A DECOMPOSITION OF DISTRIBUTIVE LATTICES*

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Let \( \mathcal{D} \) be a distributive lattice formed by subsets of a finite set \( E \) such that \( \emptyset, E \in \mathcal{D} \) with set union and intersection as the lattice operations. We define a simple split decomposition of \( \mathcal{D} \) into distributive lattices \( \mathcal{D}_i \subseteq \mathcal{D} \) (\( i = 1, 2 \)) such that \( \mathcal{D} \) is uniquely reconstructed from \( \mathcal{D}_i \) (\( i = 1, 2 \)). Based on the combinatorial decomposition theory developed by Cunningham and Edmonds, we show that \( \mathcal{D} \) can uniquely be decomposed (by repeated simple split decompositions) into a minimal collection of prime (indecomposable) distributive lattices and brittle distributive lattices, where each brittle distributive lattice corresponds to a poset represented by the Hasse diagram forming a star or a complete bipartite graph.

Introduction

Let \( \mathcal{D} \subseteq 2^E \) be a distributive lattice formed by subsets of a finite set \( E \) such that \( \emptyset, E \in \mathcal{D} \) with set union and intersection as the lattice operations. In the present paper we shall define a way of decomposing \( \mathcal{D} \) (if possible) into distributive lattices \( \mathcal{D}_i \subseteq \mathcal{D} \) (\( i = 1, 2 \)) such that \( \mathcal{D}_i \) (\( i = 1, 2 \)) are smaller than \( \mathcal{D} \) (in a sense precisely described in Section 2) and \( \mathcal{D} \) is uniquely determined by \( \mathcal{D}_i \) (\( i = 1, 2 \)). The pair \( \{\mathcal{D}_1, \mathcal{D}_2\} \) is called a simple split decomposition of \( \mathcal{D} \). The structure of the collection of distributive lattices obtained by repeated simple split decompositions will be examined in detail by employing the decomposition theory developed by Cunningham and Edmonds [3, 5].

Substitution decompositions of acyclic graphs and posets were considered, for example, in [2, 9] (also see a survey [8] for substitution decompositions) and a split decomposition of strongly connected graphs was treated in [4]. However, to the author's knowledge split decompositions of distributive lattices have not yet been considered in the literature.

The present work was motivated by the investigation of the possibility of a split decomposition of a (not necessarily symmetric) submodular system (cf. [6]). The motivation will be given in the next section.

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1. Motivation: A decomposition of submodular systems

Let $\mathcal{D}$ be the set of reals and $2^E$ be a distributive lattice with set union and intersection as the lattice operations such that $\emptyset, E \in \mathcal{D}$. Also let $f: \mathcal{D} \to \mathbb{R}$ be a \textit{submodular function} on $\mathcal{D}$, i.e., for each $X, Y \in \mathcal{D}$ we have

$$f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y). \quad (1.1)$$

Here, we assume $f(\emptyset) = 0$. The pair $(\mathcal{D}, f)$ is called a \textit{submodular system} and a polyhedron

$$B(f) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{D} : x(X) \leq f(X), x(E) = f(E)\} \quad (1.2)$$

is called the \textit{base polyhedron} associated with $(\mathcal{D}, f)$, where $\mathbb{R}^E$ is the set of all functions (or vectors) from $E$ to $\mathbb{R}$ and for each $X \in \mathcal{D}$ and $x \in \mathbb{R}^E$

$$x(X) = \sum_{e \in X} x(e). \quad (1.3)$$

It should be noted that we can not symmetrize $f$ as in [6], since $\mathcal{D}$ is not complemented in general. Therefore, we can not apply the decomposition theory in [6] to general submodular systems in an effective way.

Now, let $S = \{T_1, T_2\}$ be a dissection of $E$, i.e., a partition of $E$ into two nonempty parts, and define

$$\mathcal{D}_i = \{X \mid X \in \mathcal{D}, \text{either } X \subseteq T_i \text{ or } E - X \subseteq T_i\} \quad (1.4)$$

for each $i = 1, 2$. Also for each $i = 1, 2$ let $f_i$ be a restriction of $f$ to $\mathcal{D}_i$. Then, in general we have

$$B(f) \subseteq B(f_1) \cap B(f_2). \quad (1.5)$$

Here, it should be noted that $B(f_1) \cap B(f_2)$ is also a base polyhedron associated with a certain submodular system, since $\mathcal{D}_{12} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a crossing family and a restriction of $f$ to $\mathcal{D}_{12}$ is a submodular function on the crossing family $\mathcal{D}_{12}$ (see [7]). (We say two subsets $X, Y$ of $E$ cross if the four sets $X \cap Y, X \cap (E - Y), (E - X) \cap Y, (E - X) \cap (E - Y)$ are nonempty. A family $\mathcal{F}$ of subsets of $E$ is called a \textit{crossing family} if for each crossing pair of $X, Y \in \mathcal{F}$ we have $X \cup Y, X \cap Y \in \mathcal{F}$. A function $f: \mathcal{F} \to \mathbb{R}$ is called a \textit{submodular function on the crossing family $\mathcal{F}$} if (1.1) holds for each crossing pair of $X, Y \in \mathcal{F}$.) If (1.5) holds with equality, i.e.,

$$B(f) = B(f_1) \cap B(f_2) \quad (1.6)$$

and $|T_i| \geq 2$ ($i = 1, 2$), we say $S = \{T_1, T_2\}$ is a \textit{split} of $(\mathcal{D}, f)$ and $\{(\mathcal{D}_1, f_1), (\mathcal{D}_2, f_2)\}$ is a \textit{split decomposition} of $(\mathcal{D}, f)$. Since there is a one-to-one correspondence between base polyhedra and submodular functions $f'$ on distributive lattices $\mathcal{D}'$ with $\emptyset, E \in \mathcal{D}'$ and $f'(\emptyset) = 0$, $f$ is uniquely determined by $f_1$ and $f_2$ if $S = \{T_1, T_2\}$ is a split of $(\mathcal{D}, f)$. Equation (1.6) is equivalent to the condition that for each $X \in \mathcal{D}$

$$f(X) = \min\{f_1(X \cap T_1) + f_2(X \cap T_2), f_1(X \cup T_1) + f_2(X \cup T_2) - f(E)\}, \quad (1.7)$$
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where for each \( i = 1, 2 \) we define \( f_i(Y) = +\infty \) for \( Y \subseteq E \) with \( Y \notin \mathcal{D} \). This can be shown based on a result in [7]. We see from (1.7) that the decomposition of a matroid into connected components and 2-connected components is a special case of the split decomposition of a submodular system.

Based on the above definition of split decomposition we can develop a theory of decompositions of submodular systems. It can be shown that split decompositions of submodular systems satisfy the axioms of a decomposition frame proposed by Cunningham and Edmonds [3, 5], but the so-called intersection property [3, 5] may not hold in general. A result of the present paper implies that the intersection property also holds if \( f \) is a modular function (i.e., \( f \) satisfies (1.1) with equality for each \( X, Y \in \mathcal{D} \)). Note that, if \( f \) is modular, (1.6) or (1.7) is equivalent to the condition that for each \( X \in \mathcal{D} \)

(i) \( X \cap T_1, X \cap T_2 \in \mathcal{D} \) or

(ii) \( X \cup T_1, X \cup T_2 \in \mathcal{D} \)

(see (2.7) and (2.8)). Therefore, the problem of the decomposition of \( (\mathcal{D}, f) \) with modular \( f \) is reduced to that of the decomposition of the underlying distributive lattice \( \mathcal{D} \).

The main purpose of the present paper is to show that the split decomposition of a distributive lattice falls within the general framework of Cunningham and Edmonds [3, 5] and to characterize brittle distributive lattices (the precise definition of a brittle distributive lattice will be given later). Applications of the result of this paper to lattice-theoretical or order-theoretical questions will be left for future research.

2. Definitions and preliminaries

Let \( P \) be a finite set and \( \mathcal{P} = (P, \preceq) \) be a partially ordered set (or a poset) with a partial order \( \preceq \) among elements of \( P \). We write \( a \prec b \) if and only if \( a \preceq b \) and \( a \neq b \). Also, \( a \preceq b \) if and only if \( a < b \) and there exists no \( c \in P \) such that \( a < c < b \). A subset \( X \) of \( P \) is called a (lower) ideal of \( \mathcal{P} \) if \( a \preceq b \) in \( X \) implies \( a \in X \) for every \( a, b \in \mathcal{P} \). Let \( E \) be a finite set and denote by \( 2^E \) the set of all the subsets of \( E \). A set \( \mathcal{D} \) of subsets of \( E \) is a distributive lattice with set union and intersection as the lattice operations if for every \( X, Y \in \mathcal{D} \) we have \( X \cup Y, X \cap Y \in \mathcal{D} \). Such a \( \mathcal{D} \) is sometimes called a ring family.

