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KILLING VECTOR FIELDS AND LAGRANGIAN SUBMANIFOLDS OF THE NEARLY KAEHLER S^6

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ABSTRACT. – In this paper, we study Lagrangian submanifolds of the nearly Kähler 6-sphere $S^6(1)$. We obtain a classification of the Lagrangian submanifolds which admit a unit length Killing vector field whose integral curves are great circles by relating them to almost complex surfaces in $S^6(1)$ or holomorphic curves in $\mathbb{C}P^2(4)$.
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1. Introduction

Considering $\mathbb{R}^7$ as the imaginary Cayley numbers, it is possible to introduce a vector cross product $\times$ on $\mathbb{R}^7$, which in its turn induces an almost complex structure J on the standard unit sphere S^6 in $\mathbb{R}^7$ which is compatible with the standard metric. Details about this construction are recalled in Section 2.

With respect to the almost complex structure J on S^6, there are two natural classes of submanifolds M to be investigated. Namely those which are almost complex, *i.e.* those for which J maps the tangent space into itself and those which are totally real, *i.e.* those for which J maps the tangent space into the normal space. It is shown in [G] that if M is an almost complex submanifold then the dimension of M equals two.

Almost complex surfaces are always minimal and have ellipse of curvature a circle, *i.e.* the map $v \mapsto \alpha(v,v)$, where $v \in UM_p$ and α denotes the second fundamental form describes a circle in the normal space. If the map $v \mapsto (\nabla\alpha)(v,v,v)$, $v \in UM_p$ also describes a circle, then M^2 is called superminimal. This class of almost complex surfaces has been investigated by Bryant [B]. In particular, he showed the existence of almost complex superminimal surfaces in S^6 for every genus. Other special classes of almost complex submanifolds have been studied in [BPW] and [BVW1].

A totally real submanifold of S^6 is either 2- or 3-dimensional. Unlike almost complex surfaces, a totally real surface need not be minimal. However, if it is minimal a classification can be found in [BVW2] relating these to either:

* minimal surfaces in S^3,
* totally real minimal surfaces in $\mathbb{C}P^2$,

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* a special class of S^1-symmetric immersions of $\mathbb{R}^2$ into S^6 contained in either a totally geodesic S^4 or S^5.

Here in this paper, we want to investigate 3-dimensional totally real submanifolds of S^6. Traditionally, those submanifolds are called Lagrangian submanifolds. It was shown by Ejiri [E1] that such submanifolds are automatically minimal. Some special classes of examples of Lagrangian submanifolds were previously studied in [DV2] and [DDVV]. It turns out that all of these examples are related to either almost complex surfaces in S^6 or holomorphic surfaces in $\mathbb{C}P^2$. We investigate the Lagrangian submanifolds of S^6 which admit a unit length Killing vector field X whose integral curves are great circles in S^6. We relate these submanifolds with Hopf lifts of holomorphic curves in $\mathbb{C}P^2$ and with tubes of radius $\frac{\pi}{2}$ in the direction of the first or second normal bundle on a superminimal almost complex surface. A more precise formulation will be given in Section 3.

Finally, in Section 4, we then investigate some further properties of those holomorphic curves which are tubes of radius $\frac{\pi}{2}$ in the direction of the first normal bundle. For properties of the two other classes of Lagrangian submanifolds mentioned in the theorem, we refer to [DV2].

2. Preliminaries

We give a brief exposition of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from the Cayley multiplication. We also describe how we can use the vector cross product on $\mathbb{R}^7$ in order to define the Sasakian structure on $S^5(1)$. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [W] and [HL].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = \frac{1}{2}(uv - vu), \tag{2.1}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$(u, v) = -\frac{1}{2}(uv + vu). \tag{2.2}$$

It is now elementary [HL] to show that

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u \tag{2.3}$$

and that the triple scalar product $(u \times v, w)$ is skew symmetric in u, v, w.

Conversely, Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by:

$$(r + u)(s + v) = rs - (u, v) + rv + su + (u \times v), \quad r, s \in Re(\mathcal{O}), u, v \in Im(\mathcal{O}). \tag{2.4}$$

In view of (2.1), (2.2) and (2.4), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product.

An ordered basis $u_1, ..., u_7$ is said to be a G_2-frame if

$$u_3 = u_1 \times u_2, \quad u_5 = u_1 \times u_4, \quad u_6 = u_2 \times u_4, \quad u_7 = u_3 \times u_4. \tag{2.5}$$

For example, the standard basis $e_1, ..., e_7$ of $\mathbb{R}^7$ is a G_2-frame. Moreover, if u_1, u_2, u_4 are mutually orthogonal unit vectors with u_4 orthogonal to $u_1 \times u_2$, then u_1, u_2, u_4 determine a unique G_2-frame $u_1, ..., u_7$ and $(\mathbb{R}^7, \times)$ is generated by u_1, u_2, u_4 subject to the relations :

$$u_i \times (u_j \times u_k) + (u_i \times u_j) \times u_k = 2\delta_{ik}u_j - \delta_{ij}u_k - \delta_{jk}u_i. \tag{2.6}$$

Therefore, for any G_2-frame, we have the following very usefull multiplication table [W] :

x	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	0	u_3	$-u_2$	u_5	$-u_4$	$-u_7$	u_6
u_2	$-u_3$	0	u_1	u_6	u_7	$-u_4$	$-u_5$
u_3	u_2	$-u_1$	0	u_7	$-u_6$	u_5	$-u_4$
u_4	$-u_5$	$-u_6$	$-u_7$	0	u_1	u_2	u_3
u_5	u_4	$-u_7$	u_6	$-u_1$	0	$-u_3$	u_2
u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	0	$-u_1$
u_7	$-u_6$	u_5	u_4	$-u_3$	$-u_2$	u_1	0

