MATHEMATICS

A THEOREM ABOUT THE DETERMINATION OF A CERTAIN BEST CONSTANT IN THE APPROXIMATION BY SOME LINEAR OPERATORS

BY

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1. *Introduction*

By $C[0, 1]$ we denote the set of all real-valued functions defined and continuous on the closed interval $[0, 1]$ of the real axis. We define P to be a fixed set of parameters containing at least the following elements:

$$(1)\qquad \begin{cases} n: \text{ a positive integer;} \\ q_k = q_k(x)\ (k=0, 1, \dots n):\ q_k \in C[0, 1],\ \sum_{k=0}^{n} q_k \equiv 1; \\ \xi_k (k=0, 1, \dots n):\ \xi_k \in [0, 1],\ \xi_k < \xi_{k+1} \quad (k=0, 1, \dots, n-1). \end{cases}$$

Let L be the operator on $C[0, 1]$, with respect to P, defined by

$$(2)\qquad Lf = L\{f(t); x\} = \sum_{k=0}^{n} f(\xi_k)\, q_k(x)$$

for each $f \in C[0, 1]$. Obviously, L is a linear operator, mapping $C[0, 1]$ into itself, with the property that

$$(3)\qquad L1 \equiv 1.$$

In order to derive estimations for the difference $f - Lf$, which are uniform in x and valid for arbitrary $f \in C[0, 1]$, we introduce the norm $\|g\|$ of a bounded function g defined on $[0, 1]$ by

$$(4)\qquad \|g\| = \sup_{x \in [0, 1]} |g(x)|$$

as well as the modulus of continuity $\omega(\delta)$ of a function $f \in C[0, 1]$ for arbitrarily chosen but fixed $\delta > 0$ by

$$(5)\qquad \omega(\delta) = \max_{\substack{|x-y| \leq \delta \\ x, y \in [0, 1]}} |f(x) - f(y)|.$$

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In this paper we derive a method for determining the quantity $\varkappa$ such that

$$\varkappa = \inf K, \tag{6}$$

i.e. the infimum of all numbers K, independent of f, for which

$$\|f - Lf\| \leqslant K\omega(\delta) \tag{7}$$

holds for each $f \in C[0, 1]$, with δ a fixed positive number. From (3) we see that the case of f being constant on $[0, 1]$ needs no investigation, since $c - Lc = 0$ for any constant c. In the following we exclude these constant functions. Then – for arbitrarily chosen but fixed $\delta > 0$ – we may and do restrict ourselves without loss of generality to the functions f of the subset $C_1[0, 1]$ of $C[0, 1]$, with the property that for each $f \in C_1[0, 1]$

$$\omega(\delta) = 1. \tag{8}$$

Consequently the following problem is equivalent to the one formulated by means of (6) and (7):

Determine the quantity $\varkappa$ *such that*

$$\varkappa = \sup_{f \in C_1[0, 1]} \|f - Lf\|. \tag{9}$$

2. *A theorem and a method for the determination of* $\varkappa$ *in* (9).

Theorem 1. *Let* δ *with respect to* ξ_k $(k = 0, 1, \ldots, n)$ *in* (1) *be a fixed number with the property that*

$$\delta \geqslant \max_{k=0, 1, \ldots, n-1} (\xi_{k+1} - \xi_k),\ \xi_0,\ 1 - \xi_n; \tag{10}$$

then for each fixed $\bar{x} \in [0, 1]$ *there exists a function* $\psi \in C_1[0, 1]$ *with* $\psi(\bar{x}) = 0$, *such that*

$$\max_{f \in C_1[0, 1]} |f(\bar{x}) - L\{f(t); \bar{x}\}| = L\{\psi(t); \bar{x}\}. \tag{11}$$

The proof of this theorem will be given by means of three lemmas which will be stated and proved in 3.

As a result of theorem 1 we are able to derive a method for determining the quantity $\varkappa$ as defined in (9). This method consists of the following two steps (i) and (ii):

(i) *For all fixed* $\bar{x} \in [0, 1]$ *determine*

$$K(\bar{x}) = L\{\psi(t); \bar{x}\}, \tag{12}$$

the right hand being defined by (11).

Then $K(x)$ for $x \in [0, 1]$ is a bounded function for which

(13) $$\max_{f \in C_1[0,1]} |f(x) - L\{f(t); x\}| = K(x)$$

holds for each $x \in [0, 1]$.

