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Analytic Solutions of a Functional Equation for Invariant Curves[1]

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An iterative functional equation is deduced by C. T. Ng and W. Zhang (1997, *J. Differ. Equations Appl.* **3**, 147–168) from the problem of invariant curves. In this paper, its analytic solutions are discussed by locally reducing the equation to another functional equation without iteration and by constructing solutions in uniformly convergent power series for the latter equation. © 2001 Academic Press

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1. INTRODUCTION

The iterative functional equation

$$f(f(x)) = 2f(x) - x - \frac{1}{2}(g(f(x)) + g(x)), \qquad x \in \mathbf{R}, \quad (1.1)$$

is deduced from the problem of invariant curves (see [6]) for a delay differential equation with a piecewise constant argument (EPCA for short)

$$\frac{d^2}{dt^2}x(t) + g(x([t])) = 0, \qquad t \in \mathbf{R}, x \in \mathbf{R}, \quad (1.2)$$

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where $[t]$ denotes the greatest integer less than or equal to t and $g:\mathbf{R}\to\mathbf{R}$ is continuous (or piecewise continuous). Concerning (1.2) many nice results [2–5] on existence, periodicity, and oscillation have been given. In [6] a planar mapping $G:\mathbf{R}^2\to\mathbf{R}^2$,

$$G(x,y)=\left(y,2y-x-\frac{1}{2}(g(y)+g(x))\right), \tag{1.3}$$

was derived from (1.2), which reflects the basic dynamics of (1.2). The invariant curves of G in the form of $y=f(x)$ can be obtained by solving Eq. (1.1).

Analyticity of known and unknown functions in (1.1) was once considered in [6], where it was proved that there is an analytic function $g(x)=-2x-2x^2-8\Sigma_{k=1}^{\infty}(-1)^k x^{2^k}$ on the interval $(-1,1)$ such that Eq. (1.1) has a solution $f(x)=x^2$. It was also mentioned in [6] that we do not have to restrict ourselves to this particular f. Obviously, it is interesting to deeply study analytic solutions of (1.1) further by discussing (1.1) in the complex field $\mathbf{C}$, that is,

$$f(f(z))=2f(z)-z-\frac{1}{2}(g(f(z))+g(z)), \qquad z\in\mathbf{C}, \tag{1.4}$$

where f is unknown and g is a given function which is analytic in a neighborhood of $0\in\mathbf{C}$ such that $g(0)=0$ and its derivative $g'(0)=\xi\neq 0$.

In this paper we prove the existence of analytic solutions for (1.1) by locally reducing the equation to another functional equation without iteration

$$\phi(\lambda^2 z)=2\phi(\lambda z)-\phi(z)-\frac{1}{2}(g(\phi(\lambda z))+g(\phi(z))), \qquad z\in\mathbf{C}, \tag{1.5}$$

called the auxiliary equation of (1.1), where $\lambda\neq 0$ satisfies the algebraic equation

$$2\lambda^2-(4-\xi)\lambda+2+\xi=0, \tag{1.6}$$

and by constructing solutions in uniformly convergent power series for the auxiliary equation.

2. AUXILIARY EQUATION WHEN $|\lambda|\neq 1$

LEMMA 1. *Assume that* $0<|\lambda|\neq 1$. *Then for any* $\tau\in\mathbf{C}$, *the auxiliary equation* (1.5) *has an analytic solution* $\phi(z)$ *in a neighborhood of the origin such that* $\phi(0)=0$ *and* $\phi'(0)=\tau$.