**Proposition 2.1** ([1]). Let \( \mathcal{D} \) be a distributive lattice formed by subsets of \( E \) such that \( \emptyset, E \in \mathcal{D} \) with set union and intersection as the lattice operations. Then there uniquely exist a partition \( \Pi = \{E_1, E_2, \ldots, E_n\} \) of \( E \) and a poset \( \mathcal{P} = (\Pi, \preceq) \) such that \( X \in \mathcal{D} \) if and only if

\[
X[\Pi] = \{E_i \mid E_i \in P, E_i \subseteq X\}
\]

is an ideal of \( \mathcal{P} \) and

\[
X = \bigcup \{E_i \mid E_i \in X[\Pi]\}.
\]
Under the correspondence (2.2) we can identify the distributive lattice \( \mathcal{D} \) with the collection (denoted by \( 2^E \)) of all the ideals of \( \mathcal{P} \) in Proposition 2.1.

We suppose throughout the present paper that \( \mathcal{D} \subseteq 2^E \) is a distributive lattice for which the partition \( \Pi \) in Proposition 2.1 is given by \( \{ \{ e \} \mid e \in E \} \). Let \( P = \{ E_1, E_2, \ldots, E_k \} \) be a partition of \( E \). For any \( X \subseteq E \) we say \( X \) is compatible with the partition \( P \) if for each \( E_i \in P, E_i \cap X \neq \emptyset \) implies \( E_i \subseteq X \). A dissection of \( E \) is a partition of \( E \) into two parts and we say a dissection \( S = \{ T_1, T_2 \} \) of \( E \) is compatible with \( P \) if \( T_1 \) and then \( T_2 \) are compatible with \( P \). Then define

\[
\mathcal{D}(P) = \{ X \mid X \in \mathcal{D}, X \text{ is compatible with } P \}. 
\]

(2.3)

\( \mathcal{D}(P) \) is a (distributive) sublattice of \( \mathcal{D} \). Suppose \( T \subseteq E \) is compatible with \( P \) and define

\[
P \parallel T = \{ E_i \mid E_i \in P, E_i \cap T = \emptyset \} \cup \{ T \},
\]

(2.4)

\[
\mathcal{D}(P) \parallel T = \mathcal{D}(P \parallel T).
\]

(2.5)

We call \( P \parallel T \) (or \( \mathcal{D}(P) \parallel T \)) an aggregation of \( P \) (or \( \mathcal{D}(P) \)) by \( T \). Define

\[
T[P] = \{ E_i \mid E_i \in P, E_i \subseteq T \}.
\]

(2.6)

A dissection \( S = \{ T_1, T_2 \} \) of \( E \) is called a good dissection of \( \mathcal{D}(P) \) if \( S \) is compatible with \( P \) and for every \( X \in \mathcal{D}(P) \) at least one of the following two holds:

\[
X \cap T_1, X \cap T_2 \in \mathcal{D}(P),
\]

(2.7)

\[
X \cup T_1, X \cup T_2 \in \mathcal{D}(P).
\]

(2.8)

A good dissection \( S = \{ T_1, T_2 \} \) of \( \mathcal{D}(P) \) is called a split of \( \mathcal{D}(P) \) if

\[
|T_i[P]| \geq 2, \quad i = 1, 2,
\]

(2.9)

where \( |\cdot| \) denotes the cardinality. By definition, \( S = \{ T_1, T_2 \} \) is a split (or a good dissection) of \( \mathcal{D}(P) \) if and only if \( S \) is a split (or a good dissection) of its dual \( \mathcal{D}(P) = \{ E - X \mid X \in \mathcal{D}(P) \} \).

**Lemma 2.2.** Let \( S = \{ T_1, T_2 \} \) be a good dissection of \( \mathcal{D}(P) \) and denote \( \mathcal{D}(P) \parallel T_i \) by \( \mathcal{D}_i \) \( (i = 1, 2) \). Then \( \mathcal{D}(P) \parallel T \) is the unique minimal distributive lattice (minimal with respect to set inclusion) such that

\[
\mathcal{D}(P) \equiv \mathcal{D}_1 \cup \mathcal{D}_2.
\]

(2.10)

**Proof.** Relation (2.10) trivially holds. Let \( \mathcal{D}' \) be the unique minimal distributive lattice such that \( \mathcal{D}' \equiv \mathcal{D}_1 \cup \mathcal{D}_2 \). Then, by the definition of a good dissection, for each \( X \in \mathcal{D}(P) \) there exist \( Y_i \in \mathcal{D}_i \) \( (i = 1, 2) \) such that \( Y_1 \cup Y_2 = X \) or \( Y_1 \cap Y_2 = X \). Therefore, \( X \in \mathcal{D}' \). Consequently, \( \mathcal{D}' \equiv \mathcal{D}(P) \) and thus \( \mathcal{D}' = \mathcal{D}(P) \). \( \square \)

From Lemma 2.2, if a split \( S = \{ T_1, T_2 \} \) of \( \mathcal{D}(P) \) is given, we can construct \( \mathcal{D}(P) \parallel T_i \) \( (i = 1, 2) \). We say \( \{ \mathcal{D}(P) \parallel T_1, \mathcal{D}(P) \parallel T_2 \} \) is the simple split decomposition of \( \mathcal{D}(P) \) by the split \( S = \{ T_1, T_2 \} \). Note that \( |P \parallel T_i| < |P| \) \( (i = 1, 2) \) but that not necessarily \( |\mathcal{D}_i| < |\mathcal{D}| \) \( (i = 1, 2) \).
Denote by $\mathcal{P}(P) = (\Pi(P), \preceq_P)$ the unique poset on the partition $\Pi(P)$ of $E$ which corresponds to $\mathcal{D}(P)$ (cf. Proposition 2.1). The partition $\Pi(P)$ of $E$ is said to be induced by $\mathcal{D}(P)$. The Hasse diagram $H(\mathcal{P}(P))$ representing the poset $\mathcal{P}(P) = (\Pi(P), \preceq_P)$ is defined as a directed graph $H(\mathcal{P}(P)) = (\Pi(P), A(P))$ with a vertex set $\Pi(P)$ and an arc set

$$A(P) = \{(E_i, E_j) \mid E_i, E_j \in \Pi(P), E_i \prec_P E_j\}.$$ (2.11)

We say $\mathcal{D}(P)$ (and $\mathcal{P}(P)$) are connected if the corresponding Hasse diagram $H(\mathcal{P}(P))$ is connected. (Note that $\mathcal{D}(P)$ is connected if and only if for each $X \in \mathcal{D}(P)$ with $\emptyset \neq X \neq E$ we have $E \setminus X \in \mathcal{D}(P)$.) When $H(\mathcal{P}(P))$ is connected, a vertex $E_0$ in $H(\mathcal{P}(P))$ is called a cut vertex if deleting $E_0$ from $H(\mathcal{P}(P))$ makes $H(\mathcal{P}(P))$ disconnected. For any disjoint subsets $\hat{F}_1$ and $\hat{F}_2$ of $\Pi(P)$, we say an arc $(E_i, E_j)$ in $H(\mathcal{P}(P))$ connects $\hat{F}_1$ with $\hat{F}_2$ if "$E_i \in \hat{F}_1$ and $E_j \in \hat{F}_2$" or "$E_i \in \hat{F}_2$ and $E_j \in \hat{F}_1$".

A subgraph $H_1$ of $H(\mathcal{P}(P))$ is called a star if there exists a vertex $E_0$ in $H_1$ such that for each arc $(E_i, E_j)$ in $H_1$, either $E_i = E_0$ or $E_j = E_0$. The vertex $E_0$ is called the center of the star $H_1$. A directed path of length $k$ ($k \geq 0$) in $H(\mathcal{P}(P))$ is a sequence of vertices $E_i$ ($i = 0, 1, \ldots, k$) in $H(\mathcal{P}(P))$ such that $E_i \prec_P E_{i+1}$ ($i = 0, 1, \ldots, k-1$).

For any subsets $W_1, W_2, W_3$ of a set $V$ we say $W_3$ separates $W_1$ from $W_2$ if $W_1 \subseteq W_3 \subseteq V \setminus W_2$ or $W_2 \subseteq W_3 \subseteq V \setminus W_1$.

3. Characterization of splits

In this section we give a characterization of splits of $\mathcal{D}(P)$.

First, we consider splits $S = \{T_1, T_2\}$ which are compatible with $\Pi(P)$, where we recall that $\Pi(P)$ is a partition of $E$ induced by $\mathcal{D}(P)$. Therefore, without loss of generality we assume $P = \{\{e\} \mid e \in E\}$ and by a natural correspondence identify $\mathcal{P}(P)$ with a poset on $E$ denoted by $\mathcal{P} = (E, \preceq)$, where $\mathcal{P}(P)$ is the poset on $P$ which corresponds to $\mathcal{D}$ ($= \mathcal{D}(P)$).

**Lemma 3.1.** A dissection $S = \{T_1, T_2\}$ of $E$ is a good dissection of $\mathcal{D}$ with corresponding poset $\mathcal{P} = (E, \preceq)$ if and only if there do not exist any $X \in \mathcal{D}$ and distinct four elements $e_i \in E$ ($i = 1, 2, 3, 4$) such that

1. $e_1 \succeq e_2$ and $e_3 \succeq e_4$,
2. $T_1$ separates $\{e_1\}$ from $\{e_2\}$ and also $\{e_3\}$ from $\{e_4\}$,
3. $X$ separates $\{e_1, e_2\}$ from $\{e_3, e_4\}$.