The standard nearly Kaehler structure on $S^6(1)$ is then obtained as follows :

$$Ju = x \times u, \quad u \in T_xS^6(1), \quad x \in S^6(1).$$

It is clear that J is an orthogonal almost complex structure on $S^6(1)$. In fact J is a nearly Kähler structure in the sense that the $(2,1)$-tensor field G on $S^6(1)$ defined by

$$G(X,Y) = (\tilde{\nabla}_X J)Y,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$ is skew-symmetric. A straightforward computation also shows that

$$G(X,Y) = X \times Y - \langle x \times X, Y \rangle x.$$

For more information on the properties of ., J and G, we refer to [BVW1] and [DVV].

It is well-known (*see* for instance [B, page 32] or [DV1]) that the complex structure of $\mathbb{C}^3$ induces a Sasakian structure (φ, ξ, η, g) on $S^5(1)$ starting from $\mathbb{C}^3$. This structure can also be expressed using the vector cross product. We consider $S^5(1)$ as the hypersphere in $S^6(1) \subset \mathbb{R}^7$ given by the equation $x_4 = 0$ and define :

$$j : S^5(1) \to \mathbb{C}^3 : (x_1, x_2, x_3, 0, x_5, x_6, x_7) \mapsto (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7).$$

Then at a point $p = (x_1, x_2, x_3, 0, x_5, x_6, x_7)$, the structure vector field ξ is given by:

$$\xi(p) = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3) = e_4 \times p,$$

and for any tangent vector v, we get that

$$\varphi(v) = v \times e_4 - \langle v \times e_4, p\rangle p. \tag{2.7}$$

Following [YI], we call a submanifold M^n of S^5 (1) invariant if $\varphi(T_pM) \subset T_pM$ for every p. If n is odd, then ξ is automatically tangent to M. Assume $n = 3$. The Hopf fibration $h : S^5(1) \to \mathbb{C}P^2(4)$ annihilates ξ, *i.e.* $dh(\xi) = 0$. Then if M^3 is invariant, $h(M^3)$ is a holomorphic curve. Conversely, let $\phi : N_1 \to \mathbb{C}P^2(4)$ be a holomorphic curve, let PN_1 be the circle bundle over N_1 induced by the Hopf fibration and let ϕ be the immersion such that the following diagram commutes :

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \downarrow & & \downarrow h \\ N_1 & \xrightarrow{\phi} & \mathbb{C}^2(4) \end{array} \tag{2.8}$$

Then ψ is an invariant immersion in the Sasakian space form $S^5(1)$ with structure vector field ξ tangent along ξ.

3. Lagrangian submanifolds admitting a certain Killing vector field

Let $F : M^3 \to S^6(1)$ be a Lagrangian immersion. Then, as was shown by Ejiri [E1], M^3 is minimal and $G(X, Y)$ is orthogonal to M for any tangent vector fields X and Y. We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then respectively given by:

$$\tilde{\nabla}_X F_*(Y) = F_*(\nabla_X Y) + h(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X \eta = -F_*(A_\eta X) + \nabla^{\perp}_X \eta, \tag{3.2}$$

for tangent vector fields X and Y and normal vector η. The second fundamental form h is related to A_η by $\langle h(X, Y, \eta\rangle = \langle A_\eta X, Y\rangle$. From (3.1) and (3.2) we find that

$$\nabla^{\perp}_X JF_*(Y) = JF_*(\nabla_X Y) + G(F_*X, F_*Y) \tag{3.3}$$

$$F_*(A_{JY}X) = -Jh(X, Y). \tag{3.4}$$

The above formulas imply immediately that

$$\langle h(X, Y), JF_*Z\rangle = \langle h(X, Z), JF_*, Y\rangle, \tag{3.5}$$

i.e. $\langle h(X, Y), JF_*Z\rangle$ is totally symmetric. Whenever there is no confusion possible, we will identify M with its image in S^6 and therefore omit F_* in the equations (3.1) up to (3.5).

Now, we investigate the Lagrangian submanifolds of $S^6(1)$ who admit a unit length Killing vector field whose integral curves are great circles in $S^6(1)$. We start by introducing a class of Lagrangian submanifolds.

THEOREM 1. – *Let $\phi : N^2 \to S^6(1)$ be an almost complex surface without totally geodesic points. Define*

$$\psi : UN^2 \to S^6(1) : v \mapsto \frac{\alpha(v,v)}{\|\alpha(v,v)\|}.$$

Then, ψ defines a Lagrangian immersion if and only if φ is superminimal.

