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Property (T) and strong property (T) for unital C^*-algebras ☆

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Abstract

In this paper we will give a thorough study of the notion of property (T) for C^*-algebras (as introduced by M.B. Bekka) as well as a slightly stronger version of it, called "strong property (T)" (which is also an analogue of the corresponding concept in the case of discrete groups and type II_1-factors). More precisely, we will give some interesting equivalent formulations as well as some permanence properties for both property (T) and strong property (T). We will also relate them to certain (T)-type properties of the unitary group of the underlying C^*-algebra.

Keywords: Property (T); Unital C^*-algebras; Hilbert bimodules

1. Introduction

Property (T) for locally compact groups was first defined by D. Kazhdan in [11] and was later extended to Hausdorff topological groups. In [12], property (T) for a pair of groups $H \subseteq G$ was

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introduced. This notion was proved to be very useful and was studied by many people (see e.g. [2,4,8,10–12,15]).

In [6], A. Connes introduced the notion of property (T) for type II_1-factors and this notion was then extended to von Neumann algebras in [7]. A discrete group G has property (T) if and only if the von Neumann algebra generated by the left regular representation of G has property (T) (this was first proved in [7] for ICC groups and was generalized by P. Jolissaint in [9] to general discrete groups). The notion of property (T) for a pair of von Neumann algebras was defined by S. Popa in [14]. This notion was also proved to be very useful in the study of von Neumann algebras.

Recently, M.B. Bekka introduced in [3] the interesting notion of property (T) for a pair consisting of a unital C^*-algebra and a unital C^*-subalgebra. He showed that a countable discrete group G has property (T) if and only if its full (or equivalently reduced) group C^*-algebra has property (T). In [5], N.P. Brown did a study of property (T) for C^*-algebras and showed that a nuclear unital C^*-algebra A has property (T) if and only if $A = B \oplus C$ where B is finite dimensional and C admits no tracial state.

The aim of this paper is to give a thorough study of property (T) as well as a slightly stronger version called strong property (T) for unital C^*-algebras. On our way, we will show that our stronger version is equally good (if not a better) candidate for the notion of property (T) for a pair of unital C^*-algebras.

The paper is organised as follows. In Section 2 we will give two simple and useful reformulations of both property (T) and strong property (T). In Section 3 we consider two Kazhdan constants t_u^A and t_c^A for a C^*-algebra A which are the analogues of the Kazhdan constant for locally compact groups (see [15]). We will show that A has property (T) (respectively, strong property (T)) if and only if $t_c^A > 0$ (respectively, $t_u^A > 0$). Through them, we obtain some interesting reformulations of property (T) and strong property (T). In particular, we show that one can check property (T) by looking at just one bimodule. We will also show that one can express the gap between property (T) and strong property (T) by another Kazhdan constant t_s^A.

In Section 4 we obtain some permanence properties for property (T) and strong property (T), including quotients, direct sums, tensor products and crossed products. In Section 5 we will show that finite dimensional C^*-algebras have strong property (T). Moreover, we show that a corresponding result of Bekka concerning relation between property (T) of discrete groups and their group C^*-algebras as well as a corresponding result of Brown concerning amenable property (T) C^*-algebras also holds for strong property (T). In Section 6 we study the relation between property (T) (as well as strong property (T)) of a unital C^*-algebra A and certain (T)-type properties of the unitary group of A.

Let us first set the following notations that will be used throughout the whole paper.

Notation 1.1. (1) A is a unital C^*-algebra and $B \subseteq A$ is a C^*-subalgebra containing the identity of A. Set $A^{Dou} := A \otimes_{\max} A^{\mathrm{op}}$ (where A^{op} is the "opposite C^*-algebra" with $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$).

(2) $\mathfrak{F}(E)$ is the set of all non-empty finite subsets of a set E and $\mathfrak{S}_1(X)$ is the unit sphere of a normed space X.

(3) $U(A)$ and $S(A)$ are respectively the unitary group and the state space of A.

(4) $\mathrm{Bimod}^*(A)$ is the collection of unitary equivalence classes of unital Hilbert bimodules over A (or equivalently, non-degenerate representations of A^{Dou}). For any $H \in \mathrm{Bimod}^*(A)$, let

$$H^B := \{\xi \in H \colon b \cdot \xi = \xi \cdot b \text{ for all } b \in B\}$$

and $P_H^B : H \to H^B$ be the orthogonal projection. Elements in H^B are called *central vectors for B*. Moreover, for any $(Q,\beta) \in \mathfrak{F}(A) \times \mathbb{R}_+$, set

$$V_H(Q,\beta) := \big\{\xi \in \mathfrak{S}_1(H)\colon\ \|x\cdot\xi - \xi\cdot x\| < \beta \text{ for all } x \in Q\big\}.$$

Elements in $V_H(Q,\beta)$ are called *(Q,β)-central unit vectors*. On the other hand, a net of vectors $(\xi_i)_{i\in I}$ in $\mathfrak{S}_1(H)$ is called an *almost central unit vector for A* if $\|a\cdot\xi_i - \xi_i\cdot a\| \to 0$ for any $a \in A$.

(5) For any topological group G, we denote by $\mathrm{Rep}(G)$ the collection of all unitary equivalence classes of continuous unitary representations of G. If $(\pi, H) \in \mathrm{Rep}(G)$, we let

$$H^G := \big\{\xi \in H\colon\ \pi(s)\xi = \xi \text{ for all } s \in G\big\}$$

and $P_H^G : H \to H^G$ be the orthogonal projection. Furthermore, if $F \in \mathfrak{F}(G)$ and $\epsilon > 0$, we set

$$V_\pi(F,\epsilon) = \big\{\xi \in \mathfrak{S}_1(H)\colon\ \big\|\pi(t)\xi - \xi\big\| < \epsilon \text{ for all } t \in F\big\}.$$

(6) For any $(\mu, H), (\nu, K) \in \mathrm{Rep}(G)$, we write $(\mu, H) \leqslant (\nu, K)$ if (μ, H) is a subrepresentation of (ν, K).

2. Definitions and basic properties

Let us first recall Bekka's notion of property (T) in [3]. The pair (A, B) is said to have *property (T)* if there exist $F \in \mathfrak{F}(A)$ and $\epsilon > 0$ such that for any $H \in \mathrm{Bimod}^*(A)$, if $V_H(F,\epsilon) \neq \emptyset$, then $H^B \neq (0)$. In this case (F,ϵ) is called a *Kazhdan pair* for (A, B). Moreover, A is said to have *property (T)* if the pair (A, A) has property (T).

Note that Bekka's definition comes from the original definition of property (T) for groups (see e.g. [10, Definition 1.1(1)]). We will now give a slightly stronger version which comes from an equivalent form of property (T) for groups (see [10, Theorem 1.2(b2)]). Note that the corresponding stronger version of property (T) for type II_1-factor is also equivalent to property (T) (see e.g. [7, Proposition 1]) but we do not know if it is the case for C^*-algebras.

Definition 2.1. The pair (A, B) is said to have *strong property (T)* if for any $\alpha > 0$, there exist $Q \in \mathfrak{F}(A)$ and $\beta > 0$ such that for any $H \in \mathrm{Bimod}^*(A)$ and any $\xi \in V_H(Q,\beta)$, we have $\|\xi - P_H^B(\xi)\| < \alpha$. In this case (Q,β) is called a *strong Kazhdan pair* for (A, B, α). We say that A has strong property (T) if (A, A) has such property.

It is clear that if A has property (T) (respectively, strong property (T)) then so has the pair (A, B). Moreover, by taking $\alpha < 1/2$, we see that strong property (T) implies property (T). We will see later that strong property (T) is an equally good (if not a better) candidate for the notion of property (T) for a pair of C^*-algebras.

Let us now give the following simple reformulation of property (T) and strong property (T) which will be useful in Section 6.