**Proof.** "If" part: If $S = \{T_1, T_2\}$ is not a good dissection of $\mathcal{D}$, then there exists $X \in \mathcal{D}$ such that neither

$$X \cap T_1, X \cap T_2 \in \mathcal{D}$$ (3.1)
nor
\[ X \cup T_1, X \cup T_2 \in \mathcal{D} \] holds. Therefore, since \( X \in \mathcal{D} \),
\[ \exists e_1 \in X \cap T_1, \exists e_2 \in X \cap T_2 : e_1 \succ e_2 \text{ or } e_1 \preceq e_2, \] (3.3)
\[ \exists e_3 \in T_1 - X, \exists e_4 \in T_2 - X : e_3 \succ e_4 \text{ or } e_3 \preceq e_4. \] (3.4)
(3.3) and (3.4) imply that \( X \) and \( e_i \) \((i = 1 \sim 4)\) in (3.3) and (3.4) satisfy (1) \sim (3) with appropriate re-numbering of \( e_i \) \((i = 1 \sim 4)\).

"Only if" part: Conversely, if (1) \sim (3) hold, it is easily seen that \( S = \{ T_1, T_2 \} \) cannot be a good dissection of \( \mathcal{D} \).

**Lemma 3.2.** Suppose that for distinct four elements \( e_i \) \((i = 1 \sim 4)\) we have
\[ e_1 \succ e_2, \quad e_3 \succ e_4. \] (3.5)
Then there does not exist any \( X \in \mathcal{D} \) which separates \( \{ e_1, e_2 \} \) from \( \{ e_3, e_4 \} \) if and only if
\[ e_1 \succ e_4, \quad e_3 \succ e_2. \] (3.6)

**Proof.** Easy. □

**Lemma 3.3.** A dissection \( S = \{ T_1, T_2 \} \) of \( E \) is a good dissection of \( \mathcal{D} \) if and only if for arbitrary distinct four elements \( e_i \in E \) \((i = 1 \sim 4)\) such that
(1) \( e_1 \succ e_2, \quad e_3 \succ e_4 \),
(2) \( T_1 \) separates \( \{ e_1 \} \) from \( \{ e_3 \} \) and also \( \{ e_2 \} \) from \( \{ e_4 \} \) we have
\[ e_1 \succ e_4, \quad e_3 \succ e_2. \] (3.8)

**Proof.** Immediate from Lemmas 3.1 and 3.2. □

**Lemma 3.4.** Let \( S = \{ T_1, T_2 \} \) be a good dissection of \( \mathcal{D} \). If distinct four elements \( e_i \in E \) \((i = 1 \sim 4)\) satisfy
\[ e_1, e_3 \in T_1, \quad e_2, e_4 \in T_2, \] (3.9)
\[ e_1 \succ e_2, \quad e_3 \succ e_4, \] (3.10)
then
\[ e_1 \succ e_4, \quad e_3 \succ e_2. \] (3.11)

**Proof.** Because of Lemma 3.3 we have
\[ e_1 \succ e_4, \quad e_3 \succ e_2. \] (3.12)
Choose any directed path \( Q \) in \( H(\mathcal{D}) \) from \( e_1 \) to \( e_4 \):
\[ Q : e_1 \succ d_1 \succ d_2 \succ \cdots \succ d_k \succ e_4, \quad k \geq 0. \] (3.13)
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The path \( Q \) does not contain \( e_2 \) or \( e_3 \). For if \( e_2 \) lies on \( Q \), then \( e_2 \succ e_4 \) and from (3.12) \( e_2 \succ e_4 \succ e_4 \), which contradicts \( e_3 \succ e_4 \). Similarly, if \( e_3 \) lies on \( Q \), this leads to a contradiction to \( e_1 \succ e_2 \). Unless \( e_1 \succ e_4 \), there exist \( d_i \succ d_{i+1} \) on \( Q \) such that

\[
\begin{align*}
  e_1 &\succ d_i \succ d_{i+1} \succ e_4, \\
  d_i &\in T_1, \quad d_{i+1} \in T_2.
\end{align*}
\]  

(3.14)

(3.15)

If \( e_1 \succ d_i \succ d_{i+1} \succ e_4 \), we have \( d_i \succ e_2 \) due to Lemma 3.3 applied to four elements \( e_1, \, e_2, \, d_i, \, d_{i+1} \). Therefore, \( e_1 \succ d_i \succ e_2 \), which contradicts \( e_1 \succ e_2 \). Similarly, if \( e_1 \succ d_i \succ d_{i+1} \succ e_4 \), we reach a contradiction to \( e_2 \succ e_4 \). Consequently, we have \( e_1 \succ e_4 \). By the symmetry we also have \( e_3 \succ e_2 \). \( \square \)

**Lemma 3.5.** Let \( S = \{T_1, \, T_2\} \) be a good dissection of \( \mathcal{G} \). If distinct four elements \( e_i \in E \) (\( i = 1 \sim 4 \)) satisfy

\[
\begin{align*}
  e_1, \, e_4 &\in T_1, \quad e_2, \, e_3 \in T_2, \\
  e_1 &\succ e_2, \quad e_3 \succ e_4.
\end{align*}
\]  

(3.16)

(3.17)

then every directed path in \( H(\mathcal{G}) \) from \( e_1 \) to \( e_4 \) does not contain any vertex (or element) in \( T_2 \).

**Proof.** Note that \( e_1 \succ e_4 \) because of Lemma 3.3. Let

\[
Q : e_1 \succ d_i \succ \cdots \succ d_k \succ e_4
\]  

(3.18)

be any directed path in \( H(\mathcal{G}) \) from \( e_1 \) to \( e_4 \) (\( k \geq 0 \)). Suppose that there were \( d_i \in T_2 \) on \( Q \). Then choose such \( d_i \in T_2 \) on \( Q \) with the smallest subscript \( i \). Now,

\[
e_1 \succ d_{i-1} \succ d_i \succ e_4,
\]  

(3.19)

where \( d_{i-1} \in T_1 \), and possibly \( d_{i-1} = e_1 \) (when \( i = 1 \)). If \( d_i = e_3 \), then \( e_1 \succ e_3 \succ e_2 \) since \( e_3 \succ e_2 \) follows from (3.16), (3.17) and Lemma 3.3. This contradicts \( e_1 \succ e_2 \). Therefore,

\[
d_i \neq e_3.
\]  

(3.20)

Since \( d_{i-1} \succ d_i \) and \( e_3 \succ e_4 \), we have from (3.20) and Lemma 3.3

\[
e_3 \succ d_i.
\]  

(3.21)

It follows from (3.19) and (3.21) that

\[
e_3 \succ d_i \succ e_4,
\]  

(3.22)

which contradicts \( e_3 \succ e_4 \). Therefore, there is no \( d_i \in T_2 \) which lies on \( Q \). \( \square \)
Lemma 3.6. Let $S = \{T_1, T_2\}$ be a good dissection of $\mathcal{D}$. If there exist distinct three elements $e_i \in E$ ($i = 1, 2, 3$) such that
\begin{align*}
e_1 \in T_1, & \quad e_2, e_3 \in T_2, \\
e_3 & >_1 e_1 >_1 e_2, \quad (3.23) \\
e_3 & >_1 e_1 >_1 e_2, \quad (3.24)
\end{align*}
then for any $a \in T_1$ and $b \in T_2$ with $a < b$ or $a >_1 b$ we have $a = e_1$.

Proof. Suppose, on the contrary, that there exist $a \in T_1$ and $b \in T_2$ such that $a < b$ and $a \neq e_1$. Since $b >_1 a$ and $e_3 >_1 e_1$, from Lemma 3.4 we have $b >_1 e_1$. (Note that this is valid even if $b = e_3$.) Then, $b \neq e_2$ since $e_1 >_1 e_2$. Now, for distinct four elements $a, b, e_1, e_2$ we have $b >_1 a, e_1 >_1 e_2, a, e_1 \in T_1$ and $b, e_2 \in T_2$ and there is a directed path $b >_1 e_1 >_1 e_2$ with $e_1 \in T_1$, which contradicts Lemma 3.5. The case when $a >_1 b$ is the dual of the above case. Therefore, in both cases we have $a = e_1$. $\square$

Theorem 3.7. A dissection $S = \{T_1, T_2\}$ is a good dissection of $\mathcal{D}$ if and only if one of the following three holds:

(i) There exists no arc in $H(\mathcal{P})$ which connects $T_1$ with $T_2$.

(ii) The set of arcs in $H(\mathcal{P})$ which connect $T_1$ with $T_2$ forms a star $H$, such that $H_1$ contains a directed path $e_1 >_1 e_0 >_1 e_2$ of length 2 for some vertices $e_1, e_2$ and the center $e_0$ of the star $H$.

