Distance-Regularised Graphs Are Distance-Regular or Distance-Biregular

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One problem with the theory of distance-regular graphs is that it does not apply directly to the graphs of generalised polygons. In this paper we overcome this difficulty by introducing the class of distance-regularised graphs, a natural common generalisation. These graphs are shown to either be distance-regular or fall into a family of bipartite graphs called distance-biregular. This family includes the generalised polygons and other interesting graphs. Despite this increased generality we are also able to extend much of the basic theory of distance-regular graphs to our wider class of graphs.

1. INTRODUCTION, EXAMPLES AND DEFINITIONS

A graph $G$ is distance-regular if, for any integer $k$ and vertices $x$ and $y$, the number of vertices at distance $k$ from $x$ and adjacent to $y$ only depends on $d(x, y)$, the distance between $x$ and $y$ (note that such graphs must be regular). A generalised polygon or generalised $n$-gon is a bipartite graph of diameter $n$ with vertices in the same colour class having the same degree and with pairs of vertices less than distance $n$ apart joined by a unique shortest path.

Both distance-regular graphs and generalised polygons are important combinatorial objects. Moreover they are closely related. Any generalised polygon is by definition semiregular and determines, in a natural way, two
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Any regular generalised polygon is itself a distance-regular graph. Given the importance of these two classes of graphs, and their close connection, it seems worthwhile to look for a common framework.

Let us call a graph distance-regularised if, for any integer $k$ and any vertices $x$ and $y$, the number of vertices at distance $k$ from $x$ and adjacent to $y$ only depends on $d(x, y)$ and the vertex $x$. Clearly any distance-regular graph is distance-regularised and also any generalised polygon is distance-regularised. However, it appears that, if we sought only to catch distance-regular graphs and generalised polygons, we have cast a rather wide net.

In this paper we show that a distance-regularised graph is either distance-regular or is a bipartite semiregular graph from which we can derive two distance-regular graphs. These bipartite graphs will be called distance-biregular (thus we caught almost exactly the fish we wanted). We then present a short study of the basic theory of distance-biregular graphs, roughly corresponding to that of distance-regular graphs (for which we refer the reader to, e.g., [1]). The theory of distance-biregular graphs is also discussed at length in [5].

The existence of generalised polygons implies, by our remarks above, that some distance-regular graphs come in pairs. One interesting consequence of our work is that this pairing is a more widespread phenomenon than was previously realised.

1.1. Examples of Distance-Biregular Graphs

We have already met the generalised polygons. The complete bipartite graphs form another, somewhat trivial, family. We now present some other nontrivial examples.

**Example 1.1.1.** Consider the set $\{1, \ldots, n\}$. Let $A = \{k\text{-subsets}\}$ and $B = \{k+1\text{-subsets}\}$ where $k$ is a number less than $n$. $V_G = A \cup B$ and adjacency is defined in the natural way: $u$ is adjacent to $v$, with $u \in A$ and $v \in B$ if $u \subseteq v$.

**Example 1.1.2.** Consider an $n$-dimensional vector space over $\text{GF}(q)$, where $q$ is the power of a prime and $\text{GF}(q)$ is the (unique) Galois field of order $q$. Let $A = \{k\text{-subspaces}\}$ and $B = \{k+1\text{-subspaces}\}$ and $V_G = A \cup B$. For $u \in A$ and $v \in B$, $u$ is adjacent to $v$ if $u \subseteq v$.

**Example 1.1.3.** Let $D$ be a quasisymmetric 2-design with block intersection numbers $i_1, i_2$, with $i_2 = 0$. Then the incidence graph $G$ of $D$ is a distance-biregular graph of diameter 4. (Any 2-design with $\lambda = 1$ is an example of such a design.)
1.2. Definitions and Notation

Let $G$ be a graph. By $\mathbf{VG}$ we denote the vertex set of $G$ and by $\mathbf{EG}$ the edge set. For $u, v \in \mathbf{VG}$ we say $u$ is adjacent to $v$ if $(u, v) \in \mathbf{EG}$. With $d(u, v)$ we denote the usual distance in $G$ between vertices $u$ and $v$. For $v \in \mathbf{VG}$ and $i \in \mathbb{N}$, $G_i(v)$ denotes the set of vertices at distance $i$ from $v$. For $u \in \mathbf{VG}$ and $v \in G_i(u)$ we write $c(u, v) = |G_i(u) \cap G_i(v)|$, $b(u, v) = |G_{i+1}(u) \cap G_i(v)|$, $a(u, v) = |G_i(u) \cap G_{i+1}(v)|$ and $k_i(u) = |G_i(u)|$.