Proof. – Let $\{F_1, ..., F_7\}$ be the G_2-frame defined by $F_1 = \phi, F_2 = \phi_s V, F_3 = J\phi_*(V), F_4 = \alpha(V,V)/\mu, F_5 = \alpha(V,U)/\mu = J\alpha(V,V)/\mu = F_1 \times F_4, F_6 = F_2 \times F_4$ and $F_7 = F_3 \times F_4$. Here $\{U, V\}$ are a local orthonormal frame on N^2, α is the second fundamental form of the surface in $S^6(1)$ and $\mu = \|\alpha(V,V)\|$. We also use the formulas of Section 5 of [DV2]. It follows that we can parameterize ψ by

$$\psi(q,t) = \cos tF_4(q) + \sin tF_5(q).$$

Hence

$$\psi_*\left(\frac{\partial}{\partial t}\right) = -\sin tF_4 + \cos tF_5, \tag{3.6}$$

$$\begin{aligned}\psi_*(V) = {} & (a_2 + 2\mu_1)\psi_*\left(\frac{\partial}{\partial t}\right) - \mu(\cos tF_2 + \sin tF_3) \\ & + (a_3\cos t + (1+a_4)\sin t)F_6 + (a_4\cos t - a_3\sin t)F_7,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\psi_*(U) = {} & (a_1 - 2\mu_2)\psi_*\left(\frac{\partial}{\partial t}\right) + \mu(-\sin tF_2 + \cos tF_3) \\ & + ((1+a_4)\cos t - a_3\sin t)F_6 + (-a_3\cos t - a_4\sin t)F_7,\end{aligned} \tag{3.8}$$

where $\mu = \|h(v,v)|$. Therefore, since $\mu \neq 0$, ψ defines an immersion. First of all, we notice that

$$\psi \times \psi_*\left(\frac{\partial}{\partial t}\right) = F_1$$

is orthogonal to $\psi_*(\frac{\partial}{\partial t}), \psi_*(V)$ and $\psi_*(U)$. Next, we put

$$X = V - (a_2 + 2\mu_1)\frac{\partial}{\partial t},$$
$$Y = U - (a_1 - 2\mu_2)\frac{\partial}{\partial t}.$$

We then get that

$$\begin{aligned}\psi \times \psi_*(X) = {} & \mu\cos 2tF_6 + \mu\sin 2tF_7 + \left(a_3\cos 2t + \frac{1}{2}\sin 2t + a_4\sin 2t\right)F_2 \\ & + (a_4\cos 2t - a_3\sin 2t - \sin^2 t)F_3.\end{aligned} \tag{3.9}$$

Clearly $\psi \times \psi_*(X)$ is orthogonal to both $\psi_*(X)$ and $\psi_*(\frac{\partial}{\partial t})$. Hence $\psi : UN^2 \to S^6$ defines a Lagrangian immersion if and only if

$$0 = \langle \psi \times \psi_*(X), \psi_*(Y) \rangle. \tag{3.10}$$

Using (3.9) and (3.8), we see that (3.10) is equivalent to

$$0 = \mu(-2a_3 \sin 3t + (2a_4 + 1) \cos 3t).$$

Since this has to be valid for all values of t, we deduce that ψ is Lagrangian if and only if $a_3 = 0$ and $a_4 = -\frac{1}{2}$, *i.e.* if and only if $\phi : N^2 \to S^6(1)$ is a superminimal almost complex curve. ∎

In case a non totally geodesic surface is branched or has totally geodesic points, it is still possible to define plane bundles L_0 and L_1 such that L_0 and L_1 correspond to ϕ_* (TN^2) and the first normal space except at the isolated branch points or totally geodesic points (*see* [BW]). This allows us to extend ψ to branched points and totally geodesic points. Since for an almost complex superminimal surface, we have $a_3 = 0$ and $a_4 = -\frac{1}{2}$, it immediately follows from (3.6), (3.7) and (3.8) that if ϕ is an immersion, ψ is an immersion too.

LEMMA 3.1. – *Let $\varphi : N^2 \to S^6$ be an almost complex superminimal immersion without totally geodesic points and let ψ be as defined in Theorem 1.*

Then $\psi_*(\frac{\partial}{\partial t})$ is a unit length Killing vector field whose integral curves are great circles.

Proof. – Since

$$D_{\frac{\partial}{\partial t}} \psi_*\left(\frac{\partial}{\partial t}\right) = -\cos t F_4 - \sin t F_5 = -\psi,$$

it follows that

$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0,$$
$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0,$$
$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = 1.$$

This proves that $\psi_*(\frac{\partial}{\partial t})$ is a unit length vector field whose integral curves are great circles in $S^6(1)$. Therefore, in order to show that $\psi_*(\frac{\partial}{\partial t})$ is Killing, it is sufficient to show that :

$$\left\langle \nabla_X \frac{\partial}{\partial t}, X \right\rangle = \left\langle \nabla_Y \frac{\partial}{\partial t}, Y \right\rangle = 0, \tag{3.11}$$

$$\left\langle \nabla_X \frac{\partial}{\partial t}, Y \right\rangle + \left\langle \nabla_Y \frac{\partial}{\partial t}, X \right\rangle = 0. \tag{3.12}$$

In order to show (3.11) and (3.12), we first remark that since $\varphi : N^2 \to S^6(1)$ is superminimal, (3.7) and (3.8) reduce to:

$$\psi_*(X) = \psi_*\left(V - (a_2 + 2\mu_1)\frac{\partial}{\partial t}\right) = -\mu(\cos t F_2 + \sin t F_3) + \frac{1}{2}(\sin t F_6 - \cos t F_7), \tag{3.13}$$

$$\psi_*(Y) = \psi_*\left(U - (a_1 - 2\mu_2)\frac{\partial}{\partial t}\right) = -\mu(\sin t F_2 - \cos t F_3) + \frac{1}{2}(\cos t F_6 + \sin t F_7). \tag{3.14}$$