(iii) There exist disjoint four subsets $F_{11}, F_{12} \subseteq T_1$ and $F_{21}, F_{22} \subseteq T_2$ (some of them are possibly empty) such that
\begin{enumerate}
\item for every $e_1 \in F_{11}$ (or $F_{21}$) and $e_2 \in F_{12}$ (or $F_{22}$), $e_1 >_1 e_2$,
\item for every $e_1 \in F_{11}$ (or $F_{21}$) and $e_2 \in F_{22}$ (or $F_{12}$), $e_1 >_1 e_2$,
\item the set of all the arcs in $H(\mathcal{P})$ which connect $T_1$ with $T_2$ is given by
\begin{align*}
\{(e_1, e_2) \mid e_1 & \in F_{11}, e_2 \in F_{22}) \cup \{(e_1, e_2) \mid e_1 \in F_{21}, e_2 \in F_{12}\}.
\end{align*}
\end{enumerate}

Proof. The present theorem follows from Lemmas 3.3  3.6. $\square$

Note that in (iii) of Theorem 3.7 if, for example, $F_{12} = F_{21} = \emptyset$, $F_{22} \neq \emptyset$ and $|F_{11}| = 1$, the set of arcs which connect $T_1$ with $T_2$ forms a star which is not of the type in (ii).

From Theorem 3.7 we have

Theorem 3.8. Let $P = \{E_1, E_2, \ldots, E_k\}$ be a partition of $E$ and $S = \{T_1, T_2\}$ be a dissection of $E$ such that $S$ is compatible with $\Pi(P)$ (which is a partition of $E$ induced by $\mathcal{B}(P)$). Then $S$ is a split of $\mathcal{B}(P)$ if and only if
\begin{align*}
|T_i| & \geq 2, \quad i = 1, 2 \\
(|T_i| & >_p >_1), \quad \forall i \neq 1 \text{ and } T_i \text{ by } T_i[\Pi(P)] (i = 1, 2).
\end{align*}

and one of (i)-(iii) in Theorem 3.7 holds, where $H(\mathcal{P})$ is replaced by $H(\mathcal{P}(P))$, $>_p$ by $>_p$, $>_1$ by $>_1$ and $T_i$ by $T_i[\Pi(P)]$.
Next, we give a characterization of splits $S = \{T_1, T_2\}$ of $\mathcal{D}(P)$ which are not compatible with $\Pi(P)$.

**Lemma 3.9.** Let $S = \{T_1, T_2\}$ be a good dissection of $\mathcal{D}(P)$ which is not compatible with $\Pi(P)$. Then there exists one and only one $F_0 \in \Pi(P)$ such that

$$F_0 \cap T_i \neq \emptyset, \quad i = 1, 2. \quad (3.27)$$

**Proof.** If there exist two distinct $F_1, F_2 \in \Pi(P)$ such that

$$F_i \cap T_j \neq \emptyset, \quad i, j = 1, 2, \quad (3.28)$$

then, since there exists $X \in \mathcal{D}(P)$ which separates $F_1$ from $F_2$, $S = \{T_1, T_2\}$ cannot be a good dissection of $\mathcal{D}(P)$. $\square$

**Theorem 3.10.** Let $S = \{T_1, T_2\}$ be a dissection of $E$ which is compatible with $P$ but not with $\Pi(P)$. Then $S$ is a good dissection of $\mathcal{D}(P)$ if and only if there exists one and only one $F_0 \in \Pi(P)$ such that $F_0 \cap T_i \neq \emptyset$ ($i = 1, 2$) and either $F_0$ is a cut vertex of $H(\mathcal{D}(P))$ or one of $T_i[\Pi(P)]$ ($i = 1, 2$) is empty.

**Proof.** “If” part: Trivial.

“Only if” part: Suppose $S$ is a good dissection of $\mathcal{D}(P)$. From Lemma 3.9 there exists one and only one $F_0 \in \Pi(P)$ such that $F_0 \cap T_i \neq \emptyset$ ($i = 1, 2$). Suppose, on the contrary, that there is an arc $(F_1, F_2)$ in $H(\mathcal{D}(P))$ which connects $T_1[\Pi(P)]$ and $T_2[\Pi(P)]$. Since $F_i \neq F_0$ ($i = 1, 2$) and $F_1 \not\succ F_2$, there exists $X \in \mathcal{D}(P)$ which separates $F_0$ from $F_1 \cup F_2$, which leads to a contradiction to the fact that $S = \{T_1, T_2\}$ is a good dissection of $\mathcal{D}(P)$. $\square$

It should be noted that even if $H(\mathcal{D})$ does not contain cut vertices, some $H(\mathcal{D}(P))$ obtained by repeated simple split decompositions may contain cut vertices. Also, note that $\Pi(P) \neq P$ in general, which is explained by the following lemma.

**Lemma 3.11.** Suppose that $\mathcal{D}$ with corresponding poset $\mathcal{P} = (E, \preceq)$ is connected. Let $S = \{T_1, T_2\}$ be a split of $\mathcal{D}$, put

$$P_i = \{e \mid e \in E - T_i \cup \{T_i\}, \quad i = 1, 2 \quad (3.29)$$

and let $\mathcal{P}_i = (\Pi(P_i), \preceq_i)$ be posets corresponding to $\mathcal{D}(P_i)$ ($i = 1, 2$). Then,

$$\Pi(P_i) = P_i, \quad i = 1, 2 \quad (3.30)$$

if and only if the set of arcs in $H(\mathcal{D})$ which connect $T_1$ with $T_2$ does not form a star $H_1$ such that $H_1$ contains a directed path of length 2.

**Proof.** “If” part: Suppose that the set of arcs in $H(\mathcal{D})$ which connect $T_1$ with $T_2$ does not form a star $H_1$ such that $H_1$ contains a directed path of length 2. For an
arbitrary \( e \in T_2 \) let \( D(e) \) be the unique minimal element in \( \mathcal{D} \) which contains \( e \). If \( D(e) \cap T_1 = \emptyset \), then \( \{e\} \in \Pi(P_1) \) since \( D(e), D(e) - \{e\} \in \mathcal{D}(P_1) \) (cf. Proposition 2.1). If \( D(e) \cap T_1 \neq \emptyset \), then \( D(e) \cup T_1 \in \mathcal{D}(P_1) \). For if \( D(e) \cup T_1 \notin \mathcal{D}(P_1) \), there must be \( e_1 \in T_1 - D(e) \) and \( e_2 \in T_1 - D(e) \) such that \( e_1 \triangleright e_2 \). Since \( D(e) \cap T_1 \neq \emptyset \) and \( e_2 \notin T_1 \), there are \( e_3 \in D(e) - T_1 \) and \( e_4 \in D(e) \cap T_1 \) such that \( e_3 \triangleright e_4 \). Therefore, from Lemma 3.3 we must have \( e_3 > e_2 \) and thus \( e > e_2 \) since \( e \triangleright e_3 \), which contradicts \( e_2 \notin D(e) \). Now, \( D(e) \cup T_1 \in \mathcal{D}(P_1) \). If there exist \( e_1 \in T_1 - D(e) \) such that \( e_1 \triangleright e \), then because of Theorem 3.7 the set of arcs in \( H(\mathcal{P}) \) which connect \( T_1 \) with \( T_2 \) must form a star containing a directed path of length 2, which contradicts the assumption. Therefore, for any \( e_1 \in T_1 - D(e) \) \( e_1 \triangleright e \). Consequently, \( (D(e) - \{e\}) \cup T_1 \in \mathcal{D}(P_1) \), so that \( \{e\} \in \Pi(P_1) \). It follows that \( \Pi(P_1) = P_1 \). By the symmetry, \( \Pi(P_2) = P_2 \).

“Only if” part: Suppose \( \Pi(P_i) = P_i \) (\( i = 1, 2 \)). If the set of arcs in \( H(\mathcal{P}) \) which connect \( T_1 \) with \( T_2 \) forms a star with center \( e_0 \in T_2 \) such that for distinct two elements \( e_1, e_2 \in T_1 \) \( e_1 \triangleright e_2 \), then there exists no \( X \in \mathcal{D}(P_1) \) which separates \( \{e_0\} \) from \( T_1 \). Therefore, \( T_1 \notin \Pi(P_1) \), which is a contradiction. \( \square \)

4. Decomposition of distributive lattices

Given a distributive lattice \( \mathcal{D} \) with corresponding poset \( \mathcal{P} = (E, \leq) \), if \( \mathcal{D} \) is not connected, then \( \mathcal{D} \) can be decomposed based on the connected components of \( H(\mathcal{P}) \). This kind of decomposition is also a special case of a split decomposition and \( \mathcal{D} \) is a direct product of distributive lattices corresponding to the connected components of \( H(\mathcal{P}) \).

In this section we show that when \( \mathcal{D} \) is connected, decompositions of \( \mathcal{D} \) by splits satisfy the requirements for the decomposition theory developed by Cunningham and Edmonds \([3, 5]\).

The next lemma does not, however, require the connectedness of \( \mathcal{D} \).