Lemma 2.2. *For any $(Q,\beta) \in \mathfrak{F}(A) \times \mathbb{R}_+$, there exists $(Q',\beta') \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ such that $V_H(Q',\beta') \subseteq V_H(Q,\beta)$ for any $H \in \mathrm{Bimod}^*(A)$. Consequently, one can replace $\mathfrak{F}(A)$ by $\mathfrak{F}(U(A))$ in the definitions of both property (T) and strong property (T).*

Proof. This lemma is clear if $Q = \{0\}$. Let $Q \setminus \{0\} = \{x_1, \ldots, x_n\}$ and $M = \max\{\|x_1\|, \ldots, \|x_n\|\}$. For each $k \in \{1, \ldots, n\}$, consider $u_k, v_k \in U(A)$ such that $2x_k = \|x_k\|((u_k + u_k^*) + i(v_k + v_k^*))$. If we take Q' to be the set $\{u_1, u_1^*, v_1, v_1^*, \ldots, u_n, u_n^*, v_n, v_n^*\}$ and $\beta' = \frac{\beta}{2M}$, then $V_H(Q', \beta') \subseteq V_H(Q, \beta)$ for any $H \in \mathrm{Bimod}^*(A)$. □

Before we give a second simple reformulation, we need to set some notations. Let $S(D)$ and $S_t(D)$ be respectively the sets of all states and the set of all tracial states on a C^*-algebra D. For any $\tau \in S(D)$ and any cardinal α, we denote by M_τ the GNS construction for τ and by $M_{\tau,\alpha}$ the α-times direct sum, $\bigoplus_\alpha M_\tau$, of M_τ (we use the convention that $\bigoplus_0 M_\tau = \{0\}$).

Definition 2.3. Let

$$\mathcal{H} := \bigoplus_{\tau \in S(A^{Dou})} M_\tau \quad \text{and} \quad \mathcal{K} := \bigoplus_{\tau \in S_t(A)} M_\tau.$$

We call $\mathcal{H}$ and $\mathcal{K}$ the *universal* and the *standard* bimodules (over A) respectively. Moreover, a bimodule of the form $\bigoplus_{\tau \in S_t(A)} M_{\tau,\alpha_\tau}$ is called a *quasi-standard bimodule*.

Proposition 2.4. (a) (A, B) *has property* (T) *if and only if for any* $H \in \mathrm{Bimod}^*(A)$, *the existence of an almost central unit vector for* A *in* H *will imply that* $H^B \neq \{0\}$.

(b) *The following statement are equivalent.*

(i) (A, B) *has strong property* (T).
(ii) *For any almost central unit vector* $(\xi_i)_{i \in I}$ *for* A *in any bimodule* $H \in \mathrm{Bimod}^*(A)$, *we have* $\|\xi_i - P_H^B(\xi_i)\| \to 0$.
(iii) *For any almost central unit vector* $(\xi_i)_{i \in I}$ *for* A *in* $\mathcal{H}$ *and any* $n \in \mathbb{N}$, *there exists* $i_n \in I$ *with* $\|\xi_{i_n} - P_{\mathcal{H}}^B(\xi_{i_n})\| < 1/n$.

Proof. (a) This part is well known.

(b) It is clear that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). To obtain (iii) $\Rightarrow$ (i), we suppose, on the contrary, that (A, B) does not have strong property (T). Then one can find $\alpha_0 > 0$ such that for any $i = (Q, \beta) \in I := \mathfrak{F}(A) \times \mathbb{R}_+$, there exist $H_i \in \mathrm{Bimod}^*(A)$ and $\xi_i \in V_{H_i}(Q, \beta)$ with $\|\xi_i - P_{H_i}^B(\xi_i)\| \geqslant \alpha_0$. If $K_i = \overline{A \cdot \xi_i \cdot A}$, then $H_i = K_i \oplus K_i^\perp$ and $H_i^B = K_i^B \oplus (K_i^\perp)^B$. As $\xi_i \in K_i$, we have

$$\|\xi_i - P_{K_i}^B(\xi_i)\| = \|\xi_i - P_{H_i}^B(\xi_i)\| \geqslant \alpha_0.$$

We set $\mathcal{X} := \{K_i \colon i \in I\} \subseteq \mathrm{Bimod}^*(A)$ and $K_0 := \bigoplus_{K \in \mathcal{X}} K$. Since all bimodules in $\mathcal{X}$ are cyclic (as representations of A^{Dou}) and any two elements in $\mathcal{X}$ are inequivalent, K_0 is a Hilbert sub-bimodule of $\mathcal{H}$. Moreover, each K_i is equivalent to a unique element in $\mathcal{X}$ and this gives a canonical Hilbert bimodule embedding $\Psi_i : K_i \to K_0$. It is easy to check that $(\Psi_i(\xi_i))_{i \in I}$ is an almost central unit vector for A in $\mathcal{H}$ with $\|\Psi_i(\xi_i) - P_{\mathcal{H}}^B(\Psi_i(\xi_i))\| = \|\xi_i - P_{K_i}^B(\xi_i)\| \geqslant \alpha_0$ for every $i \in I$ (since $\Psi_i(K_i)$ is a direct summand of $\mathcal{H}$). This contradicts statement (iii). □

Since Proposition 2.4 is so fundamental to our discussions, we may use it without mentioning it explicitly throughout the whole paper.

Remark 2.5. (a) Note that if A is separable, then in the above proposition, one can replace almost central unit vector by a sequence of unit vectors that is "almost central" for A.

(b) In Proposition 2.4(b)(iii) we only need to check one bimodule (namely, the universal one) in order to verify strong property (T).

(c) One may wonder if it is possible to check whether a C^*-algebra has property (T) by looking at its universal bimodule alone. However, this cannot be done using the original formulation of property (T) because there exists a unital C^*-algebra A which does not have property (T) but $\mathcal{H}^A \neq (0)$ (i.e. A has a tracial state). Nevertheless, we will show in Theorem 3.4 below that it is possible to do so using an equivalent formulation of property (T).

3. Kazhdan constants

In this section, we will define and study some Kazhdan constants in the case when $B = A$. Let us start with the following lemma.

Lemma 3.1. *Let $H \in \mathrm{Bimod}^*(A)$. If $H_{\mathcal{C}}$ is the sub-bimodule generated by H^A (called the* centrally generated part of H*), then $H_{\mathcal{C}}$ is a quasi-standard bimodule (Definition 2.3) and $H_{\mathcal{C}}^{\perp}$ contains no non-zero central vector for A.*

Proof. Without loss of generality, we may assume that $C := \mathfrak{S}_1(H^A)$ is non-empty. Let $\mathcal{S} := \{\overline{A \cdot \xi}\colon \xi \in C\}$ and

$$\mathfrak{M} := \{\mathcal{M} \subseteq \mathcal{S}\colon K \perp L \text{ for any } K, L \in \mathcal{M}\}.$$

By the Zorn's lemma, there exists a maximal element $\mathcal{M}_0$ in $\mathfrak{M}$ and we put $H_1 := \bigoplus_{K \in \mathcal{M}_0} K$. Then clearly $H_1 \subseteq H_{\mathcal{C}}$ and $H_1^{\perp}$ contains no non-zero central vector for A. Together with the fact $H^A = H_1^A \oplus (H_1^{\perp})^A$, this shows that $H^A = H_1^A \subseteq H_1$ and hence $H_1 = H_{\mathcal{C}}$. Finally, for any $\overline{A \cdot \xi} \in \mathcal{M}_0$ with $\xi \in C$, the functional defined by $\tau(a) := \langle a\xi, \xi\rangle$ $(a \in A)$ is a tracial state and $\overline{A \cdot \xi} \cong M_\tau$. This completes the proof. □

Suppose that $H \in \mathrm{Bimod}^*(A)$ and K is a Hilbert subspace of H. For any $Q \in \mathfrak{F}(A)$, we set

$$t^A(Q; H, K) := \inf\left\{\left(\sum_{x \in Q} \|x \cdot \xi - \xi \cdot x\|^2\right)^{1/2}\colon \xi \in \mathfrak{S}_1(H \ominus K)\right\}$$

(we use the convention that the infimum over the empty set is $+\infty$).