Let $d(u) = \max \{i \mid G_i(u) \neq \emptyset \}$. We are interested in vertices $u \in \mathbf{VG}$ for which, for each $i$ ($1 \leq i < d(u)$), the numbers $b(u, v), a(u, v)$, and $c(u, v)$ are independent of the choice of $v \in G_i(u)$. In this case we say $u$ is distance-regularised and we denote $b(u, v), a(u, v)$, and $c(u, v)$ by $b_i(u), a_i(u)$, and $c_i(u)$. Then the array

\[
\begin{bmatrix}
  c_1(u), & \ldots, & c_{d(u)-1}(u), & c_{d(u)}(u) \\
  0 & a_1(u), & \ldots, & a_{d(u)-1}(u), & a_{d(u)}(u) \\
  b_0(u), & b_1(u), & \ldots, & b_{d(u)-1}(u), & * \\
\end{bmatrix}
\]

is called the intersection array for $u$, and the matrix

\[
I(u) = \begin{bmatrix}
  0 & c_1(u) & 0 & \cdots \\
  b_0(u) & a_1(u) & c_2(u) & \cdots \\
  0 & b_2(u) & a_2(u) & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  0 & 0 & 0 & b_{d(u)-1}(u) & a_{d(u)}(u) \\
  b_{d(u)-2}(u) & a_{d(u)-3}(u) & c_{d(u)}(u) & 0 & b_{d(u)-1}(u) & a_{d(u)}(u) \\
\end{bmatrix}
\]

is called the intersection matrix for $u$.

We will call a connected graph in which every vertex is distance-regularised a distance-regularised graph. The much studied distance-regular graphs are distance-regularised graphs in which all vertices have the same intersection array. Another special case of distance-regularised graphs are bipartite distance-regularised graphs in which vertices in the same colour class have the same intersection array. We call these graphs distance-biregular.

Unless explicitly stated, we use the following standardised notation for a distance-biregular graph. Sets $A$ and $B$ denote the colour partition of $\mathbf{VG}$, $d_A = d(u)$ ($u \in A$), $u$ is a vertex in $A$ and has intersection array

\[
i(A) = \begin{bmatrix}
  * & 1 & c_2 & \cdots & c_{d_A} \\
  0 & 0 & 0 & 0 & 0 \\
  r & b_1 & b_2 & \cdots & * \\
\end{bmatrix}
\]
or just
\[
\begin{bmatrix}
* & c_2 & \cdots & c_{d_A} \\
r & b_1 & b_2 & \cdots & *
\end{bmatrix},
\]
distance of \(v\) \((v \in B)\), \(v\) is a vertex in \(B\) and has intersection array
\[
i(B) = \begin{bmatrix}
* & f_2 & \cdots & f_{d_B} \\
s & e_1 & e_2 & \cdots & *
\end{bmatrix}.
\]

The corresponding intersection matrices are denoted \(I(A)\) and \(I(B)\), respectively. The diameter \(d\) of \(G\) is of course \(\max\{d_A, d_B\}\). Note that \(\deg(u) = r\) and \(\deg(v) = s\). We denote with \(k_i\), the numbers \(|G_i(u)|\) and with \(l_i\) the numbers \(|G_i(v)|\), \(i = 0, \ldots, d\). Note that \(l_{d-1} \neq 0\) and \(k_{d-1} \neq 0\) though one of \(l_d\) and \(k_d\) may be zero.

2. Distance-Regularised Graphs

We first present a lemma, which though not in itself very interesting is proved in a similar way to the main theorems and will be useful later.

**Lemma 2.1.** Let \(G\) be a distance-regularised graph. Then either \(G\) is regular or \(G\) is bipartite with vertices in the same colour class having the same degree.

**Proof.** Let \(v, v' \in VG\) with \(d(v, v') = 2\). We can therefore find \(u \in VG\) adjacent to both \(v\) and \(v'\). Then \(\deg(v) = b_1(u) + a_1(u) + c_1(u) = \deg(v')\). Let \(v\) and \(w\) be any vertices of \(G\) such that there exists a path \(v = v_1, v_2, \ldots, v_{2k+1} = w\) from \(v\) to \(w\) of even length. By the above \(\deg(v_{2i-1}) = \deg(v_{2i+1})\) for \(i = 1, \ldots, k\), and so \(\deg(v) = \deg(w)\). Assume now that \(G\) is not bipartite. In this case we can find a path of even length between any two vertices. Hence \(G\) is regular. If on the other hand \(G\) is bipartite, vertices in the same colour class are at even distance and so have the same degree. \(\blacksquare\)

We are now ready to tackle the main theorem of this section which deals with the non bipartite case.