**Lemma 4.1.** Suppose that \( S^{(1)} = \{T_1^{(1)}, T_2^{(1)}\} \) is a split of \( \mathcal{D}(P) \) and \( S^{(2)} = \{T_1^{(2)}, T_2^{(2)}\} \) is a dissection of \( E \) such that \( S^{(2)} \) is compatible with \( P \) and \( T_1^{(2)} \subseteq T_1^{(1)} \). Then \( S^{(2)} \) is a split of \( \mathcal{D}(P) \) if and only if \( S^{(2)} \) is a split of \( \mathcal{D}(P) \parallel T_2^{(1)} \).

**Proof.** “If” part: Suppose \( S^{(2)} \) is a split of \( \mathcal{D}(P) \parallel T_2^{(1)} \). Then, for any \( X \in \mathcal{D}(P) \) we have one of the following four cases:

(i) \( X \cap T_1^{(1)}, X \cap T_2^{(1)}, (X \cap T_1^{(1)}) \cap T_2^{(2)} = X \cap T_2^{(2)}), (X \cap T_1^{(1)}) \cap T_2^{(2)} \in \mathcal{D}(P), \)

(ii) \( X \cap T_1^{(1)}, X \cap T_2^{(1)}, (X \cap T_1^{(1)}) \cup T_2^{(2)} = X \cup T_2^{(2)}), (X \cap T_1^{(1)}) \cup T_2^{(2)} \in \mathcal{D}(P), \)

(iii) \( X \cup T_1^{(1)}, X \cup T_2^{(1)}, (X \cup T_1^{(1)}) \cap T_2^{(2)} = X \cap T_2^{(2)}), (X \cup T_2^{(1)}) \cap T_2^{(2)} \in \mathcal{D}(P), \)

(iv) \( X \cup T_1^{(1)}, X \cup T_2^{(1)}, (X \cup T_1^{(1)}) \cup T_2^{(2)} = X \cup T_2^{(2)}), (X \cup T_2^{(1)}) \cup T_2^{(2)} \in \mathcal{D}(P). \)
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Case (i): $X \cap T_2^{(2)} = (X \cap T_2^{(1)}) \cup ((X \cap T_1^{(1)}) \cap T_2^{(2)}) \in \mathcal{Q}(P)$.

Case (ii): $X \cup T_2^{(2)} = (X \cup T_2^{(1)}) \cup ((X \cup T_1^{(1)}) \cup T_2^{(2)}) \in \mathcal{Q}(P)$.

Case (iii): $X \cap T_2^{(2)} = (X \cup T_1^{(1)}) \cap ((X \cup T_1^{(1)}) \cap T_2^{(2)}) \in \mathcal{Q}(P)$.

Case (iv): $X \cup T_1^{(1)} = (X \cup T_1^{(1)}) \cup ((X \cup T_1^{(1)}) \cup T_2^{(2)}) \in \mathcal{Q}(P)$.

Therefore, $S^{(2)} = \{T_1^{(2)}, T_2^{(2)}\}$ is a good dissection of $\mathcal{Q}(P)$. Because $S^{(2)}$ is a split of $\mathcal{Q}(P) \parallel T_2^{(1)}$ and we also have

$$T_1^{(2)}[P] = T_1^{(1)}[P] \cup T_2^{(1)}[P],$$

$$|T_2^{(2)}[P]| > |T_2^{(1)}[P] \cup T_2^{(1)}[P]|,$$

$S^{(2)}$ is a split of $\mathcal{Q}(P)$.

"Only if" part: Since $|T_i^{(2)}[P] \cup T_j^{(1)}[P]| \geq 2 (i = 1, 2)$, if $S^{(2)} = \{T_1^{(2)}, T_2^{(2)}\}$ is a split of $\mathcal{Q}(P)$, then $S^{(2)}$ is a split of $\mathcal{Q}(P) \parallel T_2^{(1)} = \mathcal{Q}(P) \parallel T_2^{(1)}$ by the definition of a split. □

Let $\mathcal{D}$ be the family of all the distributive lattices $\mathcal{Q}(P)$ obtained from $\mathcal{Q}$ by partitions $P$ of $E$. For $\mathcal{Q}(P) \in \mathcal{D}$, we write

$$\mathcal{Q}(P) \rightarrow \{\mathcal{Q}(P_1), \mathcal{Q}(P_2)\}$$

if there exists a split $S = \{T_1, T_2\}$ of $\mathcal{Q}(P)$ and $P_i = P \parallel T_i (i = 1, 2)$. Also define

$$\phi(\mathcal{Q}(P)) = P$$

for each $\mathcal{Q}(P) \in \mathcal{D}$. (Note that $\mathcal{D}$ is a family, but not a set.)

For $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{D}$, we say $\mathcal{D}_2$ is a simple refinement of $\mathcal{D}_1$ if there exists $\mathcal{Q}(P) \in \mathcal{D}_1$ such that $\mathcal{Q}(P) \rightarrow \{\mathcal{Q}(P_1), \mathcal{Q}(P_2)\}$ and $\mathcal{D}_2 = (\mathcal{D}_1 - \{\mathcal{Q}(P)\}) \cup \{\mathcal{Q}(P_1), \mathcal{Q}(P_2)\}$. For $\mathcal{D}', \mathcal{D}'' \subseteq \mathcal{D}$, $\mathcal{D}''$ is called a refinement of $\mathcal{D}'$ if there exists a sequence of $\mathcal{D}_i \subseteq \mathcal{D}$ ($i = 1, 2, \ldots, k$) ($k > 1$) such that $\mathcal{D}_1 = \mathcal{D}'$, $\mathcal{D}_k = \mathcal{D}''$ and for each $i = 1, 2, \ldots, k - 1$, $\mathcal{D}_{i+1}$ is a simple refinement of $\mathcal{D}_i$. $\mathcal{D}''$ is called a decomposition of $\mathcal{D}$ if $\mathcal{D}''$ is a refinement of $\{\mathcal{D}\}$ or $\mathcal{D}'' = \{\mathcal{D}\}$. A decomposition $\mathcal{D}_1$ of $\mathcal{D}$ is said to be minimal with property $\alpha$ if there exists no decomposition $\mathcal{D}_2$ of $\mathcal{D}$ with property $\alpha$ such that $\mathcal{D}_1$ is a refinement of $\mathcal{D}_2$.

Now, we can easily see that $(\mathcal{D}, \phi, \rightarrow)$ satisfies the requirements for the ‘decomposition frame’ [3, 5], i.e.,

F1. If $\mathcal{Q}(P) \in \mathcal{D}$ and $\mathcal{Q}(P) \rightarrow \{\mathcal{Q}(P_1), \mathcal{Q}(P_2)\}$, then for some dissection $S = \{T_1, T_2\}$ of $E$ such that $S$ is compatible with $P$ and $|T_i[P]| \geq 2 (i = 1, 2)$ we have

$$\phi(\mathcal{Q}(P_i)) = (P - T_i[P]) \cup T_i, \quad i = 1, 2.$$  (4.5)

F2. For a split $S = \{T_1, T_2\}$ of $\mathcal{Q}(P) \in \mathcal{D}$ there exists one and only one pair of $\mathcal{Q}(P) \in \mathcal{D}$ ($i = 1, 2$) such that $\mathcal{Q}(P_i) = \mathcal{Q}(P) \parallel T_i (i = 1, 2)$.

F3. Let $S^{(1)} = \{T_1^{(1)}, T_2^{(1)}\}$ be a split of $\mathcal{Q}(P) \in \mathcal{D}$ and $S^{(2)} = \{T_1^{(2)}, T_2^{(2)}\}$ be a dissection of $E$ such that $T_1^{(2)} \neq T_1^{(1)}$. Then $S^{(2)}$ is a split of $\mathcal{Q}(P)$ if and only if $S^{(2)}$ is a split of $\mathcal{Q}(P) \parallel T_1^{(1)}$.

F4. Let $S^{(i)} = \{T_1^{(i)}, T_2^{(i)}\} (i = 1, 2)$ be splits of $\mathcal{Q}(P) \in \mathcal{D}$ such that $T_1^{(1)} \neq T_1^{(2)}$. 
Then,
\[
\begin{align*}
\mathcal{D}(P) \parallel T_2^{(2)} & = \mathcal{D}(P) \parallel T_2^{(2)}, \\
\mathcal{D}(P) \parallel T_3^{(1)} & = \mathcal{D}(P) \parallel T_3^{(2)}.
\end{align*}
\]  
(4.6)  
(4.7)

Note that F1 and F2 are tautologies by definition, F3 follows from Lemma 4.1 and F4 easily follows from the definition of an aggregation.

Next, we show that the decomposition frame \((\mathcal{D}, \phi, \rightarrow)\) satisfies the intersection property, i.e.,

\(\text{(IP)}\) 
For any two splits \(S^{(i)} = \{T_i^{(1)}, T_i^{(2)}\} (i = 1, 2)\) of \(\mathcal{D}(P)\) such that 
\[|T_i^{(1)} \cap T_i^{(2)}| > 2\text{ and } T_i^{(1)} \cup T_i^{(2)} \neq P,\]
\(S^{(3)} = \{T_1^{(1)} \cap T_1^{(2)}, T_2^{(1)} \cup T_2^{(2)}\}\) is a split of \(\mathcal{D}(P)\).