Now, it follows that

$$
\begin{aligned}
(3.15)\qquad D_X\psi_*\left(\frac{\partial}{\partial t}\right) &= D_{V-(a_2+2\mu_1)\frac{\partial}{\partial t}}\psi_*\left(\frac{\partial}{\partial t}\right)\\
&= D_{V-(a_2+2\mu_1)\frac{\partial}{\partial t}}(-\sin tF_4+\cos tF_5)\\
&= -\sin t\left(-\mu F_2+(a_2+2\mu_1)F_5-\frac{1}{2}F_7\right)\\
&\quad+\cos t\left(-\mu F_3-(a_2+2\mu_1)F_4+\frac{1}{2}F_6\right)\\
&\quad-(a_2+2\mu_1)(-\cos tF_4-\sin tF_5)\\
&= \mu(\sin tF_2-\cos tF_3)+\frac{1}{2}(\cos tF_6+\sin tF_7).
\end{aligned}
$$

Similarly, we obtain that

$$
(3.16)\qquad D_Y\psi_*\left(\frac{\partial}{\partial t}\right)=\mu(-\cos tF_2-\sin tF_3)+\frac{1}{2}(-\sin tF_6+\cos tF_7).
$$

On the other hand, (3.9) reduces to

$$
(3.17)\qquad \psi\times\psi_*(X)=\mu\cos 2tF_6+\mu\sin 2tF_7-\frac{1}{2}F_3.
$$

Using similar computations, we also find that

$$
(3.18)\qquad \psi\times\psi_*(Y)=\mu\sin 2tF_6-\mu\cos 2tF_7+\frac{1}{2}F_2.
$$

Therefore, (3.15) and (3.16) imply that :

$$
\begin{aligned}
(3.19)\qquad \nabla_X\frac{\partial}{\partial t} &= \frac{1}{(\mu^2+\frac{1}{4})}\left(-\left(\mu^2-\frac{1}{4}\right)Y\right)\\
(3.20)\qquad \nabla_Y\frac{\partial}{\partial t} &= \frac{1}{(\mu^2+\frac{1}{4})}\left(\left(\mu^2-\frac{1}{4}\right)X\right)\\
(3.21)\qquad h\left(X,\frac{\partial}{\partial t}\right) &= \frac{1}{\mu^2+\frac{1}{4}}(\mu\cos t\psi\times\psi_*(X)+\mu\sin t\psi\times\psi_*(Y))\\
(3.22)\qquad h\left(Y,\frac{\partial}{\partial t}\right) &= \frac{1}{\mu^2+\frac{1}{4}}(\mu\sin t\psi\times\psi_*(X)-\mu\cos t\psi\times\psi_*(Y)).
\end{aligned}
$$

The equations (3.11) and (3.12) now immediately follow from (3.19) and (3.20). This completes the proof of the Lemma. ■

Now, we are able to formulate the Main Theorem of this paper :

THEOREM 2. – *Let $F : M^3 \to S^6(1)$ be a Lagrangian immersion which admits a unit length Killing vector field whose integral curves are great circles. Then there exist an open dense subset U of M^3 such that each point p of U has a neighborhood V such that $F : V \to S^6(1)$ satisfies Chen's equality, or $F : V \to S^6(1)$ is obtained as in Theorem* 1.

Proof. – The proof of the theorem will be divided into different steps. We start with the construction of a orthonormal tangent frame on an open dense subset of M.

Step 1. – We choose E_3 as the unit length Killing vector field whose integral curves are great circles in S^6. Then we have

$$h(E_3, E_3) = 0. \tag{3.23}$$

Hence $A_{JE_3}E_3 = 0$ and E_3 is an eigenvector of A_{JE_3}. Denote by $U_1 = \{p \in M | A_{JE_3} \equiv 0\}$ and by $U_2 = M \setminus \bar{U}_1$. Then $U_1 \cup U_2$ is an open dense subset of M.

First, assume that $p \in U_2$. Then $A_{JE_3} = 0$. Hence it follows that $h(E_3, X) = 0$, for any tangent vector field X. From [CDVV1] and [CDVV2] we know that this implies that M satisfies Chen's equality.

Therefore, we may assume that $p \in U_1$. Since by [E1], a Lagrangian submanifold is automatically minimal, it follows that we can choose local differentiable vector fields E_1 and E_2 such that $G(E_1, E_2) = JE_3$ and

$$h(E_1, E_3) = \lambda JE_1, \tag{3.24}$$

$$h(E_2, E_3) = -\lambda JE_2, \tag{3.25}$$

where λ is a nonzero function. We then introduce local functions α and β such that

$$h(E_1, E_1) = \alpha JE_1 - \beta JE_2 + \lambda JE_3, \tag{3.26}$$

$$h(E_2, E_2) = -\alpha JE_1 + \beta JE_2 - \lambda JE_3, \tag{3.27}$$

$$h(E_1, E_2) = -\beta JE_1 - \alpha JE_2. \tag{3.28}$$

Since E_3 is a Killing vector field, we also know that we can write

$$\begin{array}{lll} \nabla_{E_3}E_1 = aE_2, & \nabla_{E_3}E_2 = -aE_1, & \nabla_{E_3}E_3 = 0, \\ \nabla_{E_1}E_3 = bE_2, & \nabla_{E_1}E_1 = cE_2, & \nabla_{E_1}E_2 = -cE_1 - bE_3, \\ \nabla_{E_2}E_3 = -bE_1, & \nabla_{E_2}E_2 = dE_1, & \nabla_{E_2}E_1 = -dE_2 + bE_3. \end{array} \tag{3.29}$$