**Theorem 2.2.** Let \(G\) be a nonbipartite distance-regularised graph, then \(G\) is distance-regular.

**Proof.** Let \(u, v \in VG\) with \(u\) adjacent to \(v\). We will prove by induction that these two vertices have the same intersection array. As \(G\) is connected the result will follow directly. Before beginning the inductive argument we
calculate the number \(|G_i(u) \cap G_i(v)|\). This is given by \(k_i(u) - r_i(u, v) - s_i(u, v)\), where \(r_i(u, v) = |G_i(u) \cap G_{i+1}(v)|\) and \(s_i(u, v) = |G_i(u) \cap G_{i-1}(v)|\). Note that \(s_i(u, v) = 1\) and \(r_i(u, v) = b_i(v)\). By counting edges between \(G_i(u) \cap G_{i-1}(v)\) and \(G_{i-1}(u) \cap G_{i-2}(v)\) \((t - 1 \leq d(v))\) we obtain

\[
c_i(u) s_i(u, v) = s_{i-1}(u, v) b_i(u)
\]
as each vertex in \(G_i(u)\) adjacent to a vertex in \(G_{i-1}(u) \cap G_{i-2}(v)\) must be in \(G_i(u) \cap G_{i-1}(v)\), while each of the \(c_{i-1}(v)\) neighbours nearer to \(v\) of a vertex in \(G_i(u) \cap G_{i-1}(v)\) must lie in \(G_{i-1}(u) \cap G_{i-2}(v)\). Hence

\[
s_i(u, v) = \frac{b_{i-1}(u) \cdots b_1(u)}{c_{i-1}(v) \cdots c_1(v)}.
\]

Similarly for \(t \leq d(u)\)

\[
r_i(u, v) = \frac{b_i(v) \cdots b_1(v)}{c_i(u) \cdots c_1(u)}.
\]

Note also that

\[
k_i(u) = \frac{b_{i-1}(u) \cdots b_0(u)}{c_i(u) \cdots c_1(u)}.
\]

We now start the induction on the columns of the intersection arrays. By Lemma 2.1 the first entry in each array is the same as \(G\) is regular. Now assume this is true for all entries up to and including the \((t - 1)\)-st column, for some \(t, 1 \leq t \leq d(u)\). In particular \(b_{t-1}(u) = b_{t-1}(v) \neq 0\), so \(d(v) \geq t\). The inductive assumption and the fact that \(t \leq d(u)\) allows us to evaluate

\[
|G_i(u) \cap G_i(v)| = k_i(u) - r_i(u, v) - s_i(u, v)
\]
as

\[
\frac{b_{t-1}(u) \cdots b_1(u)}{c_i(u) \cdots c_1(u)} \{b_0(u) - c_i(u) - b_i(v)\}. \quad (*)
\]

We consider two cases.

Case 1. \(G_i(u) \cap G_i(v) = \emptyset\).

By the above formula and the fact that \(t \leq \min\{d(u), d(v)\}\), \(c_i(u) + b_i(v) = k\), the degree of \(G\). By the symmetry of \(|G_i(u) \cap G_i(v)|\) and \(t \leq \min\{d(u), d(v)\}\), \(c_i(v) + b_i(u) = k\) and so

\[
c_i(u) + b_i(u) \mid b_i(v) \mid c_i(v) = 2k
\]
and we must have \(c_i(u) + b_i(u) = k = b_i(v) + c_i(v)\). In this case \(a_i(u) = \)
\( a_i(v) = 0 \). Note also that \( b_i(v) = k - c_i(u) = b_i(u) \), so that the arrays of \( u \) and \( v \) agree in the \( t \)-th column.

