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A COMBINATORIAL PROBLEM IN LOGIC

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This short note is an application of some theorems of graph theory to the problem of the minimum number of counter-examples needed to show that a special class of theories is complete.

0. Introduction

Let us consider a set of properties $P = \{p_1, p_2, \dots\}$ and a set of theorems of the type: "property p_i implies property p_j". These theorems can be represented by a directed graph G, with vertex set P, where (p_i, p_j) is an arc iff it follows from one or more of the given theorems that p_i implies p_j. Suppose that we want to show that no arc of the complementary graph $\bar{G}$ is good to represent a true implication of that kind: more precisely, with each arc (p, q) with $p \neq q$ and $(p, q) \notin G$, we assign a student who has to find an example where p is fulfilled but not q (i.e., a counter-example to the statement that p implies q).

In this note we determine the minimum number of students needed to show that all the possible (pairwise) implications are already represented in the graph G. In Section 2 we solve this problem under the assumption that the students work independently and in Section 3 we consider the problem without this assumption.

Consider the graph G in Fig. 1. Here it suffices to disprove the implications represented by the five arcs 3, 4, 5, 7 and 10 of $\bar{G}$ for then the falsity of the other possible implications follows. For example, we have $p_2 \not\Rightarrow p_1$ for otherwise $p_2 \Rightarrow p_1 \Rightarrow p_3$ which contradicts the statement that arc (p_2, p_3) is bad.

Let H be a graph whose vertices represent the arcs $1, 2, \dots, 10$ of $\bar{G}$ and where an arc is drawn from i to j iff "arc i is good" implies "arc j is good", see Fig. 1. In H, the set $K = \{3, 4, 5, 7, 10\}$ is a *kernel*, i.e.,

(i) every vertex of H which is not in K is the initial end of an arc going into K,

(ii) no arc connects two vertices in K.

From (i) it follows that if arcs 3, 4, 5, 7 and 10 are bad, then all the arcs $1, 2, \dots, 10$ are bad, and from (ii), K is a minimal set with this property. Since K is the only

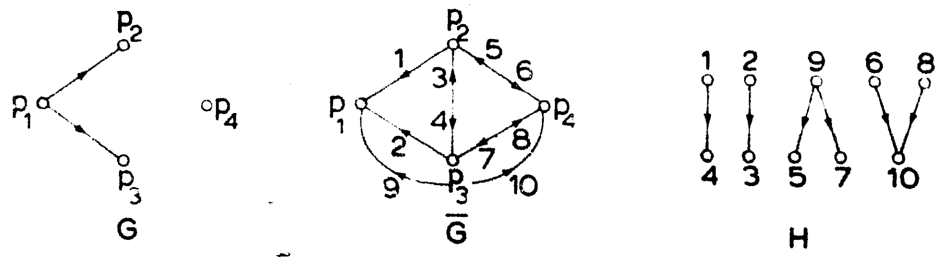

Fig. 1.

kernel of H, it follows that five counter-examples are needed to show that all the arcs of $\bar{G}$ are bad when the students work independently. Otherwise, i.e., if counter-examples to statements like "properties $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ together imply property p_j" are also considered, it is sometimes possible to do better. In the above example, to show that all the arcs of $\bar{G}$ are bad, it is enough to disprove the following three statements:

(i) p_2 and p_4 together imply p_3,

(ii) p_3 and p_4 together imply p_2,

(iii) p_1 implies p_4.

1. The anti-bases of a theory

A *theory* $T = (X, \mathscr{C})$ is defined by:

(i) a set X whose elements $x_1, x_2, \ldots$ may be thought of as propositions,

(ii) a closure relation $\mathscr{C}$ on X; for $S \subseteq X$, $\mathscr{C}(S)$ denotes the set of all the propositions in X which can be proved from the propositions in S.

For convenience we write $\mathscr{C}(s)$ instead of $\mathscr{C}(\{s\})$ for $s \in X$.

A theory $T = (X, \mathscr{C})$ is *unitary* if $x \in \mathscr{C}(S)$ implies the existence of some $s \in S$ such that $x \in \mathscr{C}(s)$. Otherwise T is *pluritary*. If a theory T is unitary, it can also be represented by a transitive graph with vertex set X, where (x, y) is an arc iff $x \in \mathscr{C}(y)$. An *axiom basis* for T is a set $B \subseteq X$ such that $\mathscr{C}(B) = X$ and which is minimal with respect to this property.

An *anti-basis* for T is a set $A \subseteq X$ such that $\mathscr{C}(x) \cap A \neq \emptyset$ for all $x \in X$ and which is minimal with respect to this property. The interpretation of this definition is that if all the propositions in A are false then all the propositions in X are false and A is minimal. The *inverse* $T' = (X, \mathscr{C}')$ of a theory $T = (X, \mathscr{C})$ is defined by: $x \in \mathscr{C}'(S)$ iff $\mathscr{C}(x) \cap S \neq \emptyset$. It can be easily checked that T' is a theory.

The closure relation $\mathscr{C}'$ can be interpreted as: if for T all the propositions in S are false, then x is false.