First, we show (IP) for two splits \(S^{(i)} = \{T_i^{(1)}, T_i^{(2)}\} (i = 1, 2)\) which are compatible with \(P\), the partition of \(E\) induced by \(\mathcal{D}(P)\). Therefore, we first show (IP) for the case when \(P = \{\{e\} | e \in E\}\). Note that the following theorem requires the connectedness of \(\mathcal{D}\).

**Theorem 4.2.** Let \(S^{(i)} = \{T_i^{(1)}, T_i^{(2)}\} (i = 1, 2)\) be splits of \(\mathcal{D}\). If \(|T_i^{(1)} \cap T_i^{(2)}| > 2\) and \(T_i^{(1)} \cup T_i^{(2)} \neq E\), then \(S^{(3)} = \{T_1^{(1)} \cap T_1^{(2)}, T_2^{(1)} \cup T_2^{(2)}\}\) is a split of \(\mathcal{D}\).

**Proof.** We can assume
\[T_1^{(1)} \cap T_2^{(2)} \neq \emptyset, \quad T_2^{(1)} \cap T_1^{(2)} \neq \emptyset.\]  
(4.9)

Let \(\mathcal{D} = (E, \prec)\) be the poset which corresponds to \(\mathcal{D}\). It is sufficient to prove that for any distinct four elements \(e_1, e'_1 \in T_1^{(1)} \cap T_1^{(2)}, e_2 \in T_2^{(1)} \cap T_2^{(2)}, e_3 \in T_1^{(1)} \cap T_2^{(2)}\) with \(e_1 \succ e_2\) or \(e_1 \prec e_2\) and \(e'_1 \succ e_2\) or \(e'_1 \prec e_2\), the requirement that \(\{T_1^{(1)} \cap T_1^{(2)}, T_2^{(1)} \cup T_2^{(2)}\}\) is a good dissection is satisfied (see Lemma 3.3). We shall use Lemmas 3.3 and 3.4 repeatedly, which will, however, not explicitly be mentioned below. Note that, because \(\mathcal{D}\) is connected, there exist \(e_5 \in T_2^{(1)} \cap T_2^{(2)}\) and \(e'' \in T_1^{(1)} \cup T_1^{(2)}\) such that \(e_4 \gg e''\) or \(e_4 \ll e''\).

Case 1: \(e_1 \ll e_2, e_1 \ll e_3\).

Case 1(i): For some \(e_2 \in T_1^{(2)} \cap T_1^{(2)}\) and \(e_4 \in T_2^{(1)} \cap T_2^{(2)}\), \(e_2 \ll e_4\).

Since \(e_3 \gg e_1\) and \(e_4 \gg e_2\), we have
\[e_4 \gg e'_1.\]  
(4.10)

Then, since \(e_2 \gg e_1\), from (4.10)
\[e_2 \gg e'_1, \quad e_4 \gg e_1.\]  
(4.11)

Furthermore, since \(e_3 \gg e'_1\) and \(e_4 \gg e_1\) from (4.11), we have \(e_3 \gg e_1\). Therefore, \(e_2 \gg e'_1\) and \(e_3 \gg e_1\).

Case 1(ii): For some \(e_2 \in T_1^{(2)} \cap T_1^{(2)}\) and \(e_4 \in T_1^{(1)} \cap T_2^{(2)}\), \(e'_2 \gg e_4\).

Since \(e_3 \gg e'_1\) and \(e'_2 \gg e_4\), because of Lemma 3.5 there exist \(x \in T_1^{(1)} \cap T_2^{(2)}\) and
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$y \in T_2^{(1)} \cap T_2^{(2)}$ such that

$$e_3 \succ x \succ y \succ e_4.$$  (4.12)

Since $x \succ y$ and $e_2 \succ e_1$,

$$x \succ e_1.$$  (4.13)

From (4.12) and (4.13) we have

$$e_3 \succ e_1.$$  (4.14)

On the other hand, regarding above $x$ and $y$ as $e'_2$ and $e_4$, respectively, by the symmetry we have

$$e_2 \succ e'_1.$$  (4.15)

Case 1(iii): For some $e''_1 \in T_1^{(1)} \cap T_1^{(2)}$ and $e_4 \in T_2^{(1)} \cap T_2^{(2)}$, $e''_1 \succ e_4$.

Since $e_2 \succ e_1$ and $e_4 \succ e''_1$,

$$e_4 \succ e_1.$$  (4.16)

Then, since $e_3 \succ e'_1$ and $e_4 \succ e_1$ from (4.16),

$$e_3 \succ e_1.$$  (4.17)

By the symmetry we also have

$$e_2 \succ e'_1.$$  (4.18)

Case 1(iv): For some $e''_1 \in T_1^{(1)} \cap T_1^{(2)}$ and $e_4 \in T_2^{(1)} \cap T_2^{(2)}$, $e''_1 \succ e_4$.

We can assume $e_1 \neq e''_1$. Then, since $e_2 \succ e_1$ and $e''_1 \succ e_4$, there exist $x \in T_2^{(1)} \cap T_1^{(2)}$ and $y \in T_2^{(1)} \cap T_2^{(2)}$ such that

$$e_2 \succ x \succ y \succ e_4.$$  (4.19)

Therefore, the present case is reduced to Case 1(ii).

Case 2: $e_1 \prec e_2$, $e'_1 \prec e_3$.

Case 2(i): For some $e'_2 \in T_2^{(1)} \cap T_1^{(2)}$ and $e_4 \in T_2^{(1)} \cap T_2^{(2)}$, $e'_2 \prec e_4$.

Since $e'_1 \succ e_3$ and $e_4 \succ e'_2$, there exist $x \in T_2^{(1)} \cap T_2^{(2)}$ and $y \in T_1^{(1)} \cap T_2^{(2)}$ such that

$$e_4 \succ x \succ y \succ e_3.$$  (4.20)

Then, since $x \succ y$ and $e_2 \succ e_1$,

$$x \succ e_1, \quad e_2 \succ y.$$  (4.21)

Therefore, from (4.20) and (4.21),

$$e_2 \succ e_3.$$  (4.22)

Moreover, since $x \succ e_1$ and $e'_1 \succ e_3$, we have

$$e'_1 \succ e_1.$$  (4.23)

Case 2(ii): For some $e'_2 \in T_2^{(1)} \cap T_1^{(2)}$ and $e_4 \in T_2^{(1)} \cap T_2^{(2)}$, $e'_2 \succ e_4$. 


Since \( e'_1 \succ e_1 \) and \( e'_2 \succ e_4 \),
\[
e'_1 \succ e_4, \quad e'_2 \succ e_3.
\] (4.24)

Then, since \( e_2 \succ e_1 \) and \( e'_2 \succ e_3 \) from (4.24),
\[
e_2 \succ e_3.
\] (4.25)

Moreover, since \( e'_1 \succ e_4 \) from (4.24) and \( e_2 \succ e_1 \),
\[
e'_1 \succ e_1.
\] (4.26)

**Case 2(ii):** For some \( e'_3 \in T_1^{(1)} \cap T_2^{(2)} \) and \( e_4 \in T_1^{(1)} \cap T_2^{(2)} \), \( e'_1 \succ e_4 \).

Since \( e'_1 \succ e_3 \) and \( e'_1 \succ e_4 \),
\[
e'_1 \succ e_4, \quad e'_1 \succ e_3.
\] (4.27)

Consequently, because of Theorem 3.7 we have
\[
e_1 \neq e'_1.
\] (4.28)

Therefore, since \( e_2 \succ e_1 \) and \( e'_1 \succ e_4 \), there exist \( x \in T_2^{(1)} \cap T_1^{(2)} \) and \( y \in T_2^{(1)} \cap T_2^{(2)} \) such that
\[
e_2 \succ x \succ y \succ e_4.
\] (4.29)

Therefore, the present case is reduced to Case 2(ii).

Note that \( S = \{T_1, T_2\} \) is a split of \( \mathcal{D} \) if and only if it is a split of the dual \( \mathcal{D} = \{X - X \mid X \in \mathcal{D}\} \) of \( \mathcal{D} \). Therefore, the above cases are exactly those which we must examine.

Now, we show (IP) in (4.8) for splits \( S^{(1)} = \{T_1^{(1)}, T_2^{(1)}\} \) (\( i = 1, 2 \)) of \( \mathcal{D}(P) \) such that \( S^{(1)} \) is not compatible with \( \Pi(P) \). We assume \( \mathcal{D}(P) \) is connected.