**Case 2.** \( G_i(u) \cap G_i(v) \neq \emptyset \).

Hence \( t \leq \min \{ d(u), d(v) \} \). Let \( w \in G_i(u) \cap G_i(v) \) and \( q_i = |G_i(w) \cap G_i(u)| \). Clearly \( q_1 = c_i(u) \) and we can readily evaluate

\[
q_i = \frac{c_i(u) \cdots c_i(t+1)(u)}{c_i(w) \cdots c_i(t+1)(w)}.
\]

Using the induction hypothesis \( q_{i-1} = c_i(u) \). But \( q_{i-1} = c_i(w) \) by definition and so \( c_i(u) = c_i(w) \). Similarly since \( w \in G_i(u) \cap G_i(v) \neq \emptyset \), \( c_i(w) - c_i(v) \) and so \( c_i(u) = c_i(v) \). Finally calculating \( |G_i(u) \cap G_i(v)| \) in two ways we have \( c_i(u) + b_i(v) = c_i(v) + b_i(u) \), so \( b_i(v) = b_i(u) \) and the \( t \)-th column of the arrays of \( u \) and \( v \) agree.

In either case the intersection arrays for \( u \) and \( v \) agree up to the \( d(u) \)-th column. But then \( b_{d(u)}(v) = 0 \) and so \( d(v) = d(u) \). Hence the arrays are identical.

We have now dealt with the nonbipartite case. To cover the bipartite case we present

**Lemma 2.3.** Let \( G \) be a bipartite distance-regularised graph with \( u, v \in V_G \) and \( u \) adjacent to \( v \). Then the intersection array for \( v \) can be computed from that of \( u \).

**Proof.** Note first that \( |d(u) - d(v)| \leq 1 \). We compute the intersection array for \( v \). We have \( G_i(v) \subseteq G_{i-1}(u) \cup G_{i+1}(u) \). Set \( x_i = |G_i(u) \cap G_{i-1}(v)| \). Thus in the notation and by the derivation in the proof of Theorem 2.2

\[
x_i = \frac{b_1(u) b_2(u) \cdots b_{i-1}(u)}{c_1(v) c_2(v) \cdots c_{i-1}(v)} \quad \text{for} \quad i = 1, \ldots, d(u).
\]

Then \( x_1 = 1, x_2 = b_1(u) \). Note also that \( k_0(v) = 1, k_1(v) = b_1(u) + 1 \). Assume now that we know \( b_i(v), c_j(v), k_i(v), j < i \) for some \( i, 1 < i \leq d(v) \). If \( i = d(u) + 1 \), then \( i = d(u) \) so \( c_i(v) = b_0(u) \), if \( d(v) \) is odd and \( c_i(v) = b_1(u) + 1 \), otherwise. Clearly \( b_i(v) = 0 \) and so we have computed the whole of \( i(v) \). Hence we can assume that \( i \leq d(u) \), enabling us to calculate \( x_i \). This also means that \( b_i(u) \) is defined (though possibly 0) and that \( c_i(v) \neq 0 \). So \( k_i(v) = k_{i-1}(u) - x_{i-1} + x_{i+1}, \) since

\[
|G_i(v) \cap G_{i-1}(u)| = |G_{i-1}(u)| - x_{i-1}.
\]

Note that \( x_{i+1} = x_i b_i(u)/c_i(v) \), which correctly computes to 0 if \( i = d(u) \).
We also have \( k_i(v) = k_{i-1}(v) \frac{b_{i-1}(v)}{c_{i-1}(v)} \). If \( k_{i-1}(u) = x_{i-1} \) then \( G_{i-1}(u) \subseteq G_{i-2}(v) \) and so \( G_i(v) = \emptyset \), as otherwise we could find a vertex in \( G_i(v) \cap G_{i-1}(u) \). Hence \( d(v) - i - 1 < i \), a contradiction. We conclude that \( k_{i-1}(u) \neq x_{i-1} \), which enables us to eliminate \( k_i(v) \) and obtain

\[
c_i(v) - (k_{i-1}(v) b_{i-1}(v) - x_i b_i(u))/(k_{i-1}(u) - x_{i-1}).
\]

We can then of course evaluate

\[
b_i(v) = b_{i-1}(u) + c_{i-1}(u) - c_i(v),
\]

while \( k_i(v) = k_{i-1}(v) \frac{b_{i-1}(v)}{c_{i-1}(v)} \). This completes the calculations of another column of the array. We can thus inductively compute the array for \( v \) to the \( d(v) \)-th column, that is we can compute the whole of \( i(v) \).

**Corollary 2.3.1.** A bipartite distance-regularised graph is distance-biregular.

**Proof.** Let \( u, w \) be vertices of a bipartite distance-regularised graph \( G \) which lie in the same colour class. Then there exists a path of even length from \( u \) to \( w \). Alternate vertices along this path have the same intersection array by the lemma. Hence \( u \) and \( w \) have the same array.