Lemma 3.2. *Let $Q \in \mathfrak{F}(A)$, $H \in \mathrm{Bimod}^*(A)$ and K be a Hilbert subspace of H. Suppose that $H = \bigoplus_{\lambda \in \Lambda} H_\lambda$ such that $K = \bigoplus_{\lambda \in \Lambda} K_\lambda$ where $K_\lambda := H_\lambda \cap K$.*

(a) *If α_λ is a cardinal for any $\lambda \in \Lambda$, and if we set $H_0 := \bigoplus_{\lambda \in \Lambda}(\bigoplus_{\alpha_\lambda} H_\lambda)$ and $K_0 := \bigoplus_{\lambda \in \Lambda}(\bigoplus_{\alpha_\lambda} K_\lambda)$, then*

$$t^A(Q; H, K)^2 \|\zeta\|^2 \leqslant \sum_{x \in Q} \|x \cdot \zeta - \zeta \cdot x\|^2 \quad (\zeta \in H_0 \ominus K_0). \tag{3.1}$$

(b) $t^A(Q; H, K) = \inf_{\lambda\in\Lambda} t^A(Q; H_\lambda, K_\lambda)$.
(c) $(t^A(Q; H, K))_{Q\in\mathfrak{F}(A)}$ *is an increasing net and* $\lim_{Q\in\mathfrak{F}(A)} t^A(Q; H, K) = 0$ *if and only if there exists an almost central unit vector for* A *in* $H \ominus K$. *In this case one can choose an almost central unit vector* $(\xi_i)_{i\in I}$ *for* A *such that for any* $i \in I$, *there exists* $\lambda_i \in \Lambda$ *with* $\xi_i \in H_{\lambda_i} \ominus K_{\lambda_i}$.

Proof. (a) For any $\zeta \in \mathfrak{S}_1(H_0 \ominus K_0)$, we have $\zeta = (\zeta_{i,\lambda})_{\lambda\in\Lambda;i\in\alpha_\lambda}$ with $\zeta_{i,\lambda} \in H_\lambda \ominus K_\lambda \subseteq H \ominus K$ and $\sum_{\lambda\in\Lambda}\sum_{i\in\alpha_\lambda} \|\zeta_{i,\lambda}\|^2 = 1$. Thus,

$$t^A(Q; H, K)^2 \leqslant \sum_{\lambda\in\Lambda}\sum_{i\in\alpha_\lambda} \|\zeta_{i,\lambda}\|^2 \sum_{x\in Q} \left\| x\cdot\frac{\zeta_{i,\lambda}}{\|\zeta_{i,\lambda}\|} - \frac{\zeta_{i,\lambda}}{\|\zeta_{i,\lambda}\|}\cdot x\right\|^2 = \sum_{x\in Q}\|x\cdot\zeta - \zeta\cdot x\|^2.$$

(b) Note that $t^A(Q; H, K) \leqslant t^A(Q; H_\lambda, K_\lambda)$ for all $\lambda \in \Lambda$ (as $H_\lambda \ominus K_\lambda \subseteq H \ominus K$). For any $\epsilon > 0$, there exists $\xi \in \mathfrak{S}_1(H \ominus K)$ such that $\sum_{x\in Q}\|x\cdot\xi - \xi\cdot x\|^2 \leqslant t^A(Q; H, K) + \epsilon$. Now, $\xi = (\xi_\lambda)_{\lambda\in\Lambda}$ with $\xi_\lambda \in H_\lambda \ominus K_\lambda$ and $\sum_{\lambda\in\Lambda}\|\xi_\lambda\|^2 = 1$. A similar argument as part (a) implies that there exists $\lambda_0 \in \Lambda$ such that

$$\sum_{x\in Q}\left\| x\cdot\frac{\xi_{\lambda_0}}{\|\xi_{\lambda_0}\|} - \frac{\xi_{\lambda_0}}{\|\xi_{\lambda_0}\|}\cdot x\right\|^2 \leqslant t^A(Q; H, K) + \epsilon.$$

(c) It is clear that $(t^A(Q; H, K))_{Q\in\mathfrak{F}(A)}$ is increasing and that $t^A(Q; H, K) = 0$ for any $Q \in \mathfrak{F}(A)$ if there exists an almost central unit vector for A in $H \ominus K$. Now, suppose that

$$\sup_{Q\in\mathfrak{F}(A)}\inf_{\lambda\in\Lambda} t^A(Q; H_\lambda, K_\lambda) = \lim_{Q\in\mathfrak{F}(A)} t^A(Q; H, K) = 0.$$

Then for any $Q \in \mathfrak{F}(A)$ and $\epsilon > 0$, there exist $\lambda_{Q,\epsilon} \in \Lambda$ and $\xi_{Q,\epsilon} \in \mathfrak{S}_1(H_{\lambda_{Q,\epsilon}} \ominus K_{\lambda_{Q,\epsilon}})$ such that $\sum_{x\in Q}\|x\cdot\xi_{Q,\epsilon} - \xi_{Q,\epsilon}\cdot x\|^2 < \epsilon^2$. It is easy to see that $(\xi_{Q,\epsilon})_{(Q,\epsilon)\in\mathfrak{F}(A)\times\mathbb{R}_+}$ is an almost central unit vector for A. $\square$

Now, we define three Kazhdan constants: for any $Q \in \mathfrak{F}(A)$, set

$$t_u^A(Q) := t^A\big(Q; \mathcal{H}, \mathcal{H}^A\big), \qquad t_c^A(Q) := t^A(Q; \mathcal{H}, \mathcal{H}_{\mathcal{C}}), \qquad t_s^A(Q) := t^A\big(Q; \mathcal{K}, \mathcal{K}^A\big)$$

(where $\mathcal{H}$ and $\mathcal{K}$ are the universal bimodule and the standard bimodule respectively) and

$$t_u^A := \sup_{Q\in\mathfrak{F}(A)} t_u^A(Q), \qquad t_c^A := \sup_{Q\in\mathfrak{F}(A)} t_c^A(Q) \quad \text{as well as} \quad t_s^A := \sup_{Q\in\mathfrak{F}(A)} t_s^A(Q).$$

Lemma 3.3. (a) *For any* $H \in \mathrm{Bimod}^*(A)$, *we have* $t_u^A(Q) \leqslant t^A(Q; H, H^A)$ *and* $t_c^A(Q) \leqslant t^A(Q; H, H_{\mathcal{C}})$. *If, in addition, H is quasi-standard, then* $t_s^A(Q) \leqslant t^A(Q; H, H^A)$.

(b) $t_u^A(Q) \leqslant \min\{t_c^A(Q), t_s^A(Q)\}$.

Proof. (a) There are cardinals α_τ ($\tau \in S(A^{Dou})$) such that $H \cong \bigoplus_{\tau\in S(A^{Dou})} M_{\alpha_\tau,\tau}$. For any $\xi \in \mathfrak{S}_1(H \ominus H^A)$, we have $\xi = (\xi_\tau)$ where $\xi_\tau \in M_{\alpha_\tau,\tau} \ominus M^A_{\alpha_\tau,\tau}$. By inequality (3.1), we have

$$t_u^A(Q)^2\|\xi_\tau\|^2 \leqslant \sum_{x\in Q} \|x\cdot\xi_\tau - \xi_\tau\cdot x\|^2$$

and so

$$t_u^A(Q)^2 \leqslant \sum_{\tau\in S(A^{Dou})}\sum_{x\in Q} \|x\cdot\xi_\tau - \xi_\tau\cdot x\|^2 = \sum_{x\in Q}\|x\cdot\xi - \xi\cdot x\|^2$$

(as $\sum_{\tau\in S(A^{Dou})}\|\xi_\tau\|^2 = \|\xi\|^2 = 1$). Thus, we have $t_u^A(Q) \leqslant t^A(Q; H, H^A)$. The arguments for the other two inequalities are similar.