**Lemma 4.3.** Let \( S^{(1)} = \{T_1^{(1)}, T_2^{(1)}\} \) (\( i = 1, 2 \)) be splits of \( \mathcal{D}(P) \) such that \( S^{(1)} \) is not compatible with \( \Pi(P) \) and
\[
T_1^{(1)} \cap T_2^{(1)} \neq \emptyset, \quad i, j = 1, 2.
\] (4.30)

Then there exists no arc \((F_1, F_2)\) in \( H(\mathcal{D}(P)) \) which connects any two from among the following four:
\[
T_1^{(1)}[\Pi(P)] \cap T_2^{(2)}[\Pi(P)], \quad i, j = 1, 2.
\] (4.31)

**Proof.** Since \( S^{(1)} \) is not compatible with \( \Pi(P) \), from Theorem 3.10 there exists one and only one \( F_i \in \Pi(P) \) such that \( F_i \cap T_1^{(1)} \neq \emptyset \) (\( i = 1, 2 \)) and there is no arc in \( H(\mathcal{D}(P)) \) which connects \( T_1^{(1)}[\Pi(P)] \) with \( T_2^{(2)}[\Pi(P)] \). If \( S^{(2)} \) is not compatible with \( \Pi(P) \), let \( F_2 \in \Pi(P) \) satisfy \( F_2 \cap T_2^{(2)} \neq \emptyset \) (\( j = 1, 2 \)) and there is no arc in \( H(\mathcal{D}(P)) \) which connects \( T_1^{(2)}[\Pi(P)] \) with \( T_2^{(2)}[\Pi(P)] \). Then it follows from connectedness of \( \mathcal{D}(P) \), (4.30) and Theorem 3.10 that \( F_1 = F_2 \). Consequently, the present lemma holds.
Now, suppose $S^{(2)}$ is compatible with $\Pi(P)$. Also suppose
\begin{equation}
F_i \subseteq T_i^{(2)}. \tag{4.32}
\end{equation}

Let $F_i$ ($i = 3 \sim 6$) be any elements in $\Pi(P)$ such that
\begin{align}
F_3 & \in (T_1^{(3)}[\Pi(P)] \cap T_1^{(2)}[\Pi(P)]) \cup \{F_1\}, \\
F_4 & \in (T_2^{(3)}[\Pi(P)] \cap T_1^{(2)}[\Pi(P)]) \cup \{F_1\}, \\
F_5 & \in T_1^{(3)}[\Pi(P)] \cap T_2^{(2)}[\Pi(P)], \\
F_6 & \in T_2^{(3)}[\Pi(P)] \cap T_2^{(2)}[\Pi(P)], \\
F_3 \uparrow & F_5 \text{ or } F_3 \downarrow F_5, \\
F_4 \uparrow & F_6 \text{ or } F_4 \downarrow F_6. \tag{4.37}
\end{align}

Then from Lemmas 3.3 ~ 3.5 and Theorem 3.10 we must have
\begin{equation}
F_3 = F_4 = F_1. \tag{4.39}
\end{equation}

Therefore, the lemma holds. \( \square \)

**Theorem 4.4.** Let $S^{(i)} = \{T_i^{(1)}, T_i^{(2)}\}$ ($i = 1, 2$) be splits of $\mathfrak{B}(P)$ such that $S^{(1)}$ is not compatible with $\Pi(P)$. If $|T_i^{(1)}[P] \cap T_i^{(2)}[P]| \geq 2$ and $T_i^{(1)}[P] \cup T_i^{(2)}[P] \neq P$, then $S^{(2)} = \{T_1^{(1)} \cap T_1^{(2)}, T_2^{(1)} \cup T_2^{(2)}\}$ is a split of $\mathfrak{B}(P)$.

**Proof.** The present theorem follows from Theorems 3.8 and 3.10 and Lemma 4.3. \( \square \)

$\mathfrak{B}(P) \in \mathfrak{D}$ is called **brittle** if $|P| \geq 4$ and every dissection $S = \{T_1, T_2\}$ of $E$ such that $S$ is compatible with $P$ and $|T_i[P]| \geq 2$, ($i = 1, 2$) is a split of $\mathfrak{B}(P)$. Also, $\mathfrak{B}(P) \in \mathfrak{D}$ is called **semi-brittle** if elements of $P$ can be indexed in such a way that $P = \{E_1, E_2, \ldots, E_k\}$ with $k \geq 4$ and the splits of $\mathfrak{B}(P)$ are exactly those dissections $S = \{T_1, T_2\}$ of $E$ which are compatible with $P$ and satisfy $|T_i[P]| \geq 2$ ($i = 1, 2$) and for which $T_1$ is given by
\begin{equation}
T_1 = \bigcup_{j=0}^{l} E_{i+j}, \tag{4.40}
\end{equation}
for some $l, m$ with $1 \leq l < l + m \leq k$. $\mathfrak{B}(P) \in \mathfrak{D}$ is called **prime** if there exists no split of $\mathfrak{B}(P)$.

**Lemma 4.5.** There exist no semi-brittle $\mathfrak{B}(P) \in \mathfrak{D}$.

**Proof.** Suppose, on the contrary, that $\mathfrak{B}(P) \in \mathfrak{D}$ is semi-brittle. Also suppose that there is $F \in \Pi(P)$ such that for distinct $E_1, E_2 \in P$ $E_1 \cup E_2 \subseteq F$. Then for any split $S = \{T_1, T_2\}$ such that $E_1 \subseteq T_1$ and $E_2 \subseteq T_2$, $S' = \{(T_1 - E_1) \cup E_2, (T_2 - E_2) \cup E_1\}$ must be a split of $\mathfrak{B}(P)$ due to Theorem 3.10. Since $|P| = 4$, this contradicts the assumption that $\mathfrak{B}(P)$ is semi-brittle. Consequently, $\Pi(P) = P$. If there exists a
semi-brittle $\mathcal{D}(P) \in \mathbb{D}$ with $\Pi(P) = P$, there exists a semi-brittle $\mathcal{D}(P') \in \mathbb{D}$ with $\Pi(P') = P'$ and $|P'| = 4$. So, suppose $\mathcal{D}$ is semi-brittle, $E = \{e_1, e_2, e_3, e_4\}$, $\{(e_1, e_2), (e_3, e_4)\}$ and $\{(e_2, e_3), (e_4, e_1)\}$ are splits of $\mathcal{D}$, and $\{(e_1, e_3), (e_2, e_3)\}$ is not a split of $\mathcal{D}$. Let $\mathcal{P} = (E, \prec)$ be the poset corresponding to $\mathcal{D}$. Since $\{(e_1, e_3), (e_2, e_4)\}$ is not a split, there are at least two arcs connecting $\{e_1, e_2\}$ with $\{e_3, e_4\}$ in $H(\mathcal{P})$. From Lemma 3.6 and by the symmetry we only consider the following two cases:

Case 1: $e_1 \succ e_2$ and $e_4 \succ e_3$. 

Case 2: $e_1 \succ e_2$ and $e_4 \prec e_3$.

For Case 1, since $\{(e_2, e_3), (e_1, e_4)\}$ is a split, we have $e_1 \succ e_3$ and $e_4 \succ e_2$. Then $\{(e_1, e_3), (e_2, e_4)\}$ is a split, which contradicts the assumption. For Case 2, since $\{(e_2, e_3), (e_1, e_4)\}$ is a split, we have $e_1 \succ e_4$ and $e_3 \succ e_2$. Then $\{(e_1, e_3), (e_2, e_4)\}$ is a split, which is a contradiction. $\square$

Lemma 4.6. Suppose $\mathcal{D}(P) \in \mathbb{D}$ with $|P| \geq 4$. Then $\mathcal{D}(P)$ is brittle if and only if

(i) $H(\mathcal{P}(P))$ is a star such that there is at most one vertex $F_0$ in $H(\mathcal{P}(P))$ with $|F_0| \geq 2$ and if such $F_0$ exists, it is the center of the star, or

(ii) $\Pi(P) = P$ and $H(\mathcal{P}(P))$ is a complete bipartite graph $(\hat{F}_1, \hat{F}_2; \hat{B})$ with end-vertex sets $\hat{F}_1$ and $\hat{F}_2$ and an arc set $\hat{B}$ given by

$$\hat{B} = \{(F_1, F_2) \mid F_1 \in \hat{F}_1, F_2 \in \hat{F}_2\}. \quad (4.41)$$

Proof. "If" part: Easy. (Cf. Lemmas 3.3 and 3.4.)

"Only if" part: Suppose that $\mathcal{D}(P)$ is brittle and that $H(\mathcal{P}(P))$ contains a vertex $F_0$ such that $|F_0| \geq 2$. Since $\mathcal{D}(P)$ is brittle, there is no vertex $F \neq F_0$ in $H(\mathcal{P}(P))$ such that $|F| \geq 2$. Let $E_1$ and $E_2$ be distinct two elements of $F_0$. Since $|P| \geq 4$ and for any $E_0 \in P - F_0$, $\{(E_0, E_1), P - \{E_0, E_1\}\}$ is a split of $\mathcal{D}(P)$ and is not compatible with $\Pi(P)$, there is no arc in $H(\mathcal{P}(P))$ which connects $E_0$ with any element in $P - F_0$. Since $E_0 \in P - F_0$ is arbitrarily chosen, $H(\mathcal{P}(P))$ must be a star such that there is only one vertex $F_0$ with $|F_0| \geq 2$ and $F_0$ is the center of $H(\mathcal{P}(P))$.