(ii) *Calculate, according to* (4)

(14) $$\varkappa = \|K\|.$$

Then from (11) and (12) it follows that $|f(x) - L\{f(t); x\}| \leqslant K(x)$ for all $f \in C_1[0, 1]$ and each $x \in [0, 1]$, from which by (14) $\|f - Lf\| \leqslant \varkappa$ holds for all $f \in C_1[0, 1]$, and obviously

(15) $$\sup_{f \in C_1[0,1]} \|f - Lf\| \leqslant \varkappa.$$

Moreover, from (14) it follows that there exists a sequence $\{x_h\}$ with $x_h \in [0, 1]$ $(h = 0, 1, \ldots)$ such that $\lim_{h \to \infty} K(x_h) = \varkappa$. Also, for each fixed h by theorem 1 and definition (12) there exists a function $\psi_h \in C_1[0, 1]$, with $\psi_h(x_h) = 0$ such that

$$K(x_h) = L\{\psi_h(t); x_h\} = |\psi_h(x_h) - L\{\psi_h(t); x_h\}|.$$

Estimating we obtain

$$\varkappa = \lim_{h \to \infty} |\psi_h(x_h) - L\{\psi_h(t); x_h\}| \leqslant \lim_{h \to \infty} \|\psi_h - L\psi_h\| \leqslant$$
$$\leqslant \lim_{h \to \infty} \sup_{f \in C_1[0,1]} \|f - Lf\| = \sup_{f \in C_1[0,1]} \|f - Lf\|.$$

This last result together with (15) shows that $\varkappa$ from (14) is the one to be found in (9).

3. *Proof of theorem* 1.

Lemma 1. *Let n, m_k, m_{kl} $(k, l = 0, 1, \ldots, n)$ be non-negative integers, and let q_k $(k = 0, 1, \ldots, n)$ be real numbers such that*

(16) $$\sum_{k=0}^{n} q_k = Q > 0.$$

Let A be the set of all $(n+1)$-tuples $\{\alpha_k\}$ of real numbers α_k $(k = 0, 1, \ldots, n)$ satisfying

(17) $$\left.\begin{array}{r} |\alpha_k| \leqslant m_k \\ |\alpha_k - \alpha_l| \leqslant m_{kl} \end{array}\right\} \quad (k, l = 0, 1, \ldots, n).$$

If $\{\beta_k\} \in A$ is such that

(18) $$\sum_{k=0}^{n} \beta_k q_k \geqslant \sum_{k=0}^{n} \alpha_k q_k$$

for all $\{\alpha_k\} \in A$, then β_k $(k = 0, 1, \ldots, n)$ is integer.

Proof. The determination of an $(n+1)$-tuple $\{\beta_k\} \in A$ which satisfies (18) is a problem of linear programming which has at least one solution in a vertex $\{\beta_k\}$ of the $(n+1)$-dimensional simplex defined by (17). Then it immediately follows that

$$\beta_k = \sigma \,(\mathrm{mod}\ 1);\ 0 \leqslant \sigma < 1 \qquad (k=0, 1, \ldots, n).$$

Assume $\sigma \neq 0$, then it follows that

$$\beta_k = n_k + \sigma \text{ with } n_k \in \{-m_k, -m_k+1, \ldots, 0, \ldots, m_k - 1\} \qquad (k=0, 1, \ldots, n).$$

We define $\bar{\beta}_k = \beta_k + 1 - \sigma$ $(k=0, 1, \ldots, n)$ and see that $\bar{\beta}_k = n_k + 1$ so that $|\bar{\beta}_k| \leqslant m_k$ $(k=0, 1, \ldots, n)$. From this result, and $|\bar{\beta}_k - \bar{\beta}_l| = |\beta_k - \beta_l| \leqslant m_{kl}$ $(k, l=0, 1, \ldots, n)$ we conclude that $\{\bar{\beta}_k\} \in A$. Consequently we can write

$$\sum_{k=0}^{n} \bar{\beta}_k q_k = \sum_{k=0}^{n} (\beta_k + 1 - \sigma)\, q_k = \sum_{k=0}^{n} \beta_k q_k + (1-\sigma)\, Q > \sum_{k=0}^{n} \beta_k q_k,$$

in which the inequality follows from (16). So the assumption $\sigma \neq 0$ leads to a contradiction of (18), with which lemma 1 has been proved.