(b) $t_u^A(Q) \leqslant t_c^A(Q)$ because $\mathcal{H}^A \subseteq \mathcal{H}_\mathcal{C}$ and $t_u^A(Q) \leqslant t_s^A(Q)$ because of part (a). □

Theorem 3.4. (a) *The following statements are equivalent.*

(i) $t_u^A > 0$.
(ii) *A has strong property* (T).
(iii) *There exists* $(Q,\delta) \in \mathfrak{F}(A)\times\mathbb{R}_+$ *such that for any* $\xi \in V_\mathcal{H}(Q,\delta)$, *we have* $P^A_\mathcal{H}(\xi) \neq 0$.

(b) *The following statements are equivalent.*

(i) $t_s^A > 0$.
(ii) *For any* $\epsilon > 0$, *there exists* $(Q,\delta) \in \mathfrak{F}(A)\times\mathbb{R}_+$ *such that for any quasi-standard bimodule* H *and any* $\xi \in V_H(Q,\delta)$, *we have* $\|\xi - P^A_H(\xi)\| < \epsilon$.
(iii) *There exists* $(Q,\delta) \in \mathfrak{F}(A)\times\mathbb{R}_+$ *such that for any* $\xi \in V_\mathcal{K}(Q,\delta)$, *we have* $P^A_\mathcal{K}(\xi) \neq 0$.

(c) *The following statements are equivalent.*

(i) $t_c^A > 0$.
(ii) *A has property* (T).
(iii) *There is* $(Q,\delta) \in \mathfrak{F}(A)\times\mathbb{R}_+$ *such that* $V_H(Q,\delta)\cap H_\mathcal{C}^\perp = \emptyset$ *for any* $H \in \mathrm{Bimod}^*(A)$.
(iv) *There exists* $(Q,\delta) \in \mathfrak{F}(A)\times\mathbb{R}_+$ *such that* $V_\mathcal{H}(Q,\delta)\cap\mathcal{H}_\mathcal{C}^\perp = \emptyset$.

(d) $t_u^A > 0$ *if and only if both* $t_c^A > 0$ *and* $t_s^A > 0$.

Proof. (a) (i) ⇒ (ii). There exists $Q \in \mathfrak{F}(A)$ with $t_u^A(Q) > 0$. Let m be the number of elements in Q and $\delta = \frac{t_u^A(Q)\epsilon}{\sqrt{m}}$. For any $H \in \mathrm{Bimod}^*(A)$ and $\tau \in S(A^{Dou})$, there is a cardinal α_τ such that $H = \bigoplus_{\tau\in S(A^{Dou})} M_{\alpha_\tau,\tau}$ (α_τ can be zero). Pick any $\xi \in V_H(Q,\delta)$ and consider $\xi'' = \xi - P^A_H(\xi) \in (H^A)^\perp$. Since $\xi'' = (\zeta_\tau)_{\tau\in S(A)}$ where $\zeta_\tau \in (M^A_{\alpha_\tau,\tau})^\perp$, we have, by inequality (3.1),

$$\|\xi''\|^2 \leqslant t_u^A(Q)^{-2}\sum_{x\in Q}\|x\cdot\xi'' - \xi''\cdot x\|^2 = t_u^A(Q)^{-2}\sum_{x\in Q}\|x\cdot\xi - \xi\cdot x\|^2 < \epsilon^2.$$

(ii) ⇒ (iii). By taking $\epsilon = 1/2$, we see that statement (iii) holds.

(iii) ⇒ (i). Suppose on the contrary that $t_u^A(Q) = 0$. Then there exists $\xi \in \mathfrak{S}_1((\mathcal{H}^A)^\perp)$ with $\sum_{x\in Q} \|x\cdot\xi - \xi\cdot x\|^2 < \delta^2$. Hence, $\xi \in V_{\mathcal{H}}(Q,\delta)$ and so $P_{\mathcal{H}}^A(\xi) \neq 0$ which contradicts the fact that $\xi \in (\mathcal{H}^A)^\perp$.

(b) The proof of this part is essentially the same as that of part (a) with $\mathcal{H}$ and t_u^A being replaced by $\mathcal{K}$ and t_s^A respectively.

(c) (i) ⇒ (ii). Let $Q \in \mathfrak{F}(A)$ such that $t_c^A(Q) > 0$. Suppose that A does not have property (T). There exists $H \in \mathrm{Bimod}^*(A)$ that contains an almost central unit vector (ξ_i) for A but $H^A = \{0\}$. Hence, $H_{\mathcal{C}}^\perp = H$, and

$$t^A(Q; H, H_{\mathcal{C}}) = \inf\left\{\left(\sum_{x\in Q} \|x\cdot\xi - \xi\cdot x\|^2\right)^{1/2} : \xi \in \mathfrak{S}_1(H)\right\} = 0.$$

Now Lemma 3.3(a) gives the contradiction that $t_c^A(Q) = 0$.

(ii) ⇒ (i). Suppose on the contrary that $t_c^A(Q) = 0$ for any $Q \in \mathfrak{F}(A)$. There exists, by Lemma 3.2(c), an almost central unit vector for A in $\mathcal{H}_{\mathcal{C}}^\perp$ which contradicts the fact that A has property (T) (because of Lemma 3.1).

(i) ⇒ (iii). Let $Q \in \mathfrak{F}(A)$ such that $t_c^A(Q) > 0$ and $0 < \delta < \frac{t_c^A(Q)}{\sqrt{m}}$ where m is the number of elements in Q. By Lemma 3.3(a), we have $t_c^A(Q) \leqslant t_c^A(Q; H, H_{\mathcal{C}})$. Thus,

$$\sum_{x\in Q} \|x\cdot\zeta - \zeta\cdot x\|^2 \geqslant t_c^A(Q) > m\delta^2$$

for any $\zeta \in \mathfrak{S}_1(H_{\mathcal{C}}^\perp)$. Suppose that there exists $\xi \in V_H(Q,\delta) \cap H_{\mathcal{C}}^\perp$. Then $m\delta^2 < \sum_{x\in Q} \|x\cdot\xi - \xi\cdot x\|^2 < m\delta^2$ which is absurd.

(iii) ⇒ (iv). This is obvious.

(iv) ⇒ (i). Suppose on the contrary that $t_c^A(Q) = 0$. Then there exists $\xi \in \mathfrak{S}_1(\mathcal{H}_{\mathcal{C}}^\perp)$ with $\sum_{x\in Q} \|x\cdot\xi - \xi\cdot x\|^2 < \delta^2$ which gives the contradiction that $\xi \in V_{\mathcal{H}}(Q,\delta) \cap \mathcal{H}_{\mathcal{C}}^\perp$.

(d) If $t_u^A > 0$, then $t_s^A > 0$ and $t_c^A > 0$ (by Lemma 3.3(b)). Conversely, suppose that $t_u^A = 0$. Then by Lemma 3.2(c), there exists an almost central unit vector $(\xi_i)_{i\in I}$ for A in

$$\mathcal{H} \ominus \mathcal{H}^A = (\mathcal{H} \ominus \mathcal{H}_{\mathcal{C}}) \oplus (\mathcal{H}_{\mathcal{C}} \ominus \mathcal{H}^A).$$

Let $\eta_i \in \mathcal{H} \ominus \mathcal{H}_{\mathcal{C}}$ and $\zeta_i \in \mathcal{H}_{\mathcal{C}} \ominus \mathcal{H}^A$ be the corresponding components of ξ_i. Then either $\eta_i \nrightarrow 0$ or $\zeta_i \nrightarrow 0$. Therefore, by rescaling, there exists an almost central unit vector for A in either $\mathcal{H} \ominus \mathcal{H}_{\mathcal{C}}$ or $\mathcal{H}_{\mathcal{C}} \ominus \mathcal{H}^A = \mathcal{H}_{\mathcal{C}} \ominus \mathcal{H}_{\mathcal{C}}^A$. In the first case we have $t_c^A = 0$ (by Lemma 3.2(c)). In the second case we have $t_s^A \leqslant \sup_{Q\in\mathfrak{F}(A)} t^A(Q; \mathcal{H}_{\mathcal{C}}, \mathcal{H}_{\mathcal{C}}^A) = 0$ (by Lemma 3.3(a), Lemma 3.1 and Lemma 3.2(c)). □

Part (a) of the above theorem tells us that in order to show that A has strong property (T), it suffices to verify a weaker condition than that of Definition 2.1 for just the universal bimodule $\mathcal{H}$.