Step 2. – Next, we investigate the Codazzi and Gauss equations, in order to obtain some relations between the functions $a, b, c, d, \alpha, \beta$ and λ and their derivatives. First, we remark that the Codazzi equation states that

$$(\nabla h)(X, Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

is symmetric in X, Y and Z. Therefore, since

$$\begin{aligned} (\nabla h)(E_1, E_3, E_3) &= -\,2h(\nabla_{E_1}E_3, E_3) = 2b\lambda JE_2, \\ (\nabla h)(E_3, E_1, E_3) &= E_3(\lambda)JE_1 + \lambda\nabla_{E_3}^{\perp}JE_1 - h(\nabla_{E_3}E_1, E_3) - h(E_1, \nabla_{E_3}E_3) \\ &= E_3(\lambda)JE_1 + \lambda JE_2 + \lambda aJE_2 + a\lambda JE_2 \\ &= E_3(\lambda)JE_1 + (2a+1)\lambda JE_2, \end{aligned}$$

and λ is a nonzero function, we deduce that :

$$E_3(\lambda) = 0, \tag{3.30}$$

$$a = b - \frac{1}{2}. \tag{3.31}$$

Similarly, we obtain from $(\nabla h)(E_2, E_1, E_3) = (\nabla h)(E_1, E_2, E_3)$ that

$$E_2(\lambda) = 2\lambda c, \tag{3.32}$$

$$E_1(\lambda) = 2\lambda d. \tag{3.33}$$

Investigating the consequences of (3.30), (3.32) and (3.33), we also find that

$$\begin{aligned} 0 &= E_1(E_3(\lambda)) - E_3(E_1(\lambda)) - (\nabla_{E_1} E_3 - \nabla_{E_3} E_1)\lambda \\ &= -2\lambda E_3(d) - 2(b-a)\lambda c. \end{aligned}$$

Hence we have

$$E_3(d) = -\frac{1}{2}c. \tag{3.34}$$

Similarly, we also obtain that

$$E_3(c) = \frac{1}{2}d. \tag{3.35}$$

We now use the Gauss equation. Since

$$\begin{aligned} R(E_1, E_3)E_3 =& E_1 - A_{h(E_1,E_3)}E_3 + A_{h(E_3,E_3)}E_1 \\ =& E_1 - \lambda A_{JE_1}E_3 \\ =& (1 - \lambda^2)E_1, \end{aligned}$$

and

$$\begin{aligned} R(E_1, E_3)E_3 =& \nabla_{E_1}\nabla_{E_3}E_3 - \nabla_{E_3}\nabla_{E_1}E_3 - \nabla_{(b-a)E_2}E_3 \\ =& - E_3(b)E_2 + baE_1 + (b-a)bE_1 \\ =& - E_3(b)E_2 + b^2E_1, \end{aligned}$$

it follows that

$$b^2 + \lambda^2 = 1. \tag{3.36}$$

From

$$\begin{aligned} R(E_3, E_1)E_2 &= -A_{h(E_2,E_3)}E_1 + A_{h(E_1,E_2)}E_3 \\ &= \lambda A_{JE_2}E_1 + A_{-\beta JE_1 - \alpha JE_2}E_3 \\ &= \lambda(-\beta E_1 - \alpha E_2) - \beta\lambda E_1 + \alpha\lambda E_2 \\ &= -2\beta\lambda E_1, \end{aligned}$$

and

$$\begin{aligned}R(E_3, E_1)E_2 &= \nabla_{E_3}\nabla_{E_1}E_2 - \nabla_{E_1}\nabla_{E_3}E_2 - \nabla_{(a-b)E_2}E_2\\ &= -E_3(c)E_1 - E_3(b)E_3 - acE_2 + E_1(a)E_1 + acE_2 - (a-b)dE_1\\ &= -\frac{1}{2}dE_1 - E_3(b)E_3 + E_1(a)E_1 + \frac{1}{2}dE_1\\ &= E_1(a)E_1 - E_3(b)E_3,\end{aligned}$$

it follows that

$$E_1(a) = -2\beta\lambda. \tag{3.37}$$

Similarly, it follows by computing $R(E_3, E_2)E_1$ in two different ways that

$$E_2(a) = -2\alpha\lambda. \tag{3.38}$$

Step 3. – An exceptional case. We interprete the above equations in the case that $\lambda = \pm 1$ on an open subset of U_1. It then follows by (3.31) and (3.36) that on this open subset, we have $b = 0$ and $a = -\frac{1}{2}$. Then (3.32), (3.33), (3.37) and (3.38), imply that $\alpha = \beta = c = d = 0$. Since now

$$(\nabla h)(E_1, E_2, E_2) = \nabla^{\perp}_{E_1}(-\lambda JE_2) = -\lambda JE_3,$$

and

$$(\nabla h)(E_2, E_1, E_2) = -h(\nabla_{E_2}E_1, E_2) - h(E_1, \nabla_{E_2}E_2) = 0,$$

a contradiction follows. Therefore, we can restrict ourselves to the open dense subset V of U_1 on which $\lambda^2 \neq 1$.