Lemma 2. *Let $\bar{x}$ be a fixed point of the interval* $[0, 1]$, *and let the quantities m_k, m_{kl} and q_k $(k, l=0, 1, \ldots, n)$ appearing in lemma* 1 *be chosen as follows*:

$$(19) \qquad \left\{\begin{aligned} q_k &= q_k(\bar{x}) \\ m_k &= 1 + \left]\frac{|\bar{x} - \xi_k|}{\delta}\right[\,^{2)} \\ m_{kl} &= 1 + \left]\frac{|\xi_k - \xi_l|}{\delta}\right[\end{aligned}\right\} \quad (k, l = 0, 1, \ldots, n),$$

with n, $q_k(x)$, ξ_k $(k=0, 1, \ldots, n)$ elements of the parameterset P defined in (1) *and δ satisfying* (10).

If $\psi(x)$ is a function defined on $[0, 1]$ *with the properties that*

$$(20) \qquad \left\{\begin{aligned} &\psi(\bar{x}) = 0 \\ &\psi(\xi_k) = \beta_k \qquad (k=0, 1, \ldots, n), \end{aligned}\right.$$

where the numbers β_k satisfy (18) *of lemma* 1, *then for each $f \in C_1[0, 1]$*

$$(21) \qquad |f(\bar{x}) - L\{f(t); \bar{x}\}| \leqslant L\{\psi(t); \bar{x}\},$$

L being the operator defined in (2).

Proof: From (1) and (2) it follows that for each $f \in C[0, 1]$

$$(22) \qquad f(\bar{x}) - L\{f(t); \bar{x}\} = \sum_{k=0}^{n} \{f(\bar{x}) - f(\xi_k)\}\, q_k(\bar{x}).$$

[2] For each real a, $]a[$ denotes the greatest integer less than a.

For each $f \in C_1[0, 1]$ the numbers $\{f(\bar{x})-f(\xi_k)\}$ $(k=0, 1, \ldots, n)$ satisfy

$$(23)[3] \quad \left\{\begin{array}{l} |f(\bar{x}) - f(\xi_k)| \leqslant 1 + \left]\dfrac{|\bar{x}-\xi_k|}{\delta}\right[= m_k \\ |f(\xi_k) - f(\xi_l)| \leqslant 1 + \left]\dfrac{|\xi_k-\xi_l|}{\delta}\right[= m_{kl} \end{array}\right\} \quad (k, l = 0, 1, \ldots, n),$$

the equal-signs following from (19). With β_k $(k=0, 1, \ldots, n)$ satisfying (17), from the assumptions of lemma 2 it follows

$$(24) \qquad \sum_{k=0}^{n} \{f(\bar{x})-f(\xi_k)\}\, q_k(\bar{x}) \leqslant \sum_{k=0}^{n} \beta_k\, q_k(\bar{x}),$$

the right hand of which by (20) is equal to $L\{\psi(t); \bar{x}\}$. Since (24) holds for each $f \in C_1[0, 1]$, it does for $-f$; consequently with (22) now (21) holds for each $f \in C_1[0, 1]$. This completes the proof of lemma 2.

Lemma 3. *There exists a function* $\psi \in C_1[0, 1]$ *which satisfies* (20), *such that lemma* 2 *holds.*

Proof: Let V_k (k a fixed non-negative integer) be a finite set of at east two points in $[0, 1]$, and δ a fixed non-negative number such that

$$(25) \quad \left\{\begin{array}{ll} \text{(i)} & a = \min \{z : z \in V_k\}, \\ \text{(ii)} & b = \max \{z : z \in V_k\}, \\ \text{(iii)} & c = \max \{(z_2 - z_1) : \{z_1, z_2\} \subset V_k,\ z_1 < z_2,\ (z_1, z_2) \cap V_k = \emptyset\}, \\ \text{(iv)} & \delta \geqslant \max \{a,\ 1-b,\ c\}. \end{array}\right.$$

Let ψ_k be a polygon on $[0, 1]$, of which the nodes are the points of V_k, with the following properties:

$$(26) \quad \left\{\begin{array}{ll} \text{(i)} & \psi_k(z) \textit{ integer for each } z \in V_k, \\ \text{(ii)} & |\psi_k(z_1) - \psi_k(z_2)| \leqslant 1 + \left]\dfrac{|z_1 - z_2|}{\delta}\right[\textit{ for each pair } \{z_1, z_2\} \subset V_k, \\ \text{(iii)} & \psi_k(x) = \psi_k\ (a) \textit{ for } x \leqslant a, \\ \text{(iv)} & \psi_k(x) = \psi_k\ (b) \textit{ for } x \geqslant b. \end{array}\right.$$

The modulus of continuity of ψ_k being $\omega(\delta)$, from the definition of ψ_k it follows that there is at least one pair $\{y_1, y_2\}$ of points in $[0, 1]$ such that

$$(27) \quad \left\{\begin{array}{ll} \text{(i)} & |\psi_k(y_1) - \psi_k(y_2)| = \omega(\delta), \\ \text{(ii)} & y_1 < y_2, \\ \text{(iii)} & \textit{for each pair } \{\bar{y}_1, \bar{y}_2\} \textit{ with } y_1 \leqslant \bar{y}_1 < \bar{y}_2 \leqslant y_2 \textit{ and} \\ & \qquad \bar{y}_2 - \bar{y}_1 < y_2 - y_1 : |\psi_k(\bar{y}_1) - \psi_k(\bar{y}_2)| < \omega(\delta), \\ \text{(iv)} & y_1 \in V_k \text{ or } y_2 \in V_k. \end{array}\right.$$

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[3]) By (5) $\omega(\delta_1) \leqslant (1 +]\delta_1/\delta[)\, \omega(\delta)$ for any non-negative δ and δ_1, and from (8): $\omega(\delta) = 1$.

We assume $y_2 \in V_k$; the case $y_1 \in V_k$ can be treated in a similar way. If also $y_1 \in V_k$, then from (26, ii) it follows that

$$|\psi_k(y_1) - \psi_k(y_2)| \leqslant 1 + \left]\frac{y_2 - y_1}{\delta}\right[= 1 \tag{28}$$

since $y_2 - y_1 \leqslant \delta$. If however $y_1 \notin V_k$, there are obviously the two possibilities

$$|\psi_k(y_1) - \psi_k(y_2)| \leqslant 1, \tag{29}$$

and

$$|\psi_k(y_1) - \psi_k(y_2)| > 1. \tag{30}$$

In the latter case we consider the pair $\{x_1, x_2\} \subset V_k$ such that $y_1 \in (x_1, x_2)$ and $(x_1, x_2) \cap V_k = \emptyset$. From (25), (26), (27) and (30) simple reasoning leads to some properties of ψ_k on $[x_1, y_2]$:

(i) $\psi_k(x_1) \neq \psi_k(x_2)$,

(ii) $y_2 - y_1 = \delta$,

(iii) $y_2 = \min \{z\colon z \in V_k,\ z > x_2,\ \psi_k(z) \neq \psi_k(x_2)\}$,

(iv) $|\psi_k(x_1) - \psi_k(y_2)| = 2$.

With $x_3 = \max \{z\colon z \in V_k,\ x_2 \leqslant z < y_2,\ \psi_k(z) = \psi_k(x_2)\}$ we conclude that essentially ψ_k on $[x_1, y_2]$ is as follows

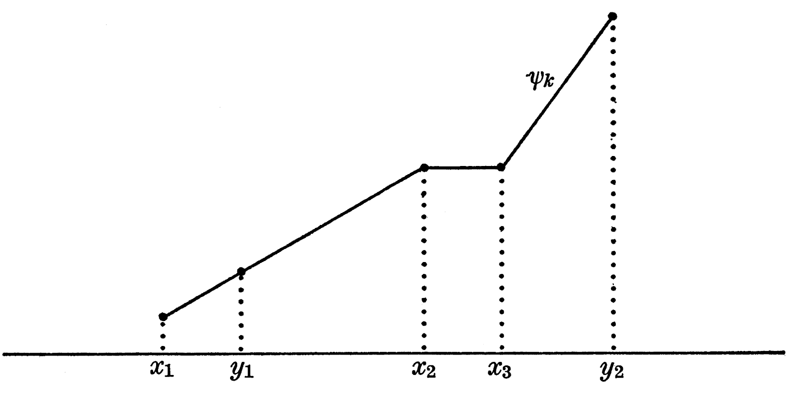

Fig. 1.

where as an example we assume $\psi_k(y_2) > \psi_k(y_1)$ and $x_3 \neq x_2$.

Let $y_3 = \max \{x_3, x_1 + \delta\}$, then consequently $x_2 \leqslant x_3 \leqslant y_3 < y_2$.

We define the continuous function ψ_{k+1} on $[0, 1]$ by

$$\begin{cases} \psi_{k+1}(x) = \psi_k(x) \ \textit{for} \ x \in [0, 1] \setminus \{(x_1, x_2) \cup (x_2, y_2)\}, \\ \qquad\qquad = \psi_k(x_2) \ \textit{for} \ x \in [y_1, y_3], \\ \qquad\qquad \textit{linear for} \ x \in [x_1, y_1) \ \textit{and} \ x \in (y_3, y_2]. \end{cases} \tag{31}$$

Taking for example ψ_k as partially described in figure 1, and $y_3 > x_3$ then by (31) ψ_{k+1} on $[x_1, y_2]$ is as follows

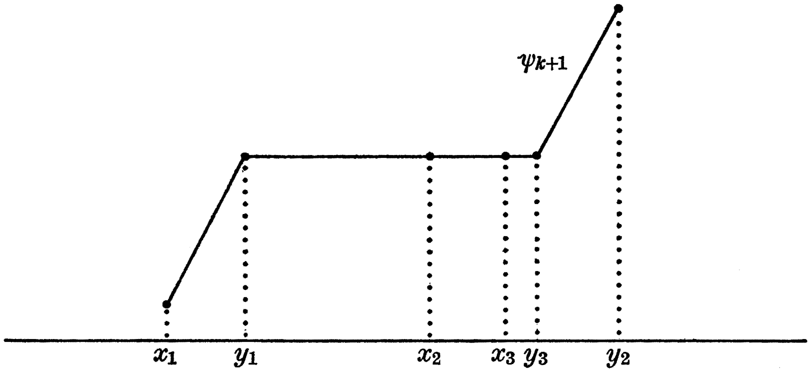

Fig. 2.

We define $V_{k+1} = V_k \cup \{y_1, y_3\}$, and write $(25)_l$, $(26)_l$ and so on when we are considering the case $k = l$. We assert that $(25)_{k+1}$ and $(26)_{k+1}$ hold; from these $(26, \text{ii})_{k+1}$ will be proved in the following, the other parts being obvious.

It is sufficient to prove

$$(32)\quad \begin{cases} \text{(i)} & |\psi_{k+1}(y_1) - \psi_{k+1}(z)| \leqslant 1 + \left]\dfrac{|y_1 - z|}{\delta}\right[, \\ \text{(ii)} & |\psi_{k+1}(y_3) - \psi_{k+1}(z)| \leqslant 1 + \left]\dfrac{|y_3 - z|}{\delta}\right[, \end{cases} \quad \textit{for each } z \in V_k,$$
$$\text{(iii)}\quad |\psi_{k+1}(y_1) - \psi_{k+1}(y_3)| \leqslant 1 + \left]\dfrac{|y_1 - y_3|}{\delta}\right[,$$

the proof of (32, i) being necessary, and that of (32, ii, iii) being necessary only if $y_3 = x_1 + \delta \neq x_3$, since otherwise $y_3 \in V_k$. The inequality (32, iii) trivially holds since $\psi_{k+1}(y_1) = \psi_{k+1}(y_3)$. We will prove (32, i) in the case of figure 2; (32, ii) and the case of $\psi_k(y_1) > \psi_k(y_2)$ can be treated analogous.

For each $z \in V_k$ with $z \geqslant x_2$

$$|\psi_{k+1}(y_1) - \psi_{k+1}(z)| = |\psi_{k+1}(x_2) - \psi_{k+1}(z)| \leqslant$$
$$\leqslant 1 + \left]\frac{z - x_2}{\delta}\right[\leqslant 1 + \left]\frac{z - y_1}{\delta}\right[.$$

For each $z \in V_k$ with $z < x_2$, i.e. $z \leqslant x_1$, it follows that if $\psi_{k+1}(y_1) - \psi_{k+1}(z) \geqslant 0$:

$$|\psi_{k+1}(y_1) - \psi_{k+1}(z)| = \{\psi_{k+1}(y_2) - 1\} - \psi_{k+1}(z) =$$
$$= |\psi_{k+1}(y_2) - \psi_{k+1}(z)| - 1 \leqslant$$
$$\leqslant 1 + \left]\frac{y_2 - z}{\delta}\right[- 1 = \left]\frac{y_1 + \delta - z}{\delta}\right[=$$
$$= 1 + \left]\frac{y_1 - z}{\delta}\right[,$$

and if $\psi_{k+1}(y_1)-\psi_{k+1}(z)<0$:

$$|\psi_{k+1}(y_1)-\psi_{k+1}(z)|=\psi_{k+1}(z)-\{\psi_{k+1}(x_1)+1\}=$$
$$=|\psi_{k+1}(x_1)-\psi_{k+1}(z)|-1\leqslant$$
$$\leqslant 1+\left]\frac{x_1-z}{\delta}\right[-1\leqslant\left]\frac{y_1-z}{\delta}\right[.$$

This completes the proof of (32, i) and as a result we may and do conclude that $(25)_{k+1}$ and $(26)_{k+1}$ hold.

From the definition of ψ_{k+1} follows that if $(30)_k$ holds

$$y_1^{k+1}\notin(y_1^k-\delta,\, y_3^k), \tag{33}$$

in case $y_1^{k+1}\in V_{k+1}$ as well as $y_2^{k+1}\in V_{k+1}$. The upper indices for a moment serve to distinguish similar symbols y_1, y_2 and y_3 for the different functions ψ_k and ψ_{k+1}. Obviously $y_3^k-y_1^k>0$, so that

$$y_3^k-(y_1^k-\delta)>\delta. \tag{34}$$

With these preparatory results we will prove the statement in lemma 3.

Let V_0 be the set $\{\bar{x}, \xi_0, \ldots, \xi_n\}$ of points in $[0, 1]$ described in the assumptions of lemma 2. We choose a δ satisfying (10), and so $(25)_0$ applies to V_0.

Let ψ_0 be the polygon on $[0, 1]$, of which the nodes are the points of the set V_0, such that it satisfies (20) for $\psi\equiv\psi_0$, and is constant for $x\geqslant\max\{\bar{x}, \xi_n\}$ and $x\leqslant\min\{\bar{x}, \xi_0\}$. Then $(26)_0$ holds.

Next, for $k=0, 1, \ldots$ we investigate which of $(28)_k$, $(29)_k$ or $(30)_k$ holds. From (33) and (34) follows that there is an $m\leqslant 1+\left]\frac{1}{\delta}\right[$ such that $(28)_m$ or $(29)_m$ holds. As a result we conclude that for ψ_m

$$\omega(\delta)\leqslant 1, \tag{35}$$

while also ψ_m satisfies (20) for $\psi\equiv\psi_m$.

Assume $\omega(\delta)<1$ in (35). Then from (26, i, ii), or by definition of ψ_0 from (20), (18) and lemma 1, it follows that

$$|\psi_m(\xi_k)-\psi_m(\xi_{k+1})|=0 \qquad (k=0, 1, \ldots, n-1),$$
$$|\psi_m(\xi_k)|=0 \qquad (k=0, 1, \ldots, n),$$

so that

$$\beta_k=0 \qquad (k=0, 1, \ldots, n). \tag{36}$$

This only is the case if for a certain integer l, with $0\leqslant l\leqslant n$

$$\begin{cases} \bar{x}=\xi_l, \\ q_l(\bar{x})=1, \\ q_k(\bar{x})=0 \qquad (k=0, 1, \ldots, n;\ k\neq l). \end{cases} \tag{37}$$

In this special case we define the polygon $\psi \in C_1[0, 1]$ by

$$(38)\quad \begin{cases} \psi(\xi)=1 \ \textit{for an arbitrary} \ \xi \in (\xi_k, \xi_{k+1}), \\ \qquad\qquad k \ \textit{arbitrary such that} \ 0 \leqslant k \leqslant n-1; \\ \psi(x)=0 \ \textit{for} \ x \in [0, 1] \backslash (\xi_k, \xi_{k+1}); \\ \psi(x) \ \textit{linear for} \ x \in [\xi_k, \xi) \ \textit{and} \ x \in (\xi, \xi_{k+1}]. \end{cases}$$

In all other cases (36) leads to a contradiction of the assumption on β_k $(k=0, 1, \ldots, n)$ since from (1) it follows that there is at least one $q_l(\bar{x})$ $(0 \leqslant l \leqslant n)$ such that $q_l(\bar{x}) > 0$, so that with $\bar{\beta}_k = \delta_{kl}$ $(k=0, 1, \ldots, n)$, δ_{kl} being the Kronecker-symbol

$$\sum_{k=0}^{n} \bar{\beta}_k \, q_k(\bar{x}) = q_1(\bar{x}) > \sum_{k=0}^{n} \beta_k \, q_k(\bar{x}) = 0.$$

So, except in the case (37), for ψ_m

$$\omega(\delta) = 1$$

holds. Defining then ψ to be identically equal to ψ_m then, in view of (38), lemma 3 has been proved.

Finally, lemma 2 and lemma 3 immediately lead to the validity of theorem 1.

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