Remark 3.5. (a) The argument of Theorem 3.4(a), together with Lemma 3.3(a), tell us that for any $Q \in \mathfrak{F}(A)$, $\delta > 0$ and $H \in \mathrm{Bimod}^*(A)$, if $\xi \in V_H(Q,\delta)$, then

$$t_u^A(Q)\|\xi - P_H^A(\xi)\| < \delta\sqrt{m}$$

(where m is the number of elements in Q).

(b) The argument of Theorem 3.4(c), together with Lemma 3.3(a), tell us that if $t_c^A(Q) > 0$, for any $Q \in \mathfrak{F}(A)$, $H \in \mathrm{Bimod}^*(A)$ and $\delta \in (0, \frac{t_c^A(Q)}{\sqrt{m}})$, we have $V_H(Q,\delta) \cap H_{\mathcal{C}}^{\perp} = \emptyset$.

(c) The gap between property (T) and strong property (T) is represented by the gap between t_c^A and t_u^A or equivalently between $\mathcal{H}^A$ and $\mathcal{H}_{\mathcal{C}}$. Note that in the case of a locally compact group G, such a gap does not exist because the set of G-invariant vectors defines a subrepresentation.

By Theorem 3.4 and Lemma 3.2(c), we have the following corollary.

Corollary 3.6. (a) *A has property (T) (respectively, strong property (T)) if and only if there is no almost central unit vector for A in $\mathcal{H}_{\mathcal{C}}^{\perp}$ (respectively, in $(\mathcal{H}^A)^{\perp}$).*

(b) *A has strong property (T) if and only if A has property (T) and $t_s^A > 0$.*

Note that one can also obtain part (b) of the above corollary by using a similar argument as that of [7, Proposition 1].

4. Some permanence properties

In this section, we study the permanence properties for property (T) and strong property (T). First of all, we have the following lemma which implies that the quotient of any pair having property (T) (respectively, strong property (T)) will have the same property. Since the proof is direct, we will omit it.

Lemma 4.1. *Let A_1 and A_2 be two unital C^*-algebras and let $B_1 \subseteq A_1$ and $B_2 \subseteq A_1$ be C^*-subalgebras containing the identities of A_1 and A_2 respectively. Suppose that $\varphi : A_1 \to A_2$ is a unital $*$-homomorphism such that $B_2 \subseteq \varphi(B_1)$. If (A_1, B_1) has property (T) (respectively, strong property (T)), then so does (A_2, B_2).*

Lemma 4.2. *Let A_1, A_2, B_1 and B_2 be the same as in Lemma 4.1. If both (A_1, B_1) and (A_2, B_2) have property (T) (respectively, strong property (T)), then so does $(A_1 \oplus A_2, B_1 \oplus B_2)$.*

Proof. The statement for property (T) is well known and we will only show the case for strong property (T). Suppose that $H \in \mathrm{Bimod}^*(A_1 \oplus A_2)$ and $e = (1_{A_1}, 0) \in A_1 \oplus A_2$. Then $H = \bigoplus_{k,l=1}^2 H_{kl}$ where H_{kl} is a non-degenerate Hilbert A_k-A_l-bimodule. Suppose that $(\xi_i)_{i\in I}$ is an almost central unit vector in H for $A_1 \oplus A_2$ and $\xi_i = \sum_{k,l=1}^2 \xi_i^{kl}$ where $\xi_i^{kl} \in H_{kl}$. Then

$$\|\xi_i^{12}\|^2 + \|\xi_i^{21}\|^2 = \|e\cdot\xi_i - \xi_i\cdot e\|^2 \to 0.$$

If $\|\xi_i^{22}\| \to 0$, then we can assume that $\|\xi_i^{11}\| > 1/2$ $(i \in I)$, and $(\frac{\xi_i^{11}}{\|\xi_i^{11}\|})_{i\in I}$ is an almost central unit vector for A_1 in H_{11}. In this case for any $\epsilon > 0$, there is $i_0 \in I$ such that $\|\xi_i^{11} - P_{H_{11}}^{B_1}(\xi_i^{11})\| < \frac{\epsilon}{2}$ $(i \geqslant i_0)$, which implies that

$$\|\xi_j - P_H^{B_1\oplus B_2}(\xi_j)\| \leqslant \sqrt{\|\xi_i^{11} - P_{H_{11}}^{B_1}(\xi_i^{11})\|^2 + \|\xi_i^{12}\|^2 + \|\xi_i^{21}\|^2 + \|\xi_i^{22}\|^2} < \epsilon$$

when j is large enough. The same conclusion holds if $\|\xi_i^{11}\| \to 0$. We consider now the case when $\|\xi_i^{11}\| \nrightarrow 0$ and $\|\xi_i^{22}\| \nrightarrow 0$. There exist a constant $\kappa > 0$ as well as subnets $(\xi_{i_k}^{11})_{k\in J_1}$ and $(\xi_{i_l}^{22})_{l\in J_2}$ such that $\|\xi_{i_k}^{11}\|, \|\xi_{i_l}^{22}\| \geqslant \kappa$ for every $k \in J_1$ and $l \in J_2$. One can show easily that $(\frac{\xi_{i_k}^{11}}{\|\xi_{i_k}^{11}\|})_{k\in J_1}$ and $(\frac{\xi_{i_l}^{22}}{\|\xi_{i_l}^{22}\|})_{l\in J_2}$ are almost central unit vectors for A_1 and A_2 in H_{11} and H_{22} respectively. Thus, for any $\epsilon > 0$, one can find $i_0 \in I$ such that

$$\left\|\xi_{i_0}^{11} - P_{H_{11}}^{B_1}\left(\xi_{i_0}^{11}\right)\right\| < \frac{\|\xi_{i_0}^{11}\|\epsilon}{2}, \qquad \left\|\xi_{i_0}^{22} - P_{H_{22}}^{B_2}\left(\xi_{i_0}^{22}\right)\right\| < \frac{\|\xi_{i_0}^{22}\|\epsilon}{2}$$

and $\|\xi_{i_0}^{12}\|^2 + \|\xi_{i_0}^{21}\|^2 < \frac{\epsilon^2}{2}$. Consequently,

$$\left\|\xi_{i_0} - P_H^{B_1\oplus B_2}(\xi_{i_0})\right\| \leqslant \sqrt{\left\|\xi_{i_0}^{11} - P_{H_{11}}^{B_1}\left(\xi_{i_0}^{11}\right)\right\|^2 + \left\|\xi_{i_0}^{22} - P_{H_{11}}^{B_1}\left(\xi_{i_0}^{11}\right)\right\|^2 + \left\|\xi_{i_0}^{12}\right\|^2 + \left\|\xi_{i_0}^{21}\right\|^2} < \epsilon.$$

In any case, $(A_1 \oplus A_2, B_1 \oplus B_2)$ has strong property (T) because of Proposition 2.4(b)(iii). □

Our next task is to consider tensor products and crossed products. Let us first recall the following useful terminology of co-rigidity from [3, Remark 19]. We will also introduce a stronger version of co-rigidity corresponding to strong property (T).

Definition 4.3. The pair (A, B) is said to be

(a) *co-rigid* if there exists $(Q, \beta) \in \mathfrak{F}(A) \times \mathbb{R}_+$ such that for any $H \in \mathrm{Bimod}^*(A)$ with $V_H(Q, \beta) \cap H^B \neq \emptyset$, we have $H^A \neq \{0\}$,
(b) *strongly co-rigid* if for any $\gamma > 0$, there exists $(Q, \delta) \in \mathfrak{F}(A) \times \mathbb{R}_+$ such that for any $H \in \mathrm{Bimod}^*(A)$ and any $\xi \in V_H(Q, \delta) \cap H^B$, we have $\|\xi - P_H^A(\xi)\| < \gamma$.