Step 4. – Since $0 \neq \lambda^2 \neq 1$, we can introduce a function s such that by (3.36) we have

$$b = \cos s, \tag{3.39}$$

$$\lambda = \sin s, \tag{3.40}$$

where $\cos s \neq 0 \neq \sin s$. From (3.31) and (3.37) (respectively (3.31) and (3.38)) it then follows that :

$$d \sin s = \beta \cos s, \tag{3.41}$$

$$c \sin s = \alpha \cos s. \tag{3.42}$$

Step 5. – The construction of the corresponding almost complex surface. We take local coordinates (t, u, v) in a neighborhood of $p \in V$ such that $\frac{\partial}{\partial t} = E_3$ and p corresponds to $(0, 0, 0)$. We identify M^3 with $I \times N^2$, where the variable t corresponds to I and we define by map $\bar{\varphi}$ from M^3 into S^6 by

$$\bar{\varphi}(t, u, v) = F \times F_*(E_3)(t, u, v).$$

Since

$$\bar{\varphi}_*\left(\frac{\partial}{\partial t}\right) = F_*(E_3) \times F_*(E_3) + F \times (D_{E_3}F_*(E_3)) = 0, \tag{3.43}$$

it follows that $\bar{\varphi}$ is independent of t. We call

$$\varphi(u,v) = \bar{\varphi}(0,u,v).$$

We find that

$$\begin{aligned}\bar{\varphi}_*(E_1) &= F_*(E_1) \times F_*(E_3) + F \times (bF_*(E_2) + \lambda F \times F_*(E_1)) \\ &= (b-1)F \times F_*(E_2) - \lambda F_*(E_1),\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\bar{\varphi}_*(E_2) &= F_*(E_2) \times F_*(E_3) + F \times (-bF_*(E_1) - \lambda F \times F_*(E_2)) \\ &= \lambda F_*(E_2) - (b-1)F \times F_*(E_1).\end{aligned} \tag{3.45}$$

From (3.43), (3.44) and (3.45) it follows that φ defines an immersion of N^2 into S^6. Defining local functions $f, g : N^2 \to \mathbb{R}$ such that $V_1 = E_1(0,u,v) + f(u,v)E_3(0,u,v)$ and $V_2 = E_2(0,u,v) + g(u,v)E_3(0,u,v)$ are tangent to N^2, we deduce from (3.44) and (3.45) that

$$\varphi_*(V_1) = (b-1)F \times F_*(E_2) - \lambda F_*(E_1)|_{t=0}, \tag{3.46}$$

$$\varphi_*(V_2) = \lambda F_*(E_2) - (b-1)F \times F_*(E_1)|_{t=0}. \tag{3.47}$$

Since

$$\begin{aligned}&(F \times F_*(E_3)) \times F_*(E_1) = F_*(E_2)\\ &(F \times F_*(E_3)) \times F_*(E_2) = -F_*(E_1)\\ &(F \times F_*(E_3)) \times (F \times F_*(E_2)) = F_*(E_3) \times F_*(E_2) = -F \times F_*(E_1)\\ &(F \times F_*(E_3)) \times (F \times F_*(E_1)) = F_*(E_3) \times F_*(E_1) = F \times F_*(E_2)\end{aligned}$$

the equations (3.46) and (3.47) imply that φ defines an almost complex surface in S^6. We now compute the first normal space. Since φ defines an almost complex surface, it suffices to compute $D_{V_1}\varphi_*(V_1)$ and $D_{V_2}\varphi_*(V_1)$. Using the equations derived in Step 3 and 4 of the proof, it follows that:

$$\begin{aligned}D_{V_1}\varphi_*(V_1) =& D_{E_1+fE_3}((b-1)F \times F_*(E_2) - \lambda F_*(E_1))\\ =& -2\frac{\sin^2 s}{\cos s} dF \times F_*(E_2) + (b-1)F_*(E_1) \times F_*(E_2) + (b-1)F \times (-cF_*(E_1)\\ & - bF_*(E_3) - \beta F \times F_*(E_1) - \alpha F \times F_*(E_2))\\ & - 2\sin s dF_*(E_1) - \sin s(cF_*(E_2) - F + \alpha F \times F_*(E_1) - \beta F \times F_*(E_2)\\ & + \lambda F \times F_*(E_3))\\ & + f(b-1)(-F \times F_*(E_1)) + (b-1)fF \times (-aF_*(E_1) - \lambda F \times F_*(E_2))\\ & - \lambda f(aF_*(E_2) + \lambda F \times F_*(E_1))\\ =& \left(-\frac{1}{2}f - \frac{c}{b}\right)\varphi_*(V_2) + \frac{(1+b)}{b} d\varphi_*(V_1)\\ & + \sin sF|_{t=0} + 2(\cos s - 1)F \times F_*(E_3)|_{t=0},\end{aligned}$$

and

$$
\begin{aligned}
D_{V_2}\varphi_*(V_1) =& D_{E_2+gE_3}((b-1)F\times F_*(E_2)-\lambda F_*(E_1))\\
=& -2\frac{\sin^2 s}{\cos s}cF\times F_*(E_2)+(b-1)F\times(dF_*(E_1)-F-\alpha F\times F_*(E_1)\\
&+\beta F\times F_*(E_2)-\lambda F\times F_*(E_3))\\
&-2\sin s cF_*(E_1)-\sin s(-dF_*(E_2)+bF\times F_*(E_3)-\beta F\times F_*(E_1)\\
&-\alpha F\times F_*(E_2))\\
&+g(b-1)F_*(E_3)\times F_*(E_2)+g(b-1)F\times(-aF_*(E_1)-\lambda F\times F_*(E_2))\\
&-\lambda g(aF_*(E_2)+\lambda F\times F_*(E_1))\\
=&\left(-\frac{1}{2}g+\frac{d}{b}\right)\varphi_*(V_2)+c\frac{(1+b)}{b}\varphi_*(V_1)\\
&+\sin sF_*(E_3)|_{t=0}.
\end{aligned}
$$