Proof. Clearly, if $\tau = 0$, (1.5) has a trivial solution $\phi(z) \equiv 0$. Assume $\tau \neq 0$. Let

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad \text{where } a_1 = \xi. \tag{2.7}$$

Without loss of generality, we assume that

$$|a_n| \le 1, \qquad n = 2, 3, \dots . \tag{2.8}$$

In fact, g is analytic in a neighborhood of $0 \in \mathbf{C}$; that is, the series

$$\xi + \sum_{n=2}^{\infty} a_n z^{n-1}$$

is uniformly convergent in a neighborhood of $0 \in \mathbf{C}$. This means that there exists a constant $\rho > 0$ such that $|a_n| \le \rho^{n-1}$, $n = 2, 3, \dots$. Then we study new functions

$$\tilde{\phi}(z) = \rho\phi(\rho^{-1}z) \qquad \text{and} \qquad \tilde{g}(z) = \rho g(\rho^{-1}z)$$

instead, because from (1.5) we see $\tilde{\phi}(z)$ satisfies

$$\tilde{\phi}(\lambda^2 z) = 2\tilde{\phi}(\lambda z) - \tilde{\phi}(z) - \frac{1}{2}\left(\tilde{g}\left(\tilde{\phi}(\lambda z)\right) + \tilde{g}\left(\tilde{\phi}(z)\right)\right), \qquad z \in \mathbf{C},$$

the same form of (1.5) but

$$\tilde{g}(z) = \rho g(\rho^{-1}z) = \xi z + \sum_{n=2}^{\infty} a_n \rho^{1-n} z^n,$$

where obviously the coefficient $|a_n \rho^{1-n}| \le 1$, $n = 2, 3, \dots$.

Furthermore, let

$$\phi(z) = \sum_{n=1}^{\infty} b_n z^n \tag{2.9}$$

be the expansion of a formal solution $\phi(z)$ of (1.5). Substituting to ϕ and g their power series (2.9) and (2.7) respectively in (1.5) we have

$$\begin{aligned} \sum_{n=1}^{\infty} b_n \lambda^{2n} z^n = {} & \sum_{n=1}^{\infty} (2\lambda^n - 1) b_n z^n \\ & - \frac{1}{2} \sum_{n=1}^{\infty} \left(\lambda^n \sum_{1 \le k \le n, (l_j) \in \mathscr{A}_n^k} a_k b_{l_1} b_{l_2} \cdots b_{l_k} \right) z^n \\ & - \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{1 \le k \le n, (l_j) \in \mathscr{A}_n^k} a_k b_{l_1} b_{l_2} \cdots b_{l_k} \right) z^n, \end{aligned} \tag{2.10}$$

where $\mathscr{A}_n^k := \{(n_1, \dots, n_k) \in \mathbf{Z}^k : n_j > 0,\ j = 1, \dots, k,\ n_1 + \cdots + n_k = n\}$. Comparing coefficients we obtain

$$(\lambda^{2n} - 2\lambda^n + 1) b_n = -\frac{1}{2}(\lambda^n + 1) \sum_{1 \le k \le n, (l_j) \in \mathscr{A}_n^k} a_k b_{l_1} b_{l_2} \cdots b_{l_k}, \quad (2.11)$$

for $n = 1, 2, \dots$. This implies that

$$\begin{cases} \left(\lambda^2 - 2\lambda + 1 + \frac{1}{2}(\lambda + 1)\xi\right) b_1 = 0 \\ \left(\lambda^{2n} - 2\lambda^n + 1 + \frac{1}{2}(\lambda^n + 1)\xi\right) b_n \\ \qquad = -\frac{1}{2}(\lambda^n + 1) \sum_{2 \le k \le n, (l_j) \in \mathscr{A}_n^k} a_k b_{l_1} b_{l_2} \cdots b_{l_k}, \end{cases} \quad (2.12)$$

for $n = 2, 3, \dots$. From (1.6) the coefficient of b_1 in the first equality of (2.12) is zero. In particular, (1.6) implies $\xi = -2(\lambda - 1)^2/(\lambda + 1)$, so the second equality of (2.12) is reduced to

$$b_n = -\frac{(\lambda + 1)(\lambda^n + 1)}{2(\lambda^n - \lambda)(\lambda^{n+1} + \lambda^n + \lambda - 3)} \sum_{2 \le k \le n, (l_j) \in \mathscr{A}_n^k} a_k b_{l_1} b_{l_2} \cdots b_{l_k}. \quad (2.13)$$

Then for arbitrarily chosen $b_1 = \tau \ne 0$, we can uniquely determine the sequence $\{b_n\}_{n=2}^{\infty}$ by (2.13) recursively.