The idea of the following result comes from [1, 2.3].

Proposition 4.4. *Suppose that B has strong property (T).*

(a) *A has property (T) if and only if (A, B) is co-rigid.*
(b) *A has strong property (T) if and only if (A, B) is strongly co-rigid.*

Proof. (a) The sufficiency is clear and we will only show the necessity. Let $(Q, r) \in \mathfrak{F}(A) \times \mathbb{R}_+$ be the pair satisfying the condition in Definition 4.3(a). Suppose that $(F, s) \in \mathfrak{F}(B) \times \mathbb{R}_+$ is the strong Kazhdan's pair for (B, B, α) where $\alpha = \min\{\frac{r}{8M}, \frac{1}{2}\}$ and $M = \max\{\|a\|: a \in Q\}$. Put $E = Q \cup F$ and $t = \min\{\frac{r}{4}, s\}$. Assume that $H \in \mathrm{Bimod}^*(A)$ with $\xi \in V_H(E, t)$. As $\xi \in V_H(F, s)$, one has $\|\xi - P_H^B(\xi)\| < \alpha$ and $\|P_H^B(\xi)\| \geqslant \frac{1}{2}$. If $\eta = \frac{P_H^B(\xi)}{\|P_H^B(\xi)\|}$, then we have, for any $a \in Q$,

$$\|a \cdot \eta - \eta \cdot a\| \leqslant \frac{\|a \cdot \xi - \xi \cdot a\| + 2\|a\|\|\xi - P_H^B(\xi)\|}{\|P_H^B(\xi)\|} < 2t + \frac{r\|a\|}{2M} \leqslant r.$$

Thus, $\eta \in V_H(Q, r) \cap H^B$ and $H^A \neq \{0\}$.

(b) Again, we only need to show the necessity. For any $\epsilon > 0$, let $(Q, r) \in \mathfrak{F}(A) \times \mathbb{R}_+$ be the pair satisfying Definition 4.3(b) for $\gamma = \frac{\epsilon}{2}$. Take a strong Kazhdan's pair $(F, s) \in \mathfrak{F}(B) \times \mathbb{R}_+$ for (B, B, α) where $\alpha = \min\{\frac{r}{8M}, \frac{1}{2}, \frac{\epsilon}{2}\}$ and $M = \max\{\|a\|: a \in Q\}$. If E and t are as in the argument of part (a), then for any $\xi \in V_H(E, t)$, we have $\eta = \frac{P_H^B(\xi)}{\|P_H^B(\xi)\|} \in V_H(Q, r) \cap H^B$ which implies that

$$\left\| P_H^B(\xi) - P_H^A(\xi) \right\| < \frac{\|P_H^B(\xi)\|\epsilon}{2} \leqslant \frac{\epsilon}{2}$$

(note that $H^A \subseteq H^B$). Since $\|\xi - P_H^B(\xi)\| < \frac{\epsilon}{2}$ as well, we see that $(E, t) \in \mathfrak{F}(A) \times \mathbb{R}_+$ is a strong Kazhdan's pair for (A, B, ϵ). □

We do not know whether B having property (T) and (A, B) being co-rigid will imply that A has property (T). If it is the case, then the statement in Theorem 4.5(a) below concerning property (T) can be improved and a similar statement as Theorem 4.6 below for property (T) will also hold.

The first application of the above proposition is the following theorem. Notice that unlike the case of type II_1-factors (see [1, 2.5]), the fact that $B \otimes_{\max} D$ having property (T) (or strong property (T)) will not imply both B and D to have property (T) (respectively, strong property (T)), but at least one of them have property (T) (respectively, strong property (T)).

Theorem 4.5. *Let B and D be two unital C^*-algebras, $A = B \otimes_{\max} D$ and $A_0 = B \otimes_{\min} D$.*

(a) *If B has strong property (T) and D has property (T) (respectively, strong property (T)), then A has property (T) (respectively, strong property (T)).*
(b) *If there is no almost central unit vector for D in any $K \in$ Bimod$^*(D)$, then A has strong property (T).*
(c) *Suppose that there exists an almost central unit vector $(\eta_j)_{j \in J}$ for D in some $K \in$ Bimod$^*(D)$. If A_0 has property (T) (respectively, strong property (T)), then so does B.*
(d) *If A_0 has property (T) (respectively, strong property (T)), then either B or D has property (T) (respectively, strong property (T)).*

Proof. (a) We show the statement for strong property (T) first. Suppose that $\alpha > 0$ and $(F, r) \in \mathfrak{F}(D) \times \mathbb{R}_+$ is the strong Kazhdan's pair for (D, D, α). Let $Q := 1 \otimes F \in \mathfrak{F}(A)$ and $H \in$ Bimod$^*(A)$. If $H^B \neq \{0\}$, then $H^B \in$ Bimod$^*(D)$ under the canonical multiplications. For any $\xi \in V_H(Q, r) \cap H^B = V_{H^B}(F, r)$, we have

$$\left\|\xi - P_H^A(\xi)\right\| \leqslant \left\|\xi - P_{H^B}^D(\xi)\right\| < \gamma$$

(note that $(H^B)^D = H^A$). Thus, (A, B) is strongly co-rigid and we can apply Proposition 4.4(b). The proof for the case of property (T) is similar.

(b) In this case there is no almost central unit vector for A in any $H \in$ Bimod$^*(A)$ and we can apply Proposition 2.4.

(c) We will establish the statement for strong property (T) and the statement for property (T) is similar (and easier). Suppose $H \in$ Bimod$^*(B)$ and there exists an almost central unit vector

$(\xi_i)_{i\in I}$ for B in H. Then $(\xi_i \otimes \eta_j)_{(i,j)\in I\times J}$ is an almost central unit vector for A_0 in $H \otimes K$. For any $\epsilon > 0$, there exists, by Proposition 2.4(b)(ii), $(i_0, j_0) \in I \times J$ such that

$$\left\| \xi_{i_0} \otimes \eta_{j_0} - P^{A_0}_{H\otimes K}(\xi_{i_0} \otimes \eta_{j_0}) \right\| < \epsilon.$$

Let $\zeta = P^{A_0}_{H\otimes K}(\xi_{i_0} \otimes \eta_{j_0}) \in (H \otimes K)^{A_0}$ and $\varphi \in K^*$ be defined by $\varphi(\eta') = \langle \eta', \eta_{j_0} \rangle$. For any $b \in B$ and any $\xi \in H$, we have

$$\big\langle b \cdot (\mathrm{id} \otimes \varphi)(\zeta), \xi \big\rangle = \big\langle (b \otimes 1) \cdot \zeta, \xi \otimes \eta_{j_0} \big\rangle = \big\langle \zeta \cdot (b \otimes 1), \xi \otimes \eta_{j_0} \big\rangle = \big\langle (\mathrm{id} \otimes \varphi)(\zeta) \cdot b, \xi \big\rangle.$$

Therefore, $(\mathrm{id} \otimes \varphi)(\zeta) \in H^B$ and

$$\left\| \xi_{i_0} - (\mathrm{id} \otimes \varphi)(\zeta) \right\| = \left\| (\mathrm{id} \otimes \varphi)\big(\xi_{i_0} \otimes \eta_{j_0} - P^{A}_{H\otimes K}(\xi_{i_0} \otimes \eta_{j_0})\big) \right\| < \epsilon.$$

This shows that B has strong property (T) (by Proposition 2.4(b)(iii)).