**Corollary 2.3.2.** Let \( G \) be a distance-biregular graph with the standard notation. Then the intersection array \( i(B) \) can be computed from the array \( i(A) \) using the following method: set \( s = b_1 + c_1, e_0 = s, f_1 = 1, e_1 = b_0 - 1, l_0 = 1, x_1 = 1, x_2 = b_1 \).

Then for \( i = 2, \ldots, \min\{d_A, d_B\} \) we have

\[
f_i = (l_{i-1} e_{i-1} - x_i b_i)/(k_{i-1} - x_{i-1}),
\]

\[
e_i = b_{i-1} + c_{i-1} - f_i,
\]

\[
x_{i+1} = x_i b_i/f_i,
\]

\[
l_i = l_{i-1} e_{i-1}/f_i,
\]

where \( d_B \) is the first \( i \) for which \( e_i = 0 \). If \( d_B > d_A \) then

\[
f_{d_B} = \begin{cases} s & \text{if } d \text{ is even} \\ r & \text{if } d \text{ is odd} \end{cases}
\]

while \( e_{d_B} = 0 \).

**Proof.** A distance-biregular graph is a bipartite distance-regularised
graph. Hence we can compute the second intersection array using the method of the lemma. The equations obtained in the lemma are those listed.

3. Feasible Arrays for a Distance-Biregular Graph

We begin by stating the main result of this section.

**Theorem 3.1.** Let $G$ be a distance-biregular graph. Then the eigenvalues of $G$ and their multiplicities can be determined from either of its two intersection arrays.

**Proof.** We begin a proof of this theorem by introducing some notation. For any square matrix $A$ we define

$$W(A, x) = \sum_{k=0}^{\infty} x^k A^k = (I - xA)^{-1}$$

and $\phi(A, x) = \det(xI - A)$. With a slight abuse of notation we write $W(G, x)$ for $W(A, x)$ and $\phi(G, x)$ for $\phi(A, x)$, where $G$ is a digraph with adjacency matrix $A$. $W(G, x)$ is called the walk generating function for $G$, while $\phi(G, x)$ is the characteristic polynomial of $G$. The basic results for walk generating functions which we will require are the following:

(i) for $v \in VG$, $W_{vv}(G, x) = (1/x) \cdot \phi(G - v, 1/x)/\phi(G, 1/x)$,

(ii) $\text{trace}(W(G, x)) = -x \cdot \phi'(G, 1/x)/\phi(G, 1/x)$.

A proof of (i) is given in [2], while (ii) is an immediate consequence of (i).

Consider a distance-biregular graph with the standard notation. Let $P$ be the intersection matrix $I(A)$ for each vertex $u$ in $A$ and $Q$ the matrix $I(B)$ for each vertex $v$ in $B$. It can be readily verified by induction that for $u \in A$, the number of walks of length $k$ in $G$ which start at a specified vertex in $G,(u)$ and finish anywhere in $G,(u)$ is $(P^k)_u$. A similar result holds for $Q$. This in turn means that $W_{uu}(G, x) = W_{00}(P, x)$ for $u \in A$ and $W_{vv}(G, x) = W_{00}(Q, x)$ for $v \in B$. Hence we can perform the following calculation:

$$-x\phi'(G, 1/x)/\phi(G, 1/x) = \text{trace}(W(G, x))$$

$$= \sum_{u \in A} W_{uu}(G, x) + \sum_{v \in B} W_{vv}(G, x)$$

$$= nW_{00}(P, x) + mW_{00}(Q, x)$$

$$= n(1/x) \phi(P - 0, 1/x)/\phi(P, 1/x) + m(1/x) \phi(Q - 0, 1/x)/\phi(Q, 1/x),$$
where $P - 0$ is the matrix obtained from $P$ by deleting the first row and column. Similarly for $Q - 0$. Replacing $x$ by $1/x$ yields:

$$\phi'(G, x)/\phi(G, x) = n\phi(P - 0, x)/\phi(P, x) + m\phi(Q - 0, x)/\phi(Q, x).$$