Lemma 1.1. *The inverse T' of a theory T is unitary.*

Proof. Let $S \subseteq X$ and $x \in \mathscr{C}'(S)$. Then $\mathscr{C}(x) \cap S \neq \emptyset$. If $s \in \mathscr{C}(x) \cap S$, then $x \in \mathscr{C}'(s)$. Thus T' is unitary.

Theorem 1.2. *In a theory T, all the anti-bases have the same cardinality.*

Proof. A set $A \subseteq X$ is an anti-basis of T iff A is a basis for the inverse T'. By Lemma 1.1, T' can be represented by a transitive graph H. Clearly a basis of T' is a kernel of H and conversely. By Corollary 1 to Theorem 3 in Chapter 14 of [1], all the kernels of H have the same cardinality. This proves the theorem.

In fact, for a transitive graph H, any kernel is obtained by choosing one vertex from each terminal strong component.

2. The graph of implications

Let G be a transitive directed graph whose vertices represent propositions and whose arcs represent implications and let $x_1, x_2, \ldots, x_m$ be the arcs of the complementary graph $\bar{G}$. Let $X = \{x_1, x_2, \ldots, x_m\}$, and for $S \subseteq X$, let $\mathscr{C}(S)$ denote the implications which can be derived from the implications in S, i.e., all the arcs of X in the transitive closure of $G + S$. The pair $T = (X, \mathscr{C})$ is a theory.

Theorem 2.1. *In the theory $T = (X, \mathscr{C})$, defined as above by a transitive graph G, all the anti-bases have the same cardinality, and this cardinality is the absorption number of the graph $H = (X, U)$ defined by: $(x, y) \in U$ iff y is an arc of the transitive closure of $G + x$. Furthermore, there is a one-to-one correspondence between the anti-bases of T and the kernels of H.*

Proof. First remark that H is a transitive graph. By Theorem 1.2, it suffices to check that this graph H represents the theory T'. Clearly, $(x, y) \in U$ iff $y \in \mathscr{C}(x)$, that is, iff $x \in \mathscr{C}'(y)$. Thus H represents T' and the theorem is proved.

This theorem gives the minimum number of students needed in the problem raised in the introduction, assuming that they work independently.

3. The unrestricted case

The problem is different if we do not assume that the students work independently. For example, consider the graph of implications G represented by the unbroken lines in Fig. 2. Its complementary graph $\bar{G}$, represented by the dotted lines, has arcs $1, 2, \ldots, 10$. The kernel of H is unique and contains four vertices: 5, 6, 7, 9; hence four counter-examples are enough to show that all the arcs of $\bar{G}$ are bad. However, there is another way to reach the same conclusions with no more than four counter-examples: If "arc 1 is bad", then either arc 9 or arc 6 is bad (because 1 is an arc of the transitive closure of $G + \{6, 9\}$). Thus if counter-examples are obtained for the implications 1, 5, 7, we need only one more counter-example to

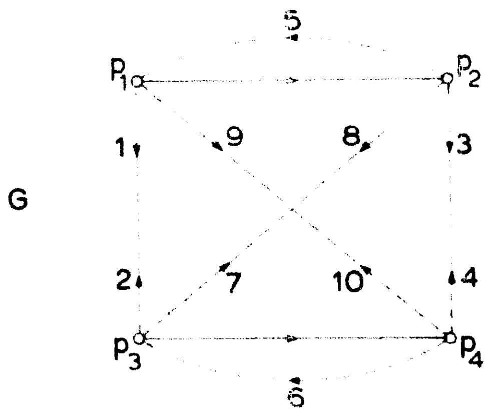

Fig. 2.

show that arcs 1, 5, 6, 7, 9 are all bad, and consequently that all the arcs in $\bar{G}$ are bad.

Now, a new problem arises: for the unrestricted case, is it true that a kernel of H gives always an optimal solution ?

As in Section 2, let $G = (P, I)$ be a transitive directed graph whose vertices represent propositions and arcs represent implications. Assuming that one counter-example can be used to disprove several implications in $\bar{G}$, we now determine the minimum number of counter-examples needed to show that all the arcs in $\bar{G}$ are bad.

From G, construct a graph G_0 as follows. The vertices of G_0 are all the nonempty subsets of P. There is an arc going from A to B in G_0 if either $A \supseteq B$ or $A = \{p_i\}$, $B = \{p_j\}$ and $(p_i, p_j) \in G$. Let G_1 be the graph obtained from G_0 by adding as many arcs as possible using the following rules repeatedly.

(i) If (A, B) and (A, C) are arcs, then $(A, B \cup C)$ is an arc.

(ii) If (A, B) and (B, C) are arcs, then (A, C) is an arc.

It is not difficult to see that G_1 gives all the implications between the various subsets of P that follow from G. Also G is (isomorphic to) a subgraph of G_1. Now construct the graphs $\bar{G}_1$ and H_1 corresponding to G_1 as in Section 2. It is easy to see that H is (isomorphic to) a subgraph of H_1. If H_2 is the subgraph of H_1 generated by the transitive closure of the vertices in H, then to show that all the arcs of $\bar{G}$ are bad, it is sufficient to disprove the implications represented by the vertices in any kernel of H_2. Again H_2 is transitive, and, consequently, all the kernels of H_2 have the same cardinality.

This gives a solution to the problem of the minimum number of students required when they work not necessarily independently.

Reference

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1974).