Since $F\times F_*(E_3)|_{t=0}=\varphi$, the above formulas imply that the first normal to the almost complex immersion φ is spanned by $F|_{t=0}$ and $F_*E_3|_{t=0}$. Since the integral curves of E_3 are great circles, we see that

$$F(t,u,v)=\cos tF|_{t=0}+\sin tF_*E_3|_{t=0}.$$

Hence F is obtained as a tube with radius $\frac{\pi}{2}$ in the direction of the first normal bundle on an almost complex surface. Applying Theorem 1 shows that φ is necessarily superminimal. This completes the proof. ∎

Remark. – The totally real submanifolds of $S^6(1)$ which satisfy Chen's equality have been classified in [DV2]. It is shown that they correspond to the Hopf lift of a holomorphic curve in $\mathbb{C}P^2$ or to a tube with radius $\frac{\pi}{2}$ in the direction of the second normal bundle on an almost complex surface in S^6. It is straightforward to compute that all examples of the first type admit a Killing vector field, whereas the examples of the second type admit a Killing vector field if and only if the almost complex surface is superminimal.

4. Further properties

In this section, we investigate some further properties of the class of Lagrangian immersions discovered in the previous section. We continue with the computations started in Lemma 3.1. Putting

$$
\begin{aligned}
E_1 &= X/\sqrt{\mu^2+\frac{1}{4}},\\
E_2 &= Y/\sqrt{\mu^2+\frac{1}{4}},\\
E_3 &= \partial/\partial t,
\end{aligned}
$$

a lengthy but straightforward computation shows that:

$$\begin{aligned}
h(E_1,E_1) =& \frac{1}{2}\mu\frac{(a_1\sin t - a_2\cos t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_1 - \frac{1}{2}\mu\frac{(a_1\cos t + a_2\sin t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_2 + \frac{\mu\cos t}{(\mu^2+\frac{1}{4})}JE_3\\
h(E_1,E_2) =& -\frac{1}{2}\mu\frac{(a_1\cos t + a_2\sin t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_1 - \frac{1}{2}\mu\frac{(a_1\sin t - a_2\cos t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_2 + \frac{\mu\sin t}{(\mu^2+\frac{1}{4})}JE_3\\
h(E_2,E_2) =& -\frac{1}{2}\mu\frac{(a_1\sin t - a_2\cos t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_1 + \frac{1}{2}\mu\frac{(a_1\cos t + a_2\sin t)}{(\mu^2+\frac{1}{4})^{\frac{3}{2}}}JE_2 - \frac{\mu\cos t}{(\mu^2+\frac{1}{4})}JE_3\\
h(E_1,E_3) =& \frac{\mu\cos t}{\mu^2+\frac{1}{4}}JE_1 + \frac{\mu\sin t}{\mu^2+\frac{1}{4}}JE_2\\
h(E_2,E_3) =& \frac{\mu\sin t}{\mu^2+\frac{1}{4}}JE_1 - \frac{\mu\cos t}{\mu^2+\frac{1}{4}}JE_2\\
h(E_3,E_3) =& 0.
\end{aligned}$$

Using the Gauss equation, it now follows that the normalized scalar curvature ρ of the immersion ψ is given by

$$\rho = 1 - \frac{\mu^2}{(\mu^2+\frac{1}{4})^2} - \frac{1}{6}\frac{\mu^2(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^3}, \tag{4.1}$$

and that the eigenvalues of the Ricci tensor are given by

$$\begin{aligned}
\lambda_1 &= 2 - \frac{2\mu^2}{(\mu^2+\frac{1}{4})^2} - \frac{1}{2}\mu^2\frac{(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^3},\\
\lambda_2 &= 2 - \frac{2\mu^2}{(\mu^2+\frac{1}{4})^2} + \sqrt{\frac{1}{16}\frac{\mu^4(a_1^2+a_2^2)^2}{(\mu^2+\frac{1}{4})^6} + \frac{\mu^2(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^5}} - \frac{1}{4}\frac{\mu^2(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^3},\\
\lambda_3 &= 2 - \frac{2\mu^2}{(\mu^2+\frac{1}{4})^2} - \sqrt{\frac{1}{16}\frac{\mu^4(a_1^2+a_2^2)^2}{(\mu^2+\frac{1}{4})^6} + \frac{\mu^2(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^5}} - \frac{1}{4}\frac{\mu^2(a_1^2+a_2^2)}{(\mu^2+\frac{1}{4})^3}.
\end{aligned}$$

Therefore, the Ricci tensor has a double eigenvalue, in which case the example can also be obtained as one of the examples in [DDVV] if and only if $a_1 = a_2 = 0$, and hence K is constant. It is well known, *see* a.o. [E2] that a superminimal almost complex surface with constant curvature, which is not totally geodesic, is G_2-congruent to the Veronese surface with constant curvature $\frac{1}{6}$ in S^6. It then follows that the corresponding Lagrangian submanifold M is an Einstein space. Since it is 3-dimensional, it has constant sectional curvature.