The matrix $P$ is the adjacency matrix of a quotient multigraph of $G$ of diameter $d_A$. Hence the eigenvalues of $P$ are eigenvalues of $G$ and $P$ has at least $d_A + 1$ distinct eigenvalues [1], so all its eigenvalues must be simple. A similar argument holds for $Q$. This means we can write:

$$1/\phi(P, x) = \sum_{\theta \in \lambda(P)} 1/((x - \theta) \phi'(P, \theta))$$

and similarly for $Q$. For the I.h.s. we have

$$\phi'(G, x)/\phi(G, x) = \sum_{\theta \in \lambda(G)} m(\theta)/(x - \theta)$$

where $m(\theta)$ is the multiplicity of $\theta$ in $G$. Hence

$$\sum_{\theta \in \lambda(G)} m(\theta)/(x - \theta) = n \sum_{\theta \in \lambda(P)} \phi(P - 0, x)/\phi'(P, \theta)(x - \theta)) + m \sum_{\theta \in \lambda(Q)} \phi(Q - 0, x)/\phi'(Q, \theta)(x - \theta))$$

equating residuals we obtain for each $\theta \in \lambda(G)$,

$$m(\theta) = n\phi(P - 0, \theta) \chi_{\lambda(P)}(\theta)/\phi'(P, \theta) + m\phi(Q - 0, \theta) \chi_{\lambda(Q)}(\theta)/\phi'(Q, \theta). \quad (***)$$

Equation (*** enables us to calculate the multiplicities for each eigenvalue of $\lambda(G)$ and also tells us we have all the eigenvalues of $G$ present on the r.h.s.:

$$\lambda(G) = \lambda(P) \cup \lambda(Q).$$

In the theory of distance-regular graphs the multiplicities of the eigenvalues are normally expressed in terms of components of the eigenvectors of the intersection array. A similar formula can be obtained in our case. To be more precise, if $t$ is an eigenvalue of $P$ and $y(x)$ a left (right) eigenvector corresponding to $t$, normalised with $x_0 = y_0 = 1$. Then

$$\phi'(P, t)/\phi(P - 0, t) = y^T x.$$ 

Of course the same holds for $Q$. For details see [5].
Theorem 3.1 makes it reasonable to define a pair of feasible arrays for a distance-biregular graph in an analogous way to feasible arrays for distance-regular graphs [11]. We give here a definition by outlining a list of conditions which the two arrays must satisfy. Any statements that have not already been proved are elementary (Proofs are given in [5]).

**Definition 3.2.** Two intersection arrays are said to be a pair of feasible arrays for a distance-biregular graph if

(i) they satisfy the following numerical conditions:

\[
\begin{align*}
    k_0 &= 1, & k_{i+1} &= k_i \cdot \frac{b_i}{c_i}, \\
    l_0 &= 1, & l_{i+1} &= l_i \cdot \frac{e_i}{f_i},
\end{align*}
\]

and the \( k_i \) and \( l_i \) are whole numbers.

Alternate (nonzero) columns in the intersection arrays sum to \( r \) and \( s \),

\[
\begin{align*}
    c_i + b_i &= \begin{cases} r & \text{if } i \text{ is even} \\ s & \text{if } i \text{ is odd} \end{cases}, \\
    e_j + f_j &= \begin{cases} r & \text{if } j \text{ is odd} \\ s & \text{if } j \text{ is even} \end{cases}, \\
    e_{i-1} &\geq b_i, & i &= 1, \ldots, d_A - 1, \\
    b_{i-1} &\geq e_i, & i &= 1, \ldots, d_B - 1, \\
    f_i &\geq c_{i-1}, & i &= 2, \ldots, d_B, \\
    c_i &\geq f_{i-1}, & i &= 2, \ldots, d_A, \\
\end{align*}
\]

\[1 + k_2 + k_4 + \cdots + k_{d'} = l_1 + l_3 + \cdots + l_{d''} = n\]

and

\[k_1 + k_3 + \cdots + k_{d'} = l_2 + l_4 + \cdots + l_{d''} = m,\]

where \( d' \) is the largest even integer less than or equal to \( d \) and \( d'' \) is the largest such odd integer. Also \( nr = ms \).

(ii) Each array can be computed from the other using the formulas of Corollary 2.3.2.

(iii) The values determined as multiplicities using Theorem 3.1 are positive integers.
4. FURTHER DIRECTIONS

There is a clear need to determine whether the classes of distance-biregular graphs mentioned in this paper exhaust, in any sense, the possibilities. One approach to finding examples would be to attempt a thorough classification of the distance-biregular graphs with small minimum valency. The results of Section 3 should be useful for this. The distance-biregular graphs with vertices of degree two are completely characterised in [3].

We have not considered any group theoretic questions in the present paper. A reasonably complete development of the theory of distance-bitransitive graphs (the distance-biregular analogue of distance-transitive graphs) is presented in [4].

REFERENCES