In what follows we prove the convergence of series (2.9) in a neighborhood of the origin. Note that $0 < |\lambda| \ne 1$. Then

$$\lim_{n \to \infty} \frac{(\lambda + 1)(\lambda^n + 1)}{2(\lambda^n - \lambda)(\lambda^{n+1} + \lambda^n + \lambda - 3)} = \begin{cases} \dfrac{\lambda + 1}{\lambda(3 - \lambda)}, & 0 < |\lambda| < 1, \\ 0, & |\lambda| > 1. \end{cases}$$

Hence there exists $M > 0$ such that

$$\left| \frac{(\lambda + 1)(\lambda^n + 1)}{2(\lambda^n - \lambda)(\lambda^{n+1} + \lambda^n + \lambda - 3)} \right| \le M, \qquad \forall n \ge 2. \quad (2.14)$$

From (2.13) and (2.8) we see

$$|b_n| \le M \sum_{2 \le k \le n, (l_j) \in \mathscr{A}_n^k} |b_{l_1}| |b_{l_2}| \cdots |b_{l_k}|, \qquad n = 2, 3, \dots . \quad (2.15)$$

To construct a governing series we consider the function

$$W(z) := \frac{1}{2(1+M)}\left\{1 + |\tau|z - \sqrt{1 - 2(2M+1)|\tau|z + |\tau|^2 z^2}\right\}, \quad (2.16)$$

which clearly satisfies the equality

$$W(z) = |\tau|z + M\frac{(W(z))^2}{1 - W(z)}. \quad (2.17)$$

Obviously, $W(z)$ is continuous and $W(0) = 0$, so $|W(z)| < 1$ and

$$W(z) = |\tau|z + M(W(z))^2 \sum_{n=0}^{\infty} (W(z))^n = |\tau|z + M \sum_{n=2}^{\infty} (W(z))^n \quad (2.18)$$

for z in a sufficiently small neighborhood $U_{r_1}(0)$ of the origin. Moreover, when

$$|z| < r_2 := |\tau|^{-1}\left(2M + 1 - 2\sqrt{M^2 + M}\right) \quad (2.19)$$

the subradical term $1 - 2(2M+1)|\tau|z + |\tau|^2 z^2 > 0$ and $W(z)$ in (2.16) is analytic in $U_{r_2}(0)$. Thus $W(z)$ in (2.16) can be expanded into a convergent series

$$W(z) = \sum_{n=1}^{\infty} B_n z^n, \quad (2.20)$$

uniformly in $U_{r_2}(0)$. Replacing (2.20) into (2.18) and comparing coefficients we obtain that

$$\begin{cases} B_1 = |\tau|, \\ B_n = M \displaystyle\sum_{2 \le k \le n, (l_j) \in \mathscr{A}_n^k} B_{l_1} B_{l_2} \cdots B_{l_k}, \qquad n = 2, 3, \ldots . \end{cases} \quad (2.21)$$

Furthermore,

$$|b_n| \le B_n, \qquad n = 1, 2, \ldots . \quad (2.22)$$

In fact $|b_1| = |\tau| = B_1$. For inductive proof we assume that $|b_j| \le B_j$, $j \le n - 1$. Observe that in (2.15), $|b_{l_j}| \le |B_{l_j}|$, $j = 1, 2, \ldots, k$, because $1 \le l_1, \ldots, l_k \le n - 1$. From (2.21) we know $|b_n| \le B_n$ and (2.22) is proved. By the convergence of (2.20) and the inequality (2.22) we see that the series (2.9) converges uniformly for $|z| \le r_3 := \min\{r_1, r_2\}$. This completes the proof. ∎

3. AUXILIARY EQUATION WHEN $|\lambda| = 1$

In this section we assume that $|\lambda| = 1$ and that

(H) λ is not a root of unity, and

$$\log \frac{1}{|\lambda^n - 1|} \leq K \log n, \qquad n = 2, 3, \ldots$$

for a constant $K > 0$. The proof of the following useful lemma can be found in [5, Chap. 6; 7, pp. 166–174].