(d) If there is no almost central unit vector for D in any $K \in \mathrm{Bimod}^*(D)$, then D has strong property (T) (by definition). Otherwise, we can apply part (b). □

Next, we will consider crossed product of C^*-algebras by actions of discrete groups. The idea of which comes from [1, 4.6]. Again, unlike the case of type II_1-factors, even if $B \times_\alpha \Gamma$ has strong property (T), this will not imply that Γ has property (T) (notice that if α is trivial and any element in $\mathrm{Bimod}^*(B)$ does not contain an almost central unit vector, then $B \times_\alpha \Gamma$ will have strong property (T) whether or not Γ have property (T)).

Theorem 4.6. *Let B be a unital C^*-algebra with an action α by a discrete group Γ and $A = B \times_\alpha \Gamma$. If Γ has property (T), then (A, B) is strongly co-rigid. Consequently if B has strong property (T) and Γ has property (T), then A has strong property (T) (and so does $B \times_{\alpha,r} \Gamma$).*

Proof. As Γ has property (T), for any $\epsilon > 0$, there exists $(F, \delta) \in \mathfrak{F}(\Gamma) \times \mathbb{R}_+$ such that for any $(K, \pi) \in \mathrm{Rep}(\Gamma)$ and any $\eta \in V_\pi(F, \delta)$, one has $\|\eta - P^\Gamma_K(\eta)\| < \epsilon$ (by [10, Theorem 1.2(b2)]). Let $\mu : B \to A$ and $u : \Gamma \to A$ be the canonical maps. For any $H \in \mathrm{Bimod}^*(A)$, we define a representation $\pi : \Gamma \to \mathcal{L}(H^B)$ by $\pi(t)\xi = u_t \cdot \xi \cdot u_t^*$ $(t \in \Gamma, \xi \in H^B)$ which is well defined because for any $b \in B$,

$$\mu(b) \cdot \pi(t)\xi = u_t \mu\big(\alpha_{t^{-1}}(b)\big) \cdot \xi \cdot u_t^* = u_t \cdot \xi \cdot \mu\big(\alpha_{t^{-1}}(b)\big) u_t^* = \pi(t)\xi \cdot \mu(b).$$

Moreover, it is easy to check that $(H^B)^\Gamma = H^A$. Thus, if $\eta \in V_H(u(F), \delta) \cap H^B = V_\pi(F, \delta)$, then $\|\eta - P^A_H(\eta)\| = \|\eta - P^\Gamma_{H^B}(\eta)\| < \epsilon$. This shows that (A, B) is strongly co-rigid. The last statement follows from Proposition 4.4(b). □

5. Some examples of strong property (T)

Our first example is finite dimensional C^*-algebras. It is easy to see that any element in $\mathrm{Bimod}^*(M_n(\mathbb{C}))$ has a non-zero central vector and so $M_n(\mathbb{C})$ has property (T). In fact, $M_n(\mathbb{C})$ also has strong property (T) but a bit more argument is needed to establish this fact.

Example 5.1. $A = M_n(\mathbb{C})$ has a unique tracial state τ and hence $\mathcal{K} = M_\tau$. If $t_s^A = 0$, there exists an almost central unit vector in $(M_\tau^A)^\perp$ for A (by Lemma 3.2(c)). As $\mathfrak{S}_1((M_\tau^A)^\perp)$ is compact, this implies the existence of a central unit vector η for A in $(M_\tau^A)^\perp$ but such $\eta \in M_n(\mathbb{C})$ should be an element of $\mathbb{C}1 = M_\tau^A$ which is absurd. Now by Corollary 3.6(b) and Lemma 4.2, we see that any finite dimensional C^*-algebra has strong property (T).

By the argument of [3, Remark 17], we know that if A does not have tracial states, then there is no almost central unit vector for A in any $H \in \mathrm{Bimod}^*(A)$. This, together with Proposition 2.4(b), give the following result (cf. [3, Remark 17]).

Proposition 5.2. *If A has no tracial states, then A has strong property (T).*

If H is any Hilbert space, $\mathcal{L}(H)$ has strong property (T) (because of Example 5.1 and Proposition 5.2) and so if $B \subseteq \mathcal{L}(H)$ is any unital C^*-subalgebra, then $(\mathcal{L}(H), B)$ has strong property (T). On the other hand, Proposition 5.2, together with Example 5.1, Lemma 4.2 and [5, 5.1], imply the following result.

Proposition 5.3. *Suppose that A is separable and amenable. Then A has strong property (T) if and only if $A = B \oplus C$ where B is finite dimensional and C has no tracial states.*

We end this section with the following analogue of [3, Theorem 7].

Proposition 5.4. *Let Γ be a countable discrete group and Λ be a subgroup of Γ. The following statements are equivalent.*

(i) (Γ, Λ) *has property* (T).
(ii) $(C^*(\Gamma), C^*(\Lambda))$ *has strong property* (T).
(iii) $(C^*(\Gamma), C^*(\Lambda))$ *has property* (T).
(iv) $(C_r^*(\Gamma), C_r^*(\Lambda))$ *has strong property* (T).
(v) $(C_r^*(\Gamma), C_r^*(\Lambda))$ *has property* (T).

Proof. It is clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) (by Lemma 4.1). The implication (v) $\Rightarrow$ (i) was proved in [3]. It remains to show that (i) $\Rightarrow$ (ii). As (Γ, Λ) has property (T), for any $\alpha > 0$, there exists $(Q, \beta) \in \mathfrak{F}(\Gamma) \times \mathbb{R}_+$ such that for any unitary representation $\pi : \Gamma \to \mathcal{L}(K)$ and any $\xi \in V_\pi(Q, \beta)$, one has $\|\xi - P_K^\Lambda(\xi)\| < \alpha$ (by [10, Theorem 1.2(b2)]). Consider $\Gamma \subseteq C^*(\Gamma)$. For any $H \in \mathrm{Bimod}^*(C^*(\Gamma))$, one can define a unitary representation $\pi_H : \Gamma \to \mathcal{L}(H)$ by $\pi_H(t)\eta = t \cdot \eta \cdot t^{-1}$ $(\eta \in H)$. If $\xi \in V_H(Q, \beta) \subseteq V_\pi(Q, \beta)$, we have $\|\xi - P_H^{C^*(\Lambda)}(\xi)\| = \|\xi - P_H^\Lambda(\xi)\| < \alpha$ as required. □

6. Property (T) for the unitary group of a C^*-algebra

It was shown in [13] that a unital C^*-algebra is amenable if and only if its unitary group under the weak topology is amenable. Motivated by this result as well as by Lemma 2.2, we study in this section the relation between property (T) and strong property (T) of a unital C^*-algebra A and certain (T)-type properties of its unitary group.

Remark 6.1. We denote by $\Phi_A : U(A) \to U(A^{Dou})$ the group homomorphism $u \mapsto u \otimes (u^*)^{\mathrm{op}}$. Note that any $K \in \mathrm{Bimod}^*(A)$ defines a non-degenerate $*$-representation μ_K of A^{Dou} and $\pi_K = \mu_K \circ \Phi_A$ is a unitary representation of $U(A)$.

The proof of the following result is more or less the same as the argument for the equivalence of (b2) and (b3) in [10].

Proposition 6.2. (a) (A, B) *has property* (T) *if and only if for any* $K \in \mathrm{Bimod}^*(A)$, *the weak containment of the trivial representation* $1_{U(A)}$ *of* $U(A)$ *in the unitary representation* π_K *will imply that* $\pi_K|_{U(B)}$ *contains the trivial representation* $1_{U(B)}$ *of* $U(B)$.

(b) (A, B) *has strong property* (T) *if and only if for any net* $(f_i)_{i\in I}$ *in* $S(A^{Dou})$ *with* $(f_i \circ \Phi_A)_{i\in I}$ *converges pointwisely to* $1_{U(A)}$, *one has* $(f_i \circ \Phi_A|_{U(B)})_{i\in I}$ *converges uniformly to* $1_{U(B)}$ *on* $U(B)$.

Proof. (a) Note that for any $K \in \mathrm{Bimod}^*(A)$ and any $\xi \in \mathfrak{S}_1(K)$, we have

$$\|t \cdot \xi - \xi \cdot t\|^2 = 2 - 2\,\mathrm{Re}\langle \pi_K(t)\xi, \xi\rangle \quad \big(t \in U(A)\big).$$

Now, this follows almost directly from Proposition 2.4(a).

(b) ($\Rightarrow$). If (H_i, μ_i, ξ_i) is the GNS representation of f_i, then $H_i \in \mathrm{Bimod}^*(A)$. For any $\epsilon > 0$, let $(Q, \beta) \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ be a strong Kazhdan's pair for $(A, B, \epsilon/2)$ (see Lemma 2.2). By the assumption of (f_i), there exists $i_0 \in I$ such that for any $i \geqslant i_0$, we have $\sup_{u\in Q} |f_i(\Phi_A(u)) - 1| < \beta^2/2$. Thus,

$$\|u \cdot \xi_i - \xi_i \cdot u\| = \big\|\mu_i\big(u \otimes (u^*)^{\mathrm{op}}\big)\xi_i - \xi_i\big\| = \sqrt{2\,\mathrm{Re}\big(1 - f_i\big(\Phi_A(u)\big)\big)} < \beta$$

for any $u \in Q$ and so $\xi_i \in V_{H_i}(Q, \beta)$. Therefore, $\|\xi_i - P^B_{H_i}(\xi_i)\| < \epsilon/2$ and for any $v \in U(B)$,

$$\big|f_i\big(\Phi_A(v)\big) - 1\big| = \big|\langle v \cdot \xi_i \cdot v^* - \xi_i, \xi_i\rangle\big| \leqslant \big\|v \cdot \xi_i \cdot v^* - v \cdot P^B_{H_i}(\xi_i) \cdot v^*\big\| + \big\|P^B_{H_i}(\xi_i) - \xi_i\big\| < \epsilon.$$

($\Leftarrow$). Suppose on the contrary that (A, B) does not have strong property (T). Let $I := \mathfrak{F}(U(A)) \times \mathbb{R}_+$. There exists $\alpha_0 > 0$ such that for any $i = (F, \epsilon) \in I$, one can find $H_i \in \mathrm{Bimod}^*(A)$ and $\xi_i \in V_{H_i}(F, \epsilon)$ with $\|\xi_i - P^B_{H_i}(\xi_i)\| > \alpha_0$. For every such H_i, let π_{H_i} be as in Remark 6.1. By [10, 2.2], we see that there exists $v_i \in U(B)$ such that

$$\big\|\pi_{H_i}(v_i)\xi_i - \xi_i\big\| > \alpha_0 \tag{6.1}$$

(note that $H_i^B = H_i^{U(B)}$). On the other hand, for any $i \in I$, we define $f_i \in S(A^{Dou})$ by

$$f_i\big(a \otimes b^{\mathrm{op}}\big) = \langle a \cdot \xi_i \cdot b, \xi_i\rangle \quad (a, b \in A).$$

For any $i = (Q, \beta) \in I$, we have $\xi_i \in V_{H_i}(Q, \beta)$ and thus,

$$\sup_{u\in Q}\big\|\pi_{H_i}(u)\xi_i - \xi_i\big\| < \beta.$$

This shows that $f_i \circ \Phi_A$ converges pointwisely to $1_{U(A)}$. Therefore, by the hypothesis, $f_i \circ \Phi_A|_{U(B)}$ converges uniformly to $1_{U(B)}$ on $U(B)$ which contradicts with (6.1) (since it implies that $\mathrm{Re}(1 - \varphi_i(v_i)) > \alpha_0^2/2$ for any $i \in I$). □

We will show in the following that A has property (T) (respectively, strong property (T)) if and only if $U(A)$ has some (T)-type property. Note that the argument of (iii) ⇒ (i) in part (b) of the following result is adapted from that of Proposition 16 of Chapter 1 of [8].

Theorem 6.3. *Let* $\mathcal{SB}(U(A)) := \{(\mu, H) \in \mathrm{Rep}(U(A))\colon\ (\mu, H) \leqslant (\pi_K, K)$ *for some* $K \in \mathrm{Bimod}^*(A)\}$ (*for* $\mathrm{Rep}(U(A))$, *we regard* $U(A)$ *as a discrete group*).

(a) *A has property (T) if and only if there exists $(F, \epsilon) \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ such that for any $K \in \mathrm{Bimod}^*(A)$ with $V_{\pi_K}(F, \epsilon) \neq \emptyset$, we have $K^{U(A)} \neq \{0\}$.*
(b) *The following statements are equivalent.*
  (i) *A has strong property (T).*
  (ii) *There exists $(F, \epsilon) \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ such that for any $(\mu, H) \in \mathcal{SB}(U(A))$ and $\xi \in V_\mu(F, \epsilon)$, we have $P_H^{U(A)}(\xi) \neq 0$.*
  (iii) *There exists $(F, \epsilon) \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ such that for any $(\mu, H) \in \mathcal{SB}(U(A))$ with $V_\mu(F, \epsilon) \neq \emptyset$, we have $H^{U(A)} \neq \{0\}$.*

Proof. (a) This follows from Lemma 2.2.

(b) (i) ⇒ (ii). Let $(F, \epsilon) \in \mathfrak{F}(U(A)) \times \mathbb{R}_+$ be a strong Kazhdan's pair for $(A, A, 1/2)$ (see Lemma 2.2). Suppose that $(\mu, H) \in \mathcal{SB}(U(A))$ and $K \in \mathrm{Bimod}^*(A)$ such that $(\mu, H) \leqslant (\pi_K, K)$. If $\xi \in V_\mu(F, \epsilon) \subseteq V_K(F, \epsilon)$, then

$$\left\|\xi - P_K^A(\xi)\right\| < 1/2.$$

As $K = H \oplus H^\perp$ and both H and $H^\perp$ are invariant under π_K, we have $K^A = K^{U(A)} = H^{U(A)} \oplus (H^\perp)^{U(A)}$. As $\xi \in H$, we know that $P_{H^\perp}^{U(A)}(\xi) = 0$ and $P_H^{U(A)}(\xi) = P_K^A(\xi) \neq 0$.

(ii) ⇒ (iii). This implication is clear.

(iii) ⇒ (i). Let (F, ϵ) be the pair satisfying the hypothesis. For any $2 \geqslant \alpha > 0$, we take $\beta := \frac{\alpha\epsilon}{2}$. Suppose that $K \in \mathrm{Bimod}^*(A)$, $H = (K^A)^\perp = (K^{U(A)})^\perp$ and $\mu = \pi_K|_H$. Then $(\mu, H) \in \mathcal{SB}(U(A))$. For any $\xi \in V_K(F, \beta)$, we have $\xi = P_K^A(\xi) + \eta$ where $\eta \in H$. Assume that $\epsilon\|\eta\| > \beta$ and put $\zeta := \eta/\|\eta\|$. As

$$\left\|\mu(v)\zeta - \zeta\right\| = \left\|\pi_K(v)\xi - \xi\right\|/\|\eta\| < \beta/\|\eta\| < \epsilon \quad (v \in F),$$

we have $\zeta \in V_\mu(F, \epsilon)$. Hence by the hypothesis, H contains a non-zero μ-invariant vector which contradicts the definition of H. Therefore, we must have $\|\eta\| \leqslant \beta/\epsilon$ and hence $\|\xi - P_K^A(\xi)\| < \alpha$. □

Remark 6.4. (a) Let $\mathcal{SB}_0(U(A)) = \{(\mu, H) \in \mathrm{Rep}(U(A))\colon\ (\mu, H) \leqslant (\pi_{\mathcal{H}}, \mathcal{H})\}$. Then using the same argument, one can show that a similar statement as Theorem 6.3(b) holds when $\mathcal{SB}(U(A))$ is replaced by $\mathcal{SB}_0(U(A))$.

(b) Note that in the proof for (i) ⇒ (ii) in Theorem 6.3(b) we only need the existence of a strong Kazhdan's pair for $(A, A, 1/2)$.

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