \| e^n_n \| \| c^n \| _3, \tag{38}
\]
where \( \tilde{a} |_K = \frac{1}{|K|} \int_K a(X, t^n) \, dx \, dy \) satisfying \( | a(X, t^n) - \tilde{a} |_K | < C h K \).

From (37) and (38), we can see that the same results are obtained by use of the two different finite element methods.

The last term of (32) can be estimated as
\[
\left| \sum_{n \in h} \int_{\partial K} a(X, t^n) \frac{\partial c^n}{\partial n} e^n_n \right| \leq C h^2 | c^n |_{3} \| e^n_n \| _h. \tag{39}
\]

From (33)–(39), we have
\[
\frac{1}{2 \Delta t} [ (e^n_{h}, e^n_{h}) - (1 + C \Delta t) (e_{h-1}^{n-1}, e_{h-1}^{n-1}) ] + a_1 \| e^n_n \|^2_h \leq C h^2 | c^n |_{3} \| e^n_n \| _h + C \left| \frac{\partial^2 c}{\partial \tau^2} \right|^2 \Delta t + C \| e^n_n \|^2 _h \\
+ \frac{C}{\Delta t} \| \mu_{1} \|^2 \| e^n_n \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h. \tag{40}
\]

Next, (40) is multiplied by 2 \Delta t, and sum them in time, we obtain
\[
\| e^n_{h} \|^2 _h \leq C \left| \frac{\partial^2 c}{\partial \tau^2} \right|^2 \Delta t^2 + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \Delta t \| c^n \| _{L^\infty(0, T; H^2)} \\
+ C \Delta t \sum_{i=1}^{n} \| e^n_{h} \|^2 _h. \tag{41}
\]

By Gronwall’s lemma, it follows that
\[
\| e^n_{h} \| \leq C h^2 \| c^n \| _{L^\infty(0, T; H^2)} + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \| \mu_{1} \|^2 \| e^n_{h} \| _h + C \Delta t \left| \frac{\partial^2 c}{\partial \tau^2} \right|^2 \| e^n_{h} \| _h. \tag{42}
\]

Note that \( c^n - c^n = e^n + \rho^n \), by (20) and (42) and the triangle inequality, we can get the desired result. \( \square \)

5. Numerical example

In order to illustrate our theoretical analysis in previous sections, we carry out two numerical simulations using the bilinear finite element and the rotated Q1 element for the Eq. (1), respectively.

Example 1. \( f(x, y, t) \) is taken such that \( c(x, y, t) = e^{-t} (1 - e^{-x(1-x)\varepsilon}) (1 - e^{-y(1-y)\varepsilon}) \) is the exact solution. \( \varepsilon \) denotes the singular perturbation parameter, when \( \varepsilon = 0.06 \), the exact solution exhibits four boundary layers as plotted in Fig. 2(a). The domain \( \Omega \) is divided into small rectangles in the following two different ways (illustrated by the Fig. 1), where \( \Omega = [0, 1] \times [0, 1] \). Mesh 1: square meshes; Mesh 2: anisotropic meshes, we subdivide the boundary of \( \Omega \) parallel to x-axis into \( n \) parts by the following \( n + 1 \) points: \( (1 - \cos(i\pi / n))/2, i = 0, 1, \ldots, n \) and the same intervals along y-axis.

Tables 1 and 2 give the numerical errors obtained with the bilinear finite element on Mesh 1 and Mesh 2, respectively. The numerical solutions on the two different meshes, for the case \( n = 16 \), \( t = h^2 \) are shown in Fig. 2(b) and (c).
Example 2. \( f(x, y, t) \) is taken such that \( c(x, y, t) = e^{-t} \sin \pi x \sin 2\pi y \) is the exact solution as plotted in Fig. 3(a). We consider the rotated \( Q_1 \) element for problem (1) on Mesh 1. The numerical errors are listed in Table 3. The numerical solution on Mesh 1 for \( t = h^2 \) is plotted in Fig. 3(b).

In Tables 1–3, \( c_h \) denotes the finite element solution of problem (1), \( \Delta t \) represents a time step, and the experiments are done with \( \Delta t = O(h^4) \). \( \alpha \) is the average convergence order for \( \|c - c_h\|_{0, \Omega} \).

Tables 1 and 2 show that the results are in good agreement with our investigation in Section 4. From the comparison of the numerical errors on the two meshes, we can see that the numerical solution based on Mesh 2 is in all cases more accurate than that based on Mesh 1 in \( L^2 \) norm, because Mesh 2 is made up of anisotropic meshes with finer mesh size in the direction of the rapid variation of the solution. For a smaller \( \epsilon \), the modified characteristic finite element scheme based on Mesh 2 is more fitted to solve the problem (1).
Fig. 3. $n = 16$; (a) Exact solution; (b) FEM solution on Mesh 1.

### Table 2

<table>
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<tr>
<th>$m \times n$</th>
<th>$t = 0.0625$</th>
<th>$\alpha$</th>
<th>$t = 0.1563$</th>
<th>$\alpha$</th>
<th>$t = 0.2188$</th>
<th>$\alpha$</th>
<th>$t = 0.2813$</th>
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<td>(O(h^2))</td>
<td>0.3728</td>
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<td>0.4514</td>
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<tr>
<td>16 $\times$ 16</td>
<td>0.0459</td>
<td>(O(h^2))</td>
<td>0.0630</td>
<td>(O(h^2))</td>
<td>0.2033</td>
<td>(O(h^2))</td>
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<td>(O(h^2))</td>
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<td>0.0271</td>
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### Table 3

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<th>$t = 0.2150$</th>
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<tr>
<td>16 $\times$ 16</td>
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<td>0.0379</td>
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From Table 3 and Fig. 3, we can see that the optimal $L^2$ norm errors estimated between the $c$ and $c_h$ with the rotated Q1 element are obtained on square element meshes, which is in good agreement with the theoretical analysis.

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### References