LEMMA 2. *Assume that* $|\lambda| = 1$ *and* **(H)** *holds. Then there is a positive number* δ *such that* $|\lambda^n - 1|^{-1} < (2n)^{\delta}$ *for* $n = 1, 2, \ldots$. *Furthermore, the sequence* $\{d_n\}_{n=1}^{\infty}$ *defined by* $d_1 = 1$ *and*

$$d_n = \frac{1}{|\lambda^{n-1} - 1|} \max_{k \geq 2, (n_j) \in \mathscr{B}_n^k} \{d_{n_1} \cdots d_{n_k}\}, \qquad n = 2, 3, \ldots,$$

where $\mathscr{B}_n^k := \{(n_1, \ldots, n_k) \in \mathbf{Z}^k : 0 < n_1 \leq \cdots \leq n_k,\ n_1 + \cdots + n_k = n\}$, *satisfies*

$$d_n \leq (2^{5\delta + 1})^{n-1} n^{-2\delta}, \qquad n = 1, 2, \ldots .$$

LEMMA 3. *Assume that* $|\lambda| = 1$ *and* **(H)** *holds. Then for any* $\tau \in \mathbf{C}$ *with* $0 < |\tau| \leq 1$, *the auxiliary equation* (1.5) *has an analytic solution* $\phi(z)$ *in a neighborhood of the origin such that* $\phi(0) = 0$ *and* $\phi'(0) = \tau$.

Proof. As in the proof of Lemma 1 we are seeking for a power series solution of (1.5) of the form (2.9). For chosen $b_1 = \tau$, using the same arguments as above we can uniquely determine the sequence $\{b_n\}_{n=2}^{\infty}$ by (2.13) recursively. Note that $\lambda = \exp(2\pi i \theta)$, $\theta \in \mathbf{R} \setminus \mathbf{Q}$, since **(H)** implies $|\lambda| = 1$ and $\lambda^n \neq 1$, $\forall n = 1, 2, \ldots$. Thus

$$|\lambda - 3| = |\cos(2\pi\theta) + i \sin(2\pi\theta) - 3| \geq 3 - \cos(2\pi\theta) \geq 2,$$

that is, $N := |\lambda - 3| - 2 > 0$. From (2.13),

$$\begin{aligned} |b_n| &\leq \frac{(|\lambda| + 1)(|\lambda|^n + 1)}{2|\lambda|\,|\lambda^{n-1} - 1|\big(|\lambda - 3| - |\lambda|^{n+1} - |\lambda|^n\big)} \\ &\quad \times \sum_{2 \leq k \leq n, (l_j) \in \mathscr{A}_n^k} |b_{l_1}|\,|b_{l_2}| \cdots |b_{l_k}| \\ &\leq \frac{2}{|\lambda^{n-1} - 1|(|\lambda - 3| - 2)} \sum_{2 \leq k \leq n, (l_j) \in \mathscr{A}_n^k} |b_{l_1}|\,|b_{l_2}| \cdots |b_{l_k}| \\ &\leq \frac{2}{N} |\lambda^{n-1} - 1|^{-1} \sum_{2 \leq k \leq n, (l_j) \in \mathscr{A}_n^k} |b_{l_1}|\,|b_{l_2}| \cdots |b_{l_k}|, \qquad \forall n \geq 2. \quad (3.23) \end{aligned}$$

To construct a governing series we consider the function

$$V(z) := \frac{N}{2(2+N)}\left\{1 + z - \sqrt{1 - 2\left(\frac{4}{N} + 1\right)z + z^2}\right\}, \quad (3.24)$$

which satisfies the equality

$$V(z) = z + \frac{2}{N}\frac{(V(z))^2}{1 - V(z)}. \quad (3.25)$$

Obviously, $V(z)$ is continuous and $V(0) = 0$, so $|V(z)| < 1$ and

$$W(z) = z + \frac{2}{N}\sum_{n=2}^{\infty}(V(z))^n \quad (3.26)$$

for z in a sufficiently small neighborhood $U_{r_4}(0)$ of the origin. Moreover, when

$$|z| < r_5 := \frac{4}{N} + 1 - 2\sqrt{\frac{4}{N^2} + \frac{2}{N}} \quad (3.27)$$

the subradical term $1 - 2(\frac{4}{N} + 1)z + z^2 > 0$ and $V(z)$ in (3.24) is analytic in $U_{r_5}(0)$. Thus $V(z)$ in (3.24) can be expanded into a convergent series

$$V(z) = \sum_{n=1}^{\infty} C_n z^n, \quad (3.28)$$

uniformly in $U_{r_5}(0)$. Replacing (3.28) into (3.26) and comparing coefficients we obtain that

$$\begin{cases} C_1 = 1, \\ C_n = \dfrac{2}{N}\displaystyle\sum_{2 \le k \le n,\,(l_j)\in\mathscr{A}_n^k} C_{l_1}C_{l_2}\cdots C_{l_k}, \qquad n = 2, 3, \dots. \end{cases} \quad (3.29)$$

Similar to (2.22) we can prove that

$$|b_n| \le C_n d_n, \qquad n = 1, 2, \dots, \quad (3.30)$$

where d_n is defined in Lemma 2. In fact, $|b_1| = |\tau| \le 1 = C_1 d_1$. For inductive proof we assume that $|b_j| \le C_j d_j$, $j \le n - 1$. From (3.23), (3.29), and Lemma 2,

$$\begin{aligned} |b_n| &\le \frac{2}{N}|\lambda^{n-1} - 1|^{-1}\sum_{2 \le k \le n,\,(l_j)\in\mathscr{A}_n^k} C_{l_1}d_{l_1}\cdot C_{l_2}d_{l_2}\cdots C_{l_k}\cdot d_{l_k} \\ &\le C_n|\lambda^{n-1} - 1|^{-1}\max_{2 \le k \le n,\,(l_j)\in\mathscr{B}_n^k}\{d_{l_1}\cdots d_{l_k}\} \\ &\le C_n d_n. \end{aligned} \quad (3.31)$$

Note that the series (3.28) converges uniformly in $U_{r_5}(0)$. Hence there is a constant $T > 0$ such that $C_n \le T^n$, $n = 1, 2, \dots$. By Lemma 2,

$$|b_n| \le T^n (2^{5\delta+1})^{n-1} n^{-2\delta}, \qquad n = 1, 2, \dots, \tag{3.32}$$

that is,

$$\limsup_{n\to\infty} (|b_n|)^{1/n} \le \limsup_{n\to\infty} T(2^{5\delta+1})^{(n-1)/n} n^{-2\delta/n} = T(2^{5\delta+1}).$$

This implies that the convergence radius of (2.9) is not less than $r_6 := (T(2^{5\delta+1}))^{-1}$. Then we ensure that the series (2.9) converges uniformly for $|z| \le \min\{r_4, r_5, r_6\}$. This completes the proof. ▮

4. EXISTENCE OF ANALYTIC SOLUTIONS

THEOREM 1. *Assume that* $0 < |\lambda| \ne 1$ *or that* $|\lambda| = 1$ *and* **(H)** *holds. Then Eq.* (1.4) *has an analytic solution of the form* $f(x) = \phi(\lambda\phi^{-1}(z))$ *in a neighborhood of the origin, where* $\phi(z)$ *is an analytic solution of the auxiliary equation* (1.5).

Proof. By Lemmas 1 and 3, we can find an analytic solution $\phi(z)$ of the auxiliary equation (1.5) in the form of (2.9) such that $\phi(0) = 0$ and $\phi'(0) = \tau \ne 0$. Clearly the inverse $\phi^{-1}(z)$ exists and is analytic in a neighborhood of the origin. Let

$$f(x) = \phi(\lambda\phi^{-1}(z)), \tag{4.33}$$

which is also analytic in a neighborhood of the origin. From (1.5), it is easy to see

$$\begin{aligned} f(f(z)) &= \phi(\lambda\phi^{-1}(\phi(\lambda\phi^{-1}(z)))) = \phi(\lambda^2\phi^{-1}(z)) \\ &= 2\phi(\lambda\phi^{-1}(z)) - \phi(\phi^{-1}(z)) \\ &\quad - \frac{1}{2}\big(g(\phi(\lambda\phi^{-1}(z))) + g(\phi(\phi^{-1}(z)))\big) \\ &= 2f(z) - z - \frac{1}{2}(g(f(z)) + g(z)), \end{aligned}$$

that is, the function f in (4.33), defined in a neighborhood of the origin, satisfies Eq. (1.4). ▮

The following example shows how to construct an analytic solution of (1.4) for a concrete function

$$g(z) = 2(1 - e^z) = -\sum_{n=1}^{\infty} \frac{2}{n!} z^n. \tag{4.34}$$

The algebraic equation corresponding to (1.6) is

$$\lambda^2 - 3\lambda = 0, \tag{4.35}$$

which has a nonzero root $\lambda_1 = 3$. By Lemma 1, the auxiliary equation

$$\phi(9z) = 2\phi(3z) - \phi(z) - \frac{1}{2}\big(g(\phi(3z)) + g(\phi(z))\big) \tag{4.36}$$

has a solution of the form (2.9) where $b_1 = \tau \neq 0$ is given arbitrarily and $b_2, b_3, \ldots$ are determined by (2.13) recursively, i.e.,

$$b_n = \frac{3^n + 1}{(3^{n-1} - 1)3^{n+1}} \sum_{2 \le k \le n, (l_j) \in \mathscr{A}_n^k} \left(\frac{1}{k!}\right) b_{l_1} b_{l_2} \cdots b_{l_k}. \tag{4.37}$$

In particular,

$$b_2 = \frac{\phi''(0)}{2!} = \frac{5}{54}\tau^2,$$

$$b_3 = \frac{\phi'''(0)}{3!} = \frac{3^3 + 1}{(3^2 - 1)3^4}\left(b_1 b_2 + \frac{b_1^3}{6}\right) = \frac{7^2}{2 \cdot 3^7}\tau^3,$$

$$\ldots \qquad \ldots .$$

Since $\phi(0) = 0$, $\phi'(0) = \tau \neq 0$, and the inverse $\phi^{-1}(z)$ is analytic near the origin, we can calculate

$$(\phi^{-1})'(0) = \frac{1}{\phi'(\phi^{-1}(0))} = \frac{1}{\phi'(0)} = \frac{1}{\tau},$$

$$(\phi^{-1})''(0) = -\frac{\phi''(\phi^{-1}(0))(\phi^{-1})'(0)}{\big(\phi'(\phi^{-1}(0))\big)^2} = -\frac{\phi''(0)(\phi^{-1})'(0)}{(\phi'(0))^2} = -\frac{5}{27\tau},$$

$$\begin{aligned}
&(\phi^{-1})'''(0)\\
&= -\frac{\Big\{\phi'''(\phi^{-1}(0))\big((\phi^{-1})'(0)\big)^2 + \phi''(\phi^{-1}(0))(\phi^{-1})''(0)\Big\}\big(\phi'(\phi^{-1}(0))\big)^2}{\big(\phi'(\phi^{-1}(0))\big)^4}\\
&\quad + \frac{\phi''(\phi^{-1}(0))(\phi^{-1})'(0) \cdot 2\phi'(\phi^{-1}(0))\phi''(\phi^{-1}(0))(\phi^{-1})'(0)}{\big(\phi'(\phi^{-1}(0))\big)^4}\\
&= -\frac{\Big\{\phi'''(0)\tau^{-2} - \phi''(0)\Big(\frac{5}{27\tau}\Big)\Big\}(\phi'(0))^2 - \phi''(0)\tau^{-1} \cdot 2\phi'(0)\phi''(0)\tau^{-1}}{(\phi'(0))^4}\\
&= \frac{26}{3^6\tau},
\end{aligned}$$

$$\ldots \qquad \ldots .$$

Furthermore, we get

$$
\begin{aligned}
f(0) &= \phi\big(3\phi^{-1}(0)\big) = \phi(0) = 0, \\
f'(0) &= \phi'\big(3\phi^{-1}(0)\big)\cdot 3(\phi^{-1})'(0) = 3\phi'(0)(\phi^{-1})'(0) = 3\tau\frac{1}{\tau} = 3, \\
f''(0) &= 9\phi''\big(3\phi^{-1}(0)\big)\big((\phi^{-1})'(0)\big)^2 + 3\phi'\big(3\phi^{-1}(0)\big)(\phi^{-1})''(0) = \frac{10}{9}, \\
f'''(0) &= 27\phi'''\big(3\phi^{-1}(0)\big)\big((\phi^{-1})'(0)\big)^3 \\
&\qquad + 18\phi''\big(3\phi^{-1}(0)\big)(\phi^{-1})'(0)(\phi^{-1})''(0) \\
&\qquad + 9\phi''\big(3\phi^{-1}(0)\big)(\phi^{-1})'(0)(\phi^{-1})''(0) \\
&\qquad + 3\phi'\big(3\phi^{-1}(0)\big)(\phi^{-1})'''(0) \\
&= 27\frac{49\tau^3}{3^6}\frac{1}{\tau^3} - 27\frac{5\tau^2}{27}\frac{1}{\tau}\frac{5}{27\tau} + 3\tau\frac{26}{3^6\tau} = \frac{28}{27}, \\
&\cdots \qquad\qquad \cdots .
\end{aligned}
$$

Thus near 0 Eq. (1.4) with g in (4.34) has an analytic solution

$$f(z) = 3z + \frac{5}{9}z^2 + \frac{17}{81}z^3 + \cdots . \tag{4.38}$$

Remark that if $g(x)$ is an analytic real function, i.e., $g(z) = \sum_{n=1}^{\infty} a_n z^n$ is a convergent series near 0 with real coefficients, and if $a_1 = \xi$ satisfies

$$\xi < 0 \qquad \text{or} \qquad \xi \geq 16, \tag{4.39}$$

then by Theorem 1, Eq. (1.4) has an analytic real solution. In fact, (4.39) guarantees that (1.6), i.e., $2\lambda^2 - (4 - \xi)\lambda + 2 + \xi = 0$, has real roots λ_1 and λ_2. Clearly by (2.13) where $\lambda = \lambda_1$ or λ_2, we can define a real sequence $\{b_n\}_{n=2}^{\infty}$ and obtain a solution $\phi(z)$ of (1.5) with real coefficients. Restricted on **R** both the function ϕ and its inverse are valued in **R**. Hence the function $f(x) = \phi(\lambda_j \phi^{-1}(z))$, $j = 1, 2$, is also a real function and Theorem 1 implies its analyticity.

The problem of analytic solutions in the case $|\lambda| = 1$ without restriction (**H**) is not solved yet. Such a difficulty was noted by C. L. Siegel when he discussed the Schröder equation in [7].

REFERENCES

1. A. R. Aftabizadeh and J. Wiener, Oscillatory and periodic solutions of systems of two first order linear differential operations with piecewise constant arguments, *Appl. Anal.* **26** (1988), 327–333.
2. A. R. Aftabizadeh, J. Wiener, and J.-M. Xu, Oscillatory and periodic properties of delay differential equations with piecewise constant arguments, *Proc. Amer. Math. Soc.* **99** (1987), 673–679.
3. K. L. Cooke and J. Wiener, Retarded differential equations with piecewise constant arguments, *J. Math. Anal. Appl.* **99** (1984), 265–297.
4. I. Gyori and G. Ladas, Linearized oscillations for equations with piecewise constant arguments, *Differential Integral Equations* **2** (1989), 123–131.
5. M. Kuczma, “Functional Equations in a Single Variable,” PWN, Warsaw, 1968.
6. C. T. Ng and W. Zhang, Invariant curves for planar mappings, *J. Differ. Equations Appl.* **3** (1997), 147–168.
7. C. L. Siegel, “Vorlesungen über Himmelsmechanik,” Springer-Verlag, Berlin, 1956.