Now, suppose that $\mathcal{D}(P)$ is brittle and $\Pi(P) = P$, i.e., for each vertex $F$ of $H(\mathcal{P}(P))$, $F \in P$. Without loss of generality we can assume $P = \{e_1 \mid e \in E\}$ and $\mathcal{D}$ corresponds to a poset $\mathcal{P} = (E, \prec)$. If $H(\mathcal{P})$ contains a directed path of length 3:

$$e_1 \succ e_2 \succ e_3 \succ e_4, \quad (4.42)$$

then any dissection $S = (T_1, T_2)$ of $F$ which separates $\{e_1, e_4\}$ and $\{e_2, e_3\}$ is not a split of $\mathcal{D}$, since $e_1 \succ e_2$ and $e_3 \succ e_4$ but $e_2 \succ e_3$. Therefore, $H(\mathcal{P})$ does not contain any directed path of length 3. Suppose $H(\mathcal{P})$ contains a directed path of length 2:

$$e_1 \succ e_2 \succ e_3. \quad (4.43)$$

If there would exist $e_4, e_5 \not\in \{e_1, e_3\}$ such that $e_4 \succ e_1, e_3$ and $e_4 \succ e_5$ or $e_4 \prec e_5$, then any dissection $S = (T_1, T_2)$ with $e_2, e_4 \in T_1$ and $e_1, e_3, e_5 \in T_2$ is not a split due to Lemma 3.6. Therefore, $H(\mathcal{P})$ must be a star with center $e_2$. If $H(\mathcal{P})$ contains
directed paths of length less than 2 only, then $H(\mathcal{P})$ is a bipartite graph $(F_1, F_2; B)$ with end-vertex sets $F_1$ and $F_2$ and an arc set $B$ such that for every $e_1, e_2 \in E$ with $e_1 \Rightarrow e_2$ we have $e_1 \in F_1$ and $e_2 \in F_2$. Since $H(\mathcal{P})$ is connected, if $|F_1| = 1$ or $|F_2| = 1$, then $H(\mathcal{P})$ is a star (as well as a complete bipartite graph), and if $|F_i| \geq 2$ ($i = 1, 2$), then $H(\mathcal{P})$ is a complete bipartite graph since $\{F_1, F_2\}$ is a split of $\mathcal{P}$ and Lemma 3.4 applies. □

From Theorem 4.4 and Lemmas 4.5 and 4.6 we have the following theorem.

**Theorem 4.7.** Let $\mathcal{D}$ be a connected distributive lattice with corresponding poset $\mathcal{D} = (E, \prec)$. Then there exists a unique minimal decomposition $\mathcal{Q}$ of $\mathcal{D}$ such that each $\mathcal{D}(P) \in \mathcal{Q}$ is prime or brittle. Here, each brittle $\mathcal{D}(P) \in \mathcal{Q}$ satisfies (i) or (ii) in Lemma 4.6.

**Proof.** Apply the result in [3, 5] together with Theorem 4.4 and Lemmas 4.5 and 4.6 □

The decomposition structure of $\mathcal{Q}$ can be represented by a tree $T$ (called a decomposition tree) with a vertex set $Q$, where $\{\mathcal{D}(P_1), \mathcal{D}(P_2)\}$ is an edge (an undirected arc) in $T$ if and only if $\mathcal{D}(P_i) \in \mathcal{Q}$ ($i = 1, 2$) and for some $\mathcal{D}(P) \in $ $\mathcal{D}(P) \rightarrow \{\mathcal{D}(P_1), \mathcal{D}(P_2)\}$ (a simple split decomposition of $\mathcal{D}(P)$). We can also obtain a hierarchical structure of $\mathcal{Q}$ similarly as in [6].

5. **Reconstruction**

For a distributive lattice $\mathcal{D}$, let $\{\mathcal{D}(P_i) \mid i \in I\}$ be a decomposition of $\mathcal{D}$. We show a relationship between $\mathcal{Q}$ and $\mathcal{D}(P_i) (i \in I)$.

**Lemma 5.1.** Suppose $\{\mathcal{D}(P_i) \mid i \in I\}$ is a decomposition of a distributive lattice $\mathcal{D}$ with corresponding poset $\mathcal{D} = (E, \prec)$. For each $e \in E$ let $D_1(e)$ ($i \in I$) and $D(e)$, respectively, denote the unique minimal elements in $\mathcal{D}(P_i)$ ($i \in I$) and $\mathcal{D}$ which contain $e$. Then,

$$D(e) = \bigcap_{i \in I} D_i(e).$$

**Proof.** It suffices if we consider the case when $I = \{1, 2\}$. Suppose that $S = \{T_1, T_2\}$ is a split of $\mathcal{Q}$,

$$P_1 = \{e \mid e \in T_2 \cup \{T_1\},$$

$$P_2 = \{e \mid e \in T_1 \cup \{T_2\}$$

(5.2)
and \( e \in T_1 \). Then, since \( D(e) \in \mathcal{D} \), one of the following two holds:

\[
D(e) \cap T_1, \quad D(e) \cap T_2 \in \mathcal{D}, \quad (5.4)\\
D(e) \cup T_1, \quad D(e) \cup T_2 \in \mathcal{D}. \quad (5.5)
\]

If (5.4) holds, then, since \( e \in T_1 \) and \( D(e) \) is minimal,

\[
D(e) \subseteq T_1. \quad (5.6)
\]

Consequently, \( D_2(e) = D(e) \) and \( T_1 \subseteq D_1(e) \), and we have (5.1) with \( I = \{1, 2\} \). If (5.5) holds, then

\[
D(e) \subseteq D_1(e) \cap D_2(e) \subseteq (D(e) \cup T_1) \cap (D(e) \cup T_2) = D(e). \quad (5.7)
\]

Therefore, we have (5.1) with \( I = \{1, 2\} \). □

Lemma 5.1 gives a way of reconstructing \( \mathcal{D} \) from \( \mathcal{D}(P_i) \) \( (i \in I) \) since \( D(e) \) \( (e \in E) \) uniquely determines \( \mathcal{D} \). In fact, \( e' > e \) in \( P = (E, \prec) \) if and only if \( e' \in D(e) - \{e\} \).

**Example 1.** Consider a distributive lattice \( \mathcal{D} \) with corresponding poset \( \mathcal{P} = (E, \prec) \) given in Fig. 1. The unique minimal decomposition \( \mathcal{Q} \) shown in Theorem 4.7 is given in Fig. 2, where each \( \mathcal{D}(P_i) \in \mathcal{Q} \) is represented by the Hasse diagram \( H(\mathcal{P}_i) \) \( (i = 1 \sim 5) \). \( \mathcal{D}(P_i) \) is brittle (see \( H(\mathcal{P}_i) \) in Fig. 1).

It is easily seen from Fig. 2 that

\[
D_1(5) = \{3 \sim 12\}, \quad D_2(5) = \{1, 2, 3, 5, 6, 8 \sim 12\}, \\
D_3(5) = \{1, 7, 3, 5, 6, 9\}, \quad D_4(5) = \{1 \sim 12\}, \quad D_5(5) = \{1 \sim 9, 12\}
\]

and

\[
D(5) = \{3, 5, 6, 9\} = \bigcap_{i=1}^{5} D_i(5).
\]

Here, it should be noted that we do not need all \( D_i(5) \) \( (i = 1 \sim 5) \) to obtain \( D(5) \). In

![Fig. 1. H(\mathcal{P}) (broken lines denote splits of \mathcal{D}).](attachment:image.png)
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Fig. 2. A decomposition tree.

Fig. 3. $H(\Phi)$ (broken lines denote splits of $\Phi$).
fact, first find $H(P_1)$ which has vertex $\{5\}$. That is $H(P_3)$. Then we see that $D_3(5) \cup \{1, 2, 3\}, \{6, 9\}$. Next, find $H(P_1)$ and $H(P_6)$ in which there are vertices including complements of $\{1, 2, 3\}$ and $\{6, 9\}$. (Follow the broken lines in Fig. 2.) They are $H(P_1)$ and $H(P_6)$. Now, since $\{3\}, \{6\}$ and $\{9\}$ are singletons, we obtain $D(5) = D_3(5) \cap D_6(5) \cap D_4(5)$.

This process will give a refinement of Lemma 5.1 but we omit the detail.

**Example 2.** Consider another example shown in Fig. 3. The unique minimal decomposition is given in Fig. 4, where note that $P_2$ is a poset which corresponds to $\mathcal{D}(P_2)$ with $P_2 = \{\{1, 2, 3\}, \{4\}, \{5, 6, 7\}, \{8, 9, 10\}\}$ and that $P(P_2) = \{\{1\} \sim 4, 8, 9, 10\}, \{5, 6, 7\}$ but that we express the vertex $\{1 \sim 4, 8, 9, 10\}$ in $H(P_2)$ as a collection of $\{1, 2, 3\}, \{4\}, \{8, 9, 10\}$ to retain the underlying partition $P_2$. It should be noted that $P(P_2) \neq P_2$ since arcs which connect $\{1, 2, 3\}$ with $\{4, 10\}$ (and $\{1 \sim 7\}$ with $\{8, 9, 10\}$) form a star containing a directed path of length $2$, where $\{\{1, 2, 3\}, \{4 \sim 10\}\}$ and $\{\{1 \sim 7\}, \{8, 9, 10\}\}$ are splits of $\mathcal{D}$. (Cf. Lemma 3.11.)

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