From Corollary 5.1, we notice that M has constant scalar curvature if and only if there exists a constant k such that

$$k\left(\mu^2+\frac{1}{4}\right)^3 = 6\mu^2\left(\mu^2+\frac{1}{4}\right) + (V(\mu))^2 + (U(\mu))^2. \tag{4.2}$$

Clearly the superminimal almost complex surface with constant Gaussian curvature $\frac{1}{6}$ satisfies (4.2). In the remainder of this section, we will show that this is actually the only almost complex superminimal surface in $S^6(1)$ which satisfies (4.2). We compute that the Gauss and Ricci equation for a superminimal almost complex surface is S^6 reduce to

$$V(\mu_2) + U(\mu_1) - \mu_1^2 - \mu_2^2 = 1 - 2\mu^2, \tag{4.3}$$

$$V(a_1) - U(a_2) + \left(6\mu^2 + \frac{5}{2}\right) + \mu_1 a_2 - \mu_2 a_1 = 0. \tag{4.4}$$

THEOREM 3. – *The Lagrangian immersion ψ, corresponding to an almost complex superminimal surface $\varphi : N^2 \to S^6$, as obtained in Theorem 1 has constant scalar curvature if and only if φ is the Veronese immersion in S^6 with constant curvature $\frac{1}{6}$.*

Proof. – Using the classification of superminimal almost complex surfaces with constant curvature, it is sufficient to show that μ is constant. We assume that this is not the case and will derive a contradiction.

We choose a local orthonormal basis such that $U(\mu) = 0$. We restrict to the open set on which $\mu \neq 0 \neq a_1$. Since μ is non-constant this is an open dense subset of N^2 and it follows that $a_2 = 0$. Recall that $V(\mu) = a_1\mu$. Since $V(U(\mu)) - U(V(\mu)) = (\nabla_V U - \nabla_U V)\mu$, we deduce that $U(a_1) = a_1\mu_1$. Combining this with (4.2) we deduce that $\mu_1 = 0$. Hence (4.4) reduces to

$$V(a_1) = \mu_2 a_1 - \left(6\mu^2 - \frac{5}{2}\right). \tag{4.5}$$

Deriving (4.2) in the direction of V, using (4.5) and (4.2) we deduce after a long but straightforward computation that

$$16a_1\mu^2\mu_2 - 32k\mu^6 - 12k\mu^4 + 40\mu^2 + \frac{k}{4} = 0. \tag{4.6}$$

We now put $\mu_2 = \frac{\rho}{\mu^2 a_1}$. From (4.6) it then follows that

$$\rho = \frac{1}{64}(-k - 160\mu^2 + 48k\mu^4 + 128k\mu^6).$$

Substituting this into (4.3), using again (4.2) we deduce after a straightforward computation that μ is constant, which is a contradiction. ■

REFERENCES

[B] D. E. BLAIR, Contact manifolds in Riemannian Geometry, Lecture Notes in Mathematics, vol. 509, Springer, Berlin, 1976.

[BPW] J. BOLTON, F. PEDIT and L. M. WOODWARD, Minimal surfaces and the affine Toda field model, *J. Reine Angew. Math.*, *459*, 1995, pp. 119-150.

[BVW1] J. BOLTON, L. VRANCKEN and L. M. WOODWARD, On almost complex curves in the nearly Kähler 6-sphere, *Quart. J. Math. Oxford Ser.*, (2) *45*, 1994, pp. 407-427.

[BVW2] J. BOLTON, L. VRANCKEN and L. M. WOODWARD, Totally real minimal surfaces with non-circular ellipse of curvature in the nearly Kähler 6-sphere, *J. London Math. Soc.* (to appear).

[BW] J. BOLTON and L. M. WOODWARD, Congruence theorems for harmonic maps from a Riemannian surface into $\mathbb{C}P^n$ and S^n, *J. London Math. Soc.*, *45*, 1992, pp. 363-376.

[Br] R. L. BRYANT, Submanifolds and special structures on the octonians, *J. Differential Geom.*, *17*, 1982, pp. 185-232.

[C] B. Y. CHEN, Some pinching and classification theorems for minimal submanifolds, *Arch. Math. (Basel) 60*, 1993, pp. 568-578.

[CDVV1] B. Y. CHEN, F. DILLEN, L. VERSTRAELEN and L. VRANCKEN, Two equivariant totally real immersions into the nearly Kähler 6-sphere and their characterization, Japanese, *J. Math. (N.S.)*, *21*, 1995, pp. 207-222.

[CDVV2] B. Y. CHEN, F. DILLEN, L. VERSTRAELEN and L. VRANCKEN, Characterizing a class of totally real submanifolds of $S^6(1)$ by their sectional curvatures, *Tôhoku Math. J.*, *47*, 1995, pp. 185-198.

[DDVV] R. DESZCZ, F. DILLEN, L. VERSTRAELEN and L. VRANCKEN, Quasi Einstein totally real submanifolds of $S^6(1)$, preprint.

[DVV] F. DILLEN, L. VERSTRAELEN and L. VRANCKEN, Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq \frac{1}{16}$, *J. Math. Soc. Japan*, *42*, 1990, pp. 565-584.

[DV1] F. DILLEN and L. VRANCKEN, C-totally real submanifolds of Sasakian space forms, *J. Math. Pures Appl.*, *69*, 1990, pp. 85-93.

[DV2] F. DILLEN and L. VRANCKEN, Totally real submanifolds in $S^6(1)$ satisfying Chen's equality, *Trans. Amer. Math. Soc.*, *348*, 1996, pp. 1633-1646.

[E1] N. EJIRI, Totally real submanifolds in a 6-sphere, *Proc. Amer. Math. Soc.*, *83*, 1981, pp. 759-763.

[E2] N. EJIRI, Equivariant minimal immersions of S^2 into S^{2m}, *Trans. Amer. Math. Soc.*, *297*, 1986, pp. 105-124.

[G] A. GRAY, Almost complex submanifolds of the six-sphere, *Proc. Amer. Math. Soc.*, 1969, pp. 277-279.

[HL] R. HARVEY and H.B. LAWSON, Calibrