Some fuzzy concepts of BCI, BCK and MV-algebras

C.S. Hoo

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received 1 March 1997; accepted 1 October 1997

Abstract

We show that in an MV-algebra $Z$, for each of the listed properties and its fuzzy analogue: implicative, prime, essential, weakly essential, and maximal, the following are equivalent: (i) the fuzzy ideal $v$ has the fuzzy property, (ii) the level ideal $Z_i$ has the property, (iii) the fuzzy ideal $Z_\mu$ has the fuzzy property. It is shown that if a non-constant fuzzy ideal $v$ of $Z$ is fuzzy weakly essential and fuzzy prime, then it is either fuzzy essential or fuzzy weakly self-reflexive. This means that if a proper ideal $I$ of $Z$ is prime and satisfies $I^\perp \subseteq B(Z)$, then either $I^\perp = \{0\}$ or $I^{1\perp} = I$. We establish a precise one-to-one correspondence between the set of fuzzy closed ideals of a quasi-commutative BCI-algebra and its set of fuzzy congruences, giving also a one-to-one correspondence between its set of closed ideals and its congruences. © 1998 Published by Elsevier Science Inc. All rights reserved.

Keywords: MV-algebra; BCI-algebra; Fuzzy ideal; Fuzzy congruence

1. Introduction

In this paper, $X,Y,Z$ will always denote a BCI, BCK and MV-algebra, respectively. Also $\lambda$ will always denote a fuzzy closed ideal of $X$ while $\mu$ and $\nu$ will always denote a fuzzy ideal of $Y$ and $Z$, respectively.

In [13,14,17,18] we obtained various results on fuzzy ideals of BCI, BCK and MV-algebras. In this paper, we systematize some of these results. Also
we “fuzzify” some of the results in [20]. In [28], Murali introduced the concept of fuzzy equivalence relations. We adopt his concept and consider fuzzy congruence relations on our algebras. We then establish results for these algebras that are analogous to the results for more traditional algebras. Finally, we consider the question of whether or not anything new about these algebras can be obtained by “fuzzifying”.

We assume knowledge of BCI, BCK and MV-algebras and refer the reader to [1,4,5,11,15,19,24–27] for full details. We shall generally adopt the notation and terminology of [11,13,14,17,18]. By the nature of these algebras, anything we prove for BCI-algebras will be true of BCK-algebras, and results for BCK-algebras will be true for MV-algebras.

We briefly review the concepts of these algebras that we need and also the basic concepts of fuzzy logic required. A BCI-algebra is a non-empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $(x * y) * (x * z) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. $x * y = 0$ and $y * x = 0$ imply that $x = y$,
5. $x * 0 = 0$ implies that $x = 0$.

We can define a partial ordering $\leq$ on $X$ by putting $x \leq y$ if and only if $x * y = 0$. Further, if every element $x$ of $X$ satisfies $x \geq 0$, then $X$ is a BCK-algebra. If a BCK-algebra satisfies the axiom $x * (x * y) = y * (y * x)$, then it is called commutative. In this case $x * (x * y)$ is the greatest lower bound $x \wedge y$ of $x$ and $y$. If a commutative BCK-algebra $Y$ has a largest element 1 (and this can also occur even if $Y$ is not commutative), then the least upper bound $x \vee y$ of $x$ and $y$ is given by $x \vee y = 1 * \{(1 * x) \wedge (1 * y)\}$. A bounded commutative BCK-algebra is then a distributive lattice. Observe that if a BCI-algebra is commutative, then it is automatically BCK. Similarly, if it is bounded, then it is also automatically BCK. We shall regard an MV-algebra as a bounded commutative BCK-algebra. The usual MV-algebra operations are given by $\bar{x} = 1 * x$, $xy = x * \bar{y}$ and $x + y = (\bar{x} \bar{y})^-$. We refer the reader to the references listed above for full details, in particular to [11,26].

An ideal of a BCI-algebra $X$ is a subset $I$ containing 0 and with the property that if $x * y \in I$ and $y \in I$, then $x \in I$. It is shown in [11], Theorem 2.1, that in the case of an MV-algebra, the ideals defined in the traditional way are precisely such ideals. An ideal $I$ is implicative if whenever $(x * y) * z \in I$ and $y * z \in I$, then $x * z \in I$. A proper ideal $I$ is maximal if whenever $I \subset J$ with $J$ being a proper ideal, then $I = J$. If $Y$ is a commutative BCK-algebra, then a proper ideal $I$ of $Y$ is prime if whenever $x \wedge y \in I$ then either $x \in I$ or $y \in I$. An ideal $I$ of a BCI-algebra $X$ is closed if whenever $x \in I$ then $0 * x \in I$. Given a non-empty subset $A$ of $X$, we denote the ideal of $X$ generated by $A$ by $\langle A \rangle$. This is the set of all $x \in X$ such that $(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0$ for some $a_1, a_2, \ldots, a_n \in A$. 


A congruence relation $C$ on $X$ is an equivalence relation on $X$ such that if $(x, y)$ and $(a, b) \in C$, then $(x * a, y * b) \in C$. In case $X$ is quasi-commutative, we shall establish a one-to-one correspondence between the closed ideals of $X$ and the congruences of $X$, analogous to the one in the case of more traditional algebras.

In general, in $Y$ we always have $(x * y) * z \leq (x * z) * (y * z)$ for all $x, y, z$. We say that $Y$ is positive implicative if we actually have $(x * y) * z = (x * z) * (y * z)$. It follows by [25], Theorem 8, that this identity holds if and only if we also have the identity $(x * y) * y = x * y$. In $X$, we can define by induction polynomials $Q_{m,n}(x,y)$ of two variables $x,y$, for all integers $m \geq 0$, $n \geq 0$ by $Q_{0,0}(x,y) = x * (x,y)$, $Q_{m-1,n}(x,y) = Q_{m,n}(x,y) * (x,y)$, and $Q_{m,n-1}(x,y) = Q_{m,n}(x,y) * (y * x)$. Then we say that $X$ is quasi-commutative of type $(i,j;m,n)$ if $Q_{i,j}(x,y) = Q_{m,n}(y,x)$ for all $x, y$ (see [8,10] for some results and other references to this topic). We shall say that $X$ is quasi-commutative if it is quasi-commutative of some type $(i,j;m,n)$. Observe that this is a generalization of commutativity since commutativity is just quasi-commutativity of type $(0,0;0,0)$. However, while commutative BCI-algebras are always BCK, there are quasi-commutative BCI-algebras that are not BCK. It is shown in [25], Proposition 5, that if $Y$ is positive implicative, then $Y$ satisfies the identity $(x * (y * x)) * (x,y) = (y * (y * x)) * (x,y)$. Since $(x * (y * x)) * (x,y) = (x * (x * y)) * (y * x)$, this means that positive implicative BCK-algebras are quasi-commutative of type $(0,1;0,1)$.

We now review some fuzzy logic concepts, referring the reader to [13,14,17,18,30] for more details. We denote the unit interval by $[0,1]$, and write $a \land b$ and $a \lor b$ for the minimum and maximum, respectively, of two real numbers $a$ and $b$. A fuzzy subset of $X$ is a function $\alpha : X \rightarrow [0,1]$. It is a fuzzy ideal if $\alpha(0) \geq \alpha(x)$ and $\alpha(x) \geq \alpha(x * y) \land \alpha(y)$ for all $x, y$ in $X$. It is fuzzy closed if $\alpha(0 * x) \geq \alpha(x)$ for all $x$ in $X$. For each $t \in [0,1]$, let $x_t = \{ x \in X | \alpha(x) \geq t \}$. Then it is shown in [30], Theorem 3, that $\alpha$ is a fuzzy ideal of $X$ if and only if each $x_t$ is either empty or is an ideal of $X$. These are the level ideals of $\alpha$. Thus given a fuzzy ideal $\lambda$ of $X$, we have, corresponding to $t = \lambda(0)$, an ideal of $X$ denoted by $X_t = \{ x \in X | \lambda(x) = \lambda(0) \}$. By a fuzzy ideal of a BCK or MV-algebra, we mean a fuzzy ideal of the underlying BCI-algebra. It is easily seen that if $\lambda$ is a fuzzy ideal of $X$ and $x \leq y$ in $X$, then $\lambda(x) \geq \lambda(y)$ (see [13], Proposition 2.1(i)).

If $E$ is a subset of $X$, we denote the characteristic function of $E$ by $\chi_E$. This is, of course, a function $\chi_E : X \rightarrow [0,1]$. If $I$ is an ideal of $X$, it is easily checked that $\chi_I$ is a fuzzy ideal of $X$ and $X_{\chi_I} = I$. If $\lambda$ satisfies $\lambda(0) = 1$, then we have $\chi_{\lambda(x)} \leq \lambda$.

Recall that $\lambda$ is fuzzy implicative if $\lambda(x * z) \geq \lambda((x * y) * z) \land \lambda(y * z)$ for all $x, y, z$ in $X$. It is shown in [30], Theorem 5, that $\lambda$ is fuzzy implicative if and only if each $\lambda_t$ is either empty or an implicative ideal of $X$. Thus if $\lambda$ is fuzzy implicative, then $X_\lambda$ is an implicative ideal of $X$. Observe that if $I$ is an implicative ideal of $X$, then $\chi_I$ is a fuzzy implicative ideal of $X$.  

If $Y$ is commutative, then $\mu$ is fuzzy prime if it is non-constant and $\mu(x \land y) = \mu(x) \lor \mu(y)$ for all $x, y$ in $Y$. It is shown in [13], Theorem 2.4, that if $Y$ is commutative and $\mu$ is non-constant, then $\mu$ is a fuzzy prime ideal if and only if each $\mu_i$ is either empty or is a prime ideal of $Y$ if it is proper. Thus if $Y$ is commutative and $\mu$ is fuzzy prime, then $Y_\mu$ is a prime ideal of $Y$. It is easily checked for such an algebra that an ideal $I$ is prime if and only if $\chi_I$ is a fuzzy prime ideal (see [13], Theorem 2.5).

2. Fuzzy ideals on MV-algebras

In this section we consider whether or not new information can be obtained about MV-algebras by “fuzzifying”. In particular, we consider the properties of ideals being implicative, prime, essential, weakly essential and maximal, and their fuzzy analogues.

We first recall that in $Z$ we always have $(x * y) \land (y * x) = 0$ for all $x, y \in Z$ (see [11], p. 566, Property (6)). In [14], Theorem 2.15, we showed that $v$ is fuzzy implicative if and only if $Z_v$ is implicative.

**Theorem 1.** $v$ is fuzzy prime if and only if $Z_v$ is prime.

**Proof.** Suppose $v$ is fuzzy prime. Then $v \neq$ constant and hence $Z_v$ is proper. Suppose that $x \land y \in Z_v$. Then $v(x) \lor v(y) = v(x \land y) = v(0)$, that is, either $v(x) = v(0)$ or $v(y) = v(0)$. Thus either $x \in Z_v$ or $y \in Z_v$. Conversely, suppose that $Z_v$ is prime. Then $Z_v$ is proper and hence $v$ is non-constant. Since $x \land y \leqslant x, y$, we have $v(x) \lor v(y) \leqslant v(x \land y)$. Also $v(x) \geqslant v((x * x) * y) \land v((x * y) * y) = v(x \land y) \land v(x * y)$. But $x * y \leq x$ and hence $v(x) \leq v(x * y)$, that is, $v(x) \geq v(x \land y) \land v(x * y) \geq v(x)$. This means that $v(x) = v(x \land y) \lor v(x * y)$. Similarly, $v(y) = v(x \land y) \lor v(x * y)$. Hence $v(x) \lor v(y) = v(x \land y) \lor (v(x * y) \lor v(x * y))$. But $(x * y) \land (y * x) = 0 \in Z_v$. Hence either $x * y \in Z_v$ or $y * x \in Z_v$, that is, $v(x * y) = v(0)$ or $v(y * x) = v(0)$. This means that $v(x) \lor v(y) = v(x \land y) \lor v(0) = v(x \land y)$.

Recall that a closed ideal $I$ of $X$ is an essential closed ideal if for every non-zero closed ideal $J$ of $X$, we have $I \cap J \neq \{0\}$. In case $I$ is an ideal of $Z$, we can re-state this in a more convenient fashion. Recall that if $A$ is a non-empty subset of $Z$, then $A^\perp = \{z \in Z | z \land a = 0 \text{ for all } a \in A\}$. This is an ideal of $Z$ (see [18], Theorem 2.2), and $A \cap A^\perp = \emptyset$ or $A \cap A^\perp = \{0\}$ if $0 \in A$. Then an ideal $I$ of $Z$ is essential if and only if $I^\perp = \{0\}$. We say that $\lambda$ is a fuzzy essential closed ideal of $X$ if for all fuzzy closed ideals $\alpha$ of $X$ such that $X_\alpha \neq \{0\}$, we have $X_\alpha \cap X_\lambda \neq \{0\}$. Then it is shown in [14], Lemma 3.2, that $\lambda$ is a fuzzy essential closed ideal if and only if $X_\lambda$ is an essential closed ideal of $X$. 


We say that \( \lambda \) is normalized if \( \lambda(0) = 1 \). The normalization of a fuzzy ideal \( \alpha \) of \( X \) is \( \overline{\alpha} \) given by \( \overline{\alpha}(x) = \alpha(x) + 1 - \alpha(0) \). This is a normalized fuzzy ideal of \( X \) and \( X_{\overline{\alpha}} = X_{\alpha} \). We define a partial ordering on the set of fuzzy ideals of \( X \) by \( \alpha \leq \beta \) if \( \alpha(x) \leq \beta(x) \) for all \( x \in X \). Then of course we always have \( \alpha \leq \overline{\alpha} \). Let \( \mathcal{F}(X) \) denote the set of all normalized fuzzy ideals \( \alpha \) of \( X \) such that \( 0 \in image(\alpha) \). We say that \( \lambda \) is a fuzzy maximal ideal of \( X \) if it is non-constant and \( \lambda \) is a maximal element of \( (\mathcal{F}(X), \leq) \). It is shown in [17], Theorem 3.9, that every fuzzy maximal ideal of \( X \) is normalized and takes only the values \( \{0, 1\} \). Further, if \( \lambda \) is fuzzy maximal, then \( \lambda = \chi_{x_{\lambda}} \) (see [17], Theorem 3.10). Observe that if \( \alpha \in \mathcal{F}(X) \), then \( X_{\overline{\alpha}} \) is a proper ideal. Also \( \overline{\alpha} = \chi \).

**Theorem 2.** Suppose that \( v \in F(Z) \). Then \( v \) is fuzzy maximal if and only if \( Z_v \) is a maximal ideal.

**Proof.** If \( v \) is fuzzy maximal, then \( Z_v \) is a maximal ideal by [17], Theorem 3.11. Conversely, suppose that \( Z_v \) is a maximal ideal of \( Z \). Then by [17], Theorem 3.24, \( v = \chi_{Z_v} \). Suppose that \( v \leq \alpha \in \mathcal{F}(Z) \). Then \( Z_v \subset Z_{\alpha} \). Since \( Z_v \) is proper, we have that \( Z_v = Z_{\alpha} \) is a maximal ideal of \( Z \). Hence \( v = \chi_{Z_v} = \chi_{Z_{\alpha}} = \alpha \). This means that \( v = \alpha \) is a maximal element of \( \mathcal{F}(Z) \).

Let \( B(Z) \) denote the Boolean subalgebra of \( Z \) consisting of all the idempotent elements, and let \( B_1(Z) = B(Z) - \{1\} \). Recall that a non-zero ideal \( I \) of \( Z \) is weakly essential if for all ideals \( J \) of \( Z \) such that \( J \cap \{Z - B_1(Z)\} \neq \emptyset \), we have \( I \cap J \neq \{0\} \). We say that \( v \) is a fuzzy weakly essential ideal of \( Z \) if for all fuzzy ideals \( \alpha \) of \( Z \) such that \( Z_v \cap \{Z - B_1(Z)\} \neq \emptyset \), we have \( Z_v \cap Z_{\alpha} \neq \{0\} \). It is easily checked that a (fuzzy) essential ideal of \( Z \) is (fuzzy) weakly essential. We showed in [14], Theorem 3.15, that if \( I \) is a proper ideal of \( Z \), then \( I \cap \{Z - B_1(Z)\} = \emptyset \) if and only if \( I \cap \inf Z = \{0\} \). Here \( \inf Z = \{z \wedge \overline{z} \text{ for all } z \in Z\} = \{z \in Z \mid z^2 = 0\} \). We showed in [21], Theorem 3.5, that \( I \neq \{0\} \) is a weakly essential ideal of \( Z \) if and only if \( I^\perp \subset B(Z) \).

**Theorem 3.** Suppose that \( v \neq \text{constant} \). Then the following are equivalent:

1. \( v \) is fuzzy weakly essential.
2. \( Z_v \) is weakly essential.
3. \( Z_v \subset B(Z) \).

**Proof.** 

(1) \( \Rightarrow \) (2): Suppose that \( v \) is fuzzy weakly essential, and let \( J \) be an ideal of \( Z \) such that \( J \cap \{Z - B_1(Z)\} \neq \emptyset \). Let \( \alpha = \chi_J \). Then \( \alpha \) is a fuzzy ideal of \( Z \), and since \( Z_v = Z_{\chi_J} = J \), we have \( Z_v \cap \{Z - B_1(Z)\} \neq \emptyset \). Hence \( Z_v \cap J = Z_v \cap Z_{\alpha} \neq \{0\} \).

(2) \( \Rightarrow \) (3): This follow from [21], Theorem 3.5.

(3) \( \Rightarrow \) (1): Suppose that \( Z_v \subset B(Z) \). Let \( \alpha \) be a fuzzy ideal such that \( Z_v \cap \{Z - B_1(Z)\} \neq \emptyset \). Then by [14], Theorem 3.15, \( Z_v \cap \inf Z \neq \{0\} \). If
$Z_r \cap Z_r = \{0\}$, then $Z_r \subseteq Z_r^\perp \subseteq B(Z)$. This means that there exists $z \in B(Z) \cap \text{Inf } Z$ with $z \neq 0$. But $B(Z) \cap \text{Inf } Z = \{0\}$, a contradiction. Hence $Z_r \cap Z_r \neq \{0\}$.

Thus the properties of $Z_r$ of being implicative, prime, essential, weakly essential or maximal are completely reflected in the corresponding fuzzy properties of $\nu$. Similarly, we can compare $\nu$ with the fuzzy ideal $\mathcal{J}_{\nu}$. Recall that if $\nu$ is fuzzy maximal, then $\nu = \mathcal{J}_{\nu}$ ([17], Theorem 3.10).

**Theorem 4.** Let $I$ be a proper ideal of $Z$. Then we have the following:

1. $I$ is implicative if and only if $\mathcal{J}_I$ is fuzzy implicative. In fact, this is true if $I$ is a proper ideal of $X$.
2. $I$ is prime if and only if $\mathcal{J}_I$ is fuzzy prime.
3. $I$ is essential if and only if $\mathcal{J}_I$ is fuzzy essential.
4. $I$ is maximal if and only if $\nu$ is fuzzy maximal.
5. $I$ is weakly essential if and only if $\mathcal{J}_I$ is fuzzy weakly essential.

**Proof.** (1) This is Lemma 2.7 (3) of [14].

(2) This is Theorem 2.5 of [13].

(3) If $I$ is essential, then $\mathcal{J}_I$ is fuzzy essential by [14], Lemma 3.2(2). Conversely, suppose that $\mathcal{J}_I$ is fuzzy essential. Then $I = Z_{\mathcal{J}_I}$ is essential by [14], Lemma 3.2(1).

(4) Suppose that $I$ is maximal. Observe that $\mathcal{J}_I \in \mathcal{F}(Z)$. Suppose that $\mathcal{J}_I \nleq \alpha \in \mathcal{F}(Z)$. Then $I = Z_{\mathcal{J}_I} \subseteq Z_\alpha$. Since $Z_\alpha$ is proper, we have $I = Z_\alpha$ and hence $Z_{\mathcal{J}_I} = Z_\alpha = \alpha$. Thus $\mathcal{J}_I$ is fuzzy maximal. Conversely, suppose that $\mathcal{J}_I$ is fuzzy maximal. Let $J$ be a proper ideal such that $I \subseteq J$. Then $\mathcal{J}_I \leq \mathcal{J}_J \in \mathcal{F}(Z)$. Hence $\mathcal{J}_I = \mathcal{J}_J$ and $I = Z_{\mathcal{J}_I} = Z_{\mathcal{J}_J} = J$.

(5) Since $I = Z_{\mathcal{J}_I}$ and $\mathcal{J}_I \neq \text{constant}$, the result follows from Theorem 3.

Thus, once again, there is no difference between the listed fuzzy properties of $\nu$ and $\mathcal{J}_{\nu}$. There is then the question if whether “fuzzifying” achieves anything. In some cases, it makes it easier to prove or discover results. In [13], we proved that $\mathcal{J}$ is always fuzzy weakly implicative, that is, it satisfies $\lambda((x*y)*z) \geq \lambda((x*y)*z) \land \lambda(y*z)$ for all $x, y, z \in X$. This implies that every ideal $I$ of $X$ is weakly implicative, that is, if $(x*y)*z \in I$ and $y*z \in I$, then $(x*z)*z \in I$. The fuzzy result follows immediately from the definition of fuzzy ideals.

Subsequent to this, we re-derived the corresponding result for ideals of $X$ by “reverse-engineering”, as for example, we did in [15], Theorem 2.3, for $Z$.

We conclude this section with some further fuzzy properties of $\nu$. We define $\nu^\perp : Z \rightarrow [0, 1]$ by $\nu^\perp = \mathcal{J}_{\nu(\mathcal{J}_\nu)}$. Then $\nu^\perp \in \mathcal{F}(Z)$ and takes on only the values $\{0, 1\}$. It is easily checked that $Z_{\nu^\perp} = (Z_{\nu})^\perp$ and $Z_{\nu}^\perp = (Z_{\nu^\perp})^\perp = (Z_{\nu})^{\perp\perp}$. Finally observe that $\nu^{\perp\perp} = \mathcal{J}_{\nu(\mathcal{J}_{\nu^\perp})} = \mathcal{J}_{\nu^\perp} = \mathcal{J}_{\nu(\mathcal{J}_\nu)^\perp}$. 

Definition 1. (i) An ideal $I$ of $Z$ is self-reflexive if $I^\perp = I$.
(ii) $v$ is fuzzy self-reflexive if $v^\perp = v$.
(iii) $v$ is fuzzy weakly self-reflexive if $Z_v^\perp = Z_v$, that is, if $Z_v$ is self-reflexive.

Theorem 5. Suppose that $v \neq \text{constant.}$ If $v$ is fuzzy weakly essential and fuzzy prime, then $v$ is either fuzzy essential or fuzzy weakly self-reflexive.

Proof. By Theorem 1, $Z_v$ is prime. Suppose that $v$ is not fuzzy essential. Then by [14], Lemma 3.2, $Z_v$ is not essential and hence $Z_v^\perp \neq \{0\}$. It then follows by [20], Theorem 2.3, that $Z_v^\perp$ is linearly ordered. Since $Z_v^\perp \subset B(Z)$ by Theorem 3, it follows that $Z_v^\perp = \{0,e\}$ for some $e \in B(Z)$ with $e \neq 0$ by [11], Lemma 5.1. Hence $Z_v^\perp = \langle e \rangle$ and $Z_v^{\perp \perp} = \langle e \rangle^{\perp} = \langle \bar{e} \rangle$. Let $z \in Z_v$. Then $z \land e = 0$, that is, $ze = 0$. Hence $z \leq \bar{e}$, showing that $Z_v \subset \langle \bar{e} \rangle = \langle e \rangle^{\perp}$. On the other hand, $e \land \bar{e} = 0 \in Z_v = a$ prime ideal. Since $e \notin Z_v$ because $e \neq 0$, we have $\bar{e} \in Z_v$. Thus $\langle \bar{e} \rangle \subset Z_v$, that is, we have $Z_v = \langle \bar{e} \rangle = \langle e \rangle^{\perp}$. This means that $Z_v = Z_v^{\perp \perp} = Z_v^{\perp}$, proving that $v$ is fuzzy weakly self-reflexive.

Corollary 1. Suppose that $I$ is a proper ideal of $Z$. If $I$ is weakly essential and prime, then either $I$ is essential or self-reflexive.

Proof. Take $v = Z_I$ and apply Theorem 5 and the above correspondences between the properties of $Z_v$ and the fuzzy properties of $v$.

Remark 1. Compare this result with Theorem 3.14 of [20] where we showed that if $Z$ has no idempotent atoms, then any ideal that is weakly essential and prime is essential.

3. Fuzzy congruences

Following [28], we make the following definition.

Definition 2. A fuzzy equivalence relation on $X$ is a fuzzy subset $\varphi$ of $X \times X$ satisfying
1. $\varphi(x,x) = \sup \{ \varphi(y,z) \text{ for all } y,z \in X \}$ (reflexive).
2. $\varphi(x,y) = \varphi(y,x)$ (symmetric).
3. $\varphi(x,y) \geq \varphi(x,z) \land \varphi(z,y)$ for all $z \in X$ (transitive).

Obviously $\varphi(x,x) = \varphi(y,y)$ for all $x,y \in X$. Also, of course, $\varphi(x,y) \geq \sup \{ \varphi(x,z) \land \varphi(z,y) \text{ for all } z \in X \} \geq \varphi(x,y) \land \varphi(y,y) = \varphi(x,y)$, that is, $\varphi(x,y) = \sup \{ \varphi(x,z) \land \varphi(z,y) \text{ for all } z \in X \}$. Given fuzzy equivalences $\varphi, \psi$ on $X$, we write $\varphi \leq \psi$ if $\varphi(x,y) \leq \psi(x,y)$ for all $x,y \in X$. 
Definition 3. A fuzzy equivalence relation \( \phi \) on \( X \) is a fuzzy congruence relation on \( X \) if for all \( x, y, w, z \in X \), we have \( \phi(x \ast y, w \ast z) \geq \phi(x, w) \land \phi(y, z) \).

Remark 2. This means that if \( \phi \) is a fuzzy congruence relation on \( X \), then \( \phi(x \ast y, 0) \geq \phi(x, z) \land \phi(y, z) \) for all \( x, y, z \in X \) since we can take \( z \ast z \) for \( 0 \). In particular, taking \( y = z \), we have \( \phi(x \ast y, 0) \geq \phi(x, y) \land \phi(y, y) \), that is, \( \phi(x \ast y, 0) \geq \phi(x, y) \) for all \( x, y \in X \).

If \( C \) is a congruence relation on \( X \), then it is easily checked that \( \chi_C \) is a fuzzy congruence relation on \( X \). More generally, if \( E \subseteq X \times X \) is an equivalence relation on \( X \), then \( \chi_E \) is a fuzzy equivalence relation on \( X \). Of course one would like to see other fuzzy congruence relations on \( X \) besides the characteristic functions.

Theorem 6. Let \( \phi \) be a fuzzy equivalence relation on \( X \). Then \( \phi \) is a fuzzy congruence relation if and only if \( \phi(x \ast a, y \ast a) \geq \phi(x, y) \) and \( \phi(b \ast x, b \ast y) \geq \phi(x, y) \) for all \( x, y, a, b \in X \).

Proof. If \( \phi \) is a fuzzy congruence relation, then \( \phi(x \ast a, y \ast a) \geq \phi(x, y) \land \phi(a, a) = \phi(x, y), \) and similarly \( \phi(b \ast x, b \ast y) \geq \phi(x, y) \). Conversely, suppose that \( \phi(x \ast a, y \ast a) \geq \phi(x, y) \) and \( \phi(b \ast x, b \ast y) \geq \phi(x, y) \) for all \( x, y, a, b \in X \). Then for all \( x, y, w, z \in X \) we have \( \phi(x \ast y, w \ast z) \geq \phi(x, y, x \ast z) \land \phi(z \ast y) \) since \( (x \ast y) \ast (z \ast y) \leq x \ast z \). Similarly \( \phi(y \ast x) \geq \phi(y \ast z) \land \phi(z \ast x) \). Hence \( \phi(x \ast z) \land \phi(z \ast x) \geq \phi(z \ast x) \land \phi(x \ast z) \) for all \( z \in X \). Thus \( \chi_E \) is a fuzzy equivalence relation on \( X \). We now check \( \chi_E(x \ast a, y \ast a) \) and \( \chi_E(a \ast x, a \ast y) \). We have \( \chi_E(x \ast a, y \ast a) \geq \phi((x \ast a) \ast (y \ast a)) \land \phi((y \ast a) \ast (x \ast a)) \geq \phi(x \ast y) \land \phi(y \ast x) = \chi_E(x, y) \). Similarly \( \chi_E(a \ast x, a \ast y) \geq \phi(x \ast y, 0) \geq \phi(x, y) \land \phi(0, y) \land \phi(x \ast y, 0) = \phi(x, y) \). Thus by Theorem 6, \( \chi_E \) is a fuzzy congruence on \( X \).

Theorem 7. \( \mathcal{C}(\lambda) : X \times X \to [0, 1] \) given by \( \mathcal{C}(\lambda)(x, y) = \lambda(x \ast y) \land \lambda(y \ast x) \) is a fuzzy congruence relation on \( X \).

Proof. We first show that it is a fuzzy equivalence relation on \( X \). We have \( \mathcal{C}(\lambda)(x, y) = \lambda(0) = \sup \{ \mathcal{C}(\lambda)(y, z) \text{ for all } y, z \in X \} \) since \( \lambda(0) \geq \lambda(y) \) for all \( y \in X \). Also \( \mathcal{C}(\lambda)(x, y) = \mathcal{C}(\lambda)(y, x) \). Since \( \lambda \) is a fuzzy ideal, we have \( \lambda(x \ast y) \geq \lambda\{x \ast (y \ast z)\} \land \lambda(z \ast y) \) since \( x \ast (y \ast z) \leq (x \ast z) \). Similarly \( \lambda(y \ast x) \geq \lambda(y \ast z) \land \lambda(z \ast x) \). Hence \( \mathcal{C}(\lambda)(x, y) \geq \mathcal{C}(\lambda)(x, z) \land \mathcal{C}(\lambda)(z, y) \) for all \( z \in X \). Thus \( \mathcal{C}(\lambda) \) is a fuzzy equivalence relation on \( X \). Now we check \( \mathcal{C}(\lambda)(x \ast a, y \ast a) \) and \( \mathcal{C}(\lambda)(a \ast x, a \ast y) \). We have \( \mathcal{C}(\lambda)(x \ast a, y \ast a) = \lambda((x \ast a) \ast (y \ast a)) \land \lambda((y \ast a) \ast (x \ast a)) \geq \lambda(x \ast y) \land \lambda(y \ast x) = \mathcal{C}(\lambda)(x, y) \). Similarly \( \mathcal{C}(\lambda)(a \ast x, a \ast y) \geq \mathcal{C}(\lambda)(x, y) \). Thus by Theorem 6, \( \mathcal{C}(\lambda) \) is a fuzzy congruence on \( X \).

Theorem 8. Let \( \phi \) be a fuzzy congruence relation on \( X \). Then \( \mathcal{I}(\phi) : X \to [0, 1] \) defined by \( \mathcal{I}(\phi)(x) = \phi(x, 0) \) is a fuzzy closed ideal on \( X \).

Proof. We have \( \mathcal{I}(\phi)(0) = \phi(0, 0) \geq \phi(x, 0) = \mathcal{I}(\phi)(x) \) for all \( x \in X \). Also \( \mathcal{I}(\phi)(x) = \phi(x, 0) \geq \phi(x, x \ast y) \land \phi(x \ast y, 0) \geq \phi(x, x) \land \phi(0, y) \land \phi(x \ast y, 0) = \phi(x, y) \).
\( \phi(y, 0) \land \phi(x \ast y, 0) = \mathcal{J}(\phi)(x \ast y) \). Finally, \( \mathcal{J}(\phi)(0 \ast x) = \phi(0 \ast x, 0) \geq \phi(0, 0) \land \phi(x, 0) = \phi(x, 0) = \mathcal{J}(\phi)(x) \). Thus \( \mathcal{J}(\phi) \) is a fuzzy closed ideal on \( X \).

**Theorem 9.** \( \mathcal{C}(\mathcal{C}(\lambda)) = \lambda \).

**Proof.** \( \mathcal{C}(\mathcal{C}(\lambda))(x) = \mathcal{C}(\lambda)(x, 0) = \lambda(x \ast 0) \land \lambda(0 \ast x) = \lambda(x) \).

**Theorem 10.** If \( \phi \) is a fuzzy congruence on \( X \), then \( \mathcal{C}(\mathcal{J}(\phi)) \geq \phi \).

**Proof.**

\[
\mathcal{C}(\mathcal{J}(\phi))(x, y) = \mathcal{J}(\phi)(x \ast y) \land \mathcal{J}(\phi)(y \ast x) \\
= \phi(x \ast y, 0) \land \phi(y \ast x, 0) \\
\geq \phi(x, y) \land \phi(y, x)
\]

by the remarks after Definition 3. Thus \( \mathcal{C}(\mathcal{J}(\phi)) \geq \phi \).

**Theorem 11.** If \( \phi \) is a fuzzy congruence on \( X \), then \( \phi(x, \mathcal{Q}_{m,n}(x, y)) \geq \phi(x \ast y, 0) \land \phi(y \ast x, 0) \geq \phi(x, y) \) for all \( x, y \in X \) and integers \( m, n \geq 0 \).

**Proof.** We observe that \( \phi(x, \mathcal{Q}_{0,0}(x, y)) = \phi(x, x \ast (x \ast y)) \geq \phi(x, x) \land \phi(0, x \ast y) = \phi(x, y, 0) \geq \phi(x, y) \). Suppose that \( \phi(x, \mathcal{Q}_{m,0}(x, y)) \geq \phi(x, y, 0) \land \phi(y, x, 0) \geq \phi(x, y) \) for some integer \( m \geq 0 \). Then \( \phi(x, \mathcal{Q}_{m+1,0}(x, y)) = \phi(x, \mathcal{Q}_{m,0}(x, y) \ast (x \ast y)) \geq \phi(x, \mathcal{Q}_{m,0}(x, y)) \land \phi(0, x \ast y) \geq \phi(x, y, 0) \land \phi(y \ast x, 0) \geq \phi(x, y) \). Thus, we have shown by induction on \( m \) that \( \phi(x, \mathcal{Q}_{m}(x, y)) \geq \phi(x \ast y, 0) \land \phi(y \ast x, 0) \geq \phi(x, y) \) for all integers \( m \geq 0 \). Fix an integer \( m \geq 0 \) and suppose that \( \phi(x, \mathcal{Q}_{m}(x, y)) \geq \phi(x \ast y, 0) \land \phi(y \ast x, 0) \geq \phi(x, y) \) for some integer \( n \geq 0 \). Then \( \phi(x, \mathcal{Q}_{m,n+1}(x, y)) = \phi(x, \mathcal{Q}_{m,n}(x, y) \ast (y \ast x)) \geq \phi(x, \mathcal{Q}_{m,n}(x, y)) \land \phi(0, y \ast x) \geq \phi(x, y, 0) \land \phi(y \ast x, 0) \geq \phi(x, y) \). This proves the result.

**Theorem 12.** If \( X \) is quasi-commutative, then for each fuzzy congruence \( \phi \) on \( X \), we have \( \mathcal{C}(\mathcal{J}(\phi)) = \phi \).

**Proof.** We have seen that \( \mathcal{C}(\mathcal{J}(\phi)) \geq \phi \). Suppose that \( X \) is quasi-commutative of type \( (m, n; i, j) \). Then \( \phi(x, y) \geq \phi(x, \mathcal{Q}_{m,n}(x, y)) \land \phi(\mathcal{Q}_{m,n}(x, y), y) = \phi(x, \mathcal{Q}_{m,n}(x, y)) \land \phi(\mathcal{Q}_{i,j}(y, x), y) \geq \phi(x \ast y, 0) \land \phi(y \ast x, 0) = \mathcal{J}(\phi)(x \ast y) \land \mathcal{J}(\phi)(y \ast x) = \mathcal{C}(\mathcal{J}(\phi))(x, y) \). This proves that \( \mathcal{C}(\mathcal{J}(\phi)) = \phi \).

Because of Theorems 9 and 12, we have the following result.

**Theorem 13.** If \( X \) is a quasi-commutative BCI-algebra, then there is a one-to-one correspondence between the set of fuzzy closed ideals of \( X \) and the set of fuzzy
congruence relations on $X$ given by $\mathcal{G} : \{\text{Fuzzy closed ideals of } X\} \rightarrow \{\text{Fuzzy congruences on } X\}$ with inverse given by $I : \{\text{Fuzzy congruences on } X\} \rightarrow \{\text{Fuzzy closed ideals of } X\}$.

**Remarks 3.** (1) If $Y$ is commutative, then it is, of course, quasi-commutative, and hence the one-to-one correspondence of Theorem 3.10 of [17] applies.

(2) If $Y$ is positive implicative, then it is shown in [25], Proposition 5, that it is quasi-commutative of type $(0, 1; 0, 1)$.

(3) Of course, every MV-algebra $Z$ is commutative and hence quasi-commutative.

**Theorem 14.** If $\varphi$ is a fuzzy congruence on $Z$, then we have $\varphi(x \lor y, 0) = \varphi(x, 0) \land \varphi(y, 0)$ and $\varphi(nx, 0) = \varphi(x, 0)$ for all $x, y \in Z$ and all positive integers $n$.

**Proof.** $\mathcal{I}(\varphi)$ is a fuzzy ideal on $Z$ and hence the result follows from Theorem 3.3 and Corollary 3.4 of [13].

The corresponding theory for closed ideals and congruences on $X$ is as follows. If $I$ is a closed ideal of $X$, we define $\mathcal{G}(I) = \{(x, y) \in X \times X \mid x * y, y * x \in I \}$.

**Theorem 15.** $\mathcal{G}(I)$ is a congruence relation on $X$.

**Proof.** Clearly $(x, x) \in \mathcal{G}(I)$. If $(x, y) \in \mathcal{G}(I)$ then $(y, x) \in \mathcal{G}(I)$. Suppose that $(x, y) \in \mathcal{G}(I)$ and $(y, z) \in \mathcal{G}(I)$. Then $x * y, y * x, y * z, z * y \in I$. Since $(x * z) * (x * y) \leq y * z \in I$ and $(z * x) * (z * y) \leq y * x \in I$, it follows that $x * z \in I$ and $z * x \in I$, that is, $(x, z) \in \mathcal{G}(I)$. Thus $\mathcal{G}(I)$ is an equivalence relation on $X$. Now suppose that $(x, y) \in \mathcal{G}(I)$ and $(w, z) \in \mathcal{G}(I)$. Since $(x * w) * (y * w) \leq x * y \in I$ and $\{(x * w) * (y * w)\} \leq (y * w) * (y * z) \leq z * w \in I$. It follows that $(x * w) * (y * z) \in I$. Similarly we have $(y * z) * (x * w) \leq w * z \in I$ and $\{(y * z) * (x * w)\} \leq (y * w) * (x * w) \leq y * x \in I$. Hence $(y * z) * (x * w) \in I$. This means that $(x * w, y * z) \in \mathcal{G}(I)$, proving that $\mathcal{G}(I)$ is a congruence relation on $X$.

Now suppose that $C$ is a congruence relation on $X$. We define $\mathcal{I}(C) = \{x \in X \mid (x, 0) \in C\}$.

**Theorem 16.** $\mathcal{I}(C)$ is a closed ideal of $X$.

**Proof.** Clearly $0 \in \mathcal{I}(C)$. Now suppose that $x * y, y \in \mathcal{I}(C)$, that is, $(x * y, 0) \in C$ and $(y, 0) \in C$. Since $(x, x) \in C$, we also have $(x * y, x * 0) \in C$, that is, $(x * y, x) \in C$. Hence $(x, 0) \in C$. This means that $x \in \mathcal{I}(C)$ and proves
that \( \mathcal{I}(C) \) is an ideal of \( X \). Suppose that \( x \in \mathcal{I}(C) \). Then \((x,0) \in C\). Since \((0,0) \in C\), we have \((0 \ast x,0 \ast 0) \in C\), that is, \((0 \ast x,0) \in C\). Thus \(0 \ast x \in \mathcal{I}(C)\), proving that \( \mathcal{I}(C) \) is a closed ideal of \( X \).

**Theorem 17.** If \( I \) is a closed ideal of \( X \), then \( \mathcal{I}(\mathcal{G}(I)) = I \).

**Proof.** \( x \in \mathcal{I}(\mathcal{G}(I)) \iff (x,0) \in \mathcal{G}(I) \iff x,0 \ast x \in I \iff x \in I \).

**Theorem 18.** If \( C \) is a congruence relation on \( X \), then \( C \subset \mathcal{G}(\mathcal{I}(C)) \).

**Proof.** Suppose that \((x,y) \in C\). Then \((x \ast y,0) \in C\) and \((y \ast x,0) \in C\). This means that \(x \ast y, y \ast x \in \mathcal{I}(C)\), and hence \((x,y) \in \mathcal{G}(\mathcal{I}(C))\).

**Theorem 19.** Suppose that \( C \) is a congruence relation on \( X \). If \((x \ast y,0) \in C\) and \((y \ast x,0) \in C\), then for all integers \( i \geq 0, j \geq 0 \), we have \((Q_{i,j}(x,y),x) \in C\).

**Proof.** Since \( Q_{0,0}(x,y) = x \ast (x \ast y) \), and we have \((x \ast y,0) \in C\) and \((x,x) \in C\), it follows that \((Q_{i,0}(x,y),x) \in C\). Suppose that for some integer \( i \geq 0 \), we have \((Q_{i,0}(x,y),x) \in C\). Then, since \( Q_{i+1,0}(x,y) = Q_{i,0}(x,y) \ast (x \ast y) \) and \((x \ast y,0) \in C\), we have \((Q_{i+1,0}(x,y),Q_{i,0}(x,y)) \in C\) and hence \((Q_{i+1,0}(x,y),x) \in C\). Thus \((Q_{i,0}(x,y),x) \in C\) for all integers \( i \geq 0 \). Now fix an integer \( i \), and suppose that \((Q_{i,j}(x,y),x) \in C\) for some integer \( j \geq 0 \). Then since \( Q_{i+1,0}(x,y) = Q_{i,j}(x,y) \ast (y \ast x) \) and \((y \ast x,0) \in C\), we have \((Q_{i+1,0}(x,y),Q_{i,j}(x,y)) \in C\), and hence \((Q_{i+1,0}(x,y),x) \in C\). This proves the theorem.

**Theorem 20.** Suppose that \( X \) is quasi-commutative. Then for each congruence relation \( C \) on \( X \), we have \( \mathcal{G}(\mathcal{I}(C)) = C \).

**Proof.** We need only show that \( \mathcal{G}(\mathcal{I}(C)) \subset C \). Suppose that \((x,y) \in \mathcal{G}(\mathcal{I}(C))\). Then \( x \ast y, y \ast x \in \mathcal{I}(C) \) and hence \((x \ast y,0) \in C\) and \((y \ast x,0) \in C\). Suppose that \( X \) is quasi-commutative of type \((i,j; m, n)\). Then we have \((Q_{i,j}(x,y),x) \in C\) and \((Q_{m,n}(y,x),y) \in C\). Since \( Q_{i,j}(x,y) = Q_{m,n}(y,x) \), we have \((x,y) \in C\).

As a result of Theorems 17 and 20 we have the following result.

**Theorem 21.** Suppose that \( X \) is quasi-commutative. Then there is a one-to-one correspondence between the set of closed ideals in \( X \) and the set of congruences on \( X \) given by \( \mathcal{G} : \{ \text{closed ideals of } X \} \rightarrow \{ \text{congruences on } X \} \) with inverse given by \( \mathcal{I} : \{ \text{congruences on } X \} \rightarrow \{ \text{closed ideals of } X \} \).

**Theorem 22.** Suppose that \( I \) is a closed ideal of \( X \) and \( C \) is a congruence on \( X \). Then
1. \( \mathcal{G}(z_c) = z_{e,C} \),
2. \( \mathcal{I}(z_c) = z_{r,C} \),
3. \( \mathcal{I}(\mathcal{G}(z_c)) = \mathcal{I}(z_{e,C}) = z_{x(e,C)} = z_c \),
4. \( \mathcal{G}(\mathcal{I}(z_c)) = \mathcal{G}(z_{r,C}) = z_{x(r,C)} \geq z_c \).

If further, \( X \) is quasi-commutative, then \( \mathcal{G}(\mathcal{I}(z_c)) = z_c \).

**Proof.** Eqs. (3) and (4) follow from Eqs. (1) and (2) and Theorem 21. For Eq. (1), we have

\[
\mathcal{G}(z_c)(x,y) = \chi_J(x \ast y) \land \chi_J(y \ast x)
\]

\[
= \begin{cases} 
1 & \text{if } x \ast y, y \ast x \in I, \\
0 & \text{otherwise},
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } (x,y) \in \mathcal{G}(I), \\
0 & \text{if } (x,y) \notin \mathcal{G}(I),
\end{cases}
\]

\[= z_{e,I}(x,y).\]

For Eq. (2), we have

\[
\mathcal{I}(z_c)(x) = \chi_{z_c}(x,0)
\]

\[
= \begin{cases} 
1 & \text{if } (x,0) \in C, \\
0 & \text{if } (x,0) \notin C,
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } x \in \mathcal{I}(C), \\
0 & \text{if } x \notin \mathcal{I}(C),
\end{cases}
\]

\[= z_{r,C}(x).\]

Thus if we consider the subsets \{characteristic functions of closed ideals of \( X \}\} \subset \{fuzzy closed ideals of \( X \}\}, \{characteristic functions of congruences on \( X \}\} \subset \{fuzzy congruences on \( X \}\}, and apply Theorem 13 in case \( X \) is quasi-commutative, we have the following result.

**Theorem 18.** Suppose that \( X \) is quasi-commutative. Then the one-to-one correspondence \( \mathcal{G} : \{\text{fuzzy closed ideals of } X\} \rightarrow \{\text{fuzzy congruences on } X\} \) and its inverse \( \mathcal{I} : \{\text{fuzzy congruences on } X\} \rightarrow \{\text{fuzzy closed ideals of } X\} \) restrict to the subsets to give a one-to-one correspondence \( \mathcal{G} : \{\text{characteristic functions of closed ideals of } X\} \rightarrow \{\text{characteristic functions of congruences on } X\} \) with inverse \( \mathcal{I} : \{\text{characteristic functions of congruences on } X\} \rightarrow \{\text{characteristic functions of closed ideals of } X\} \).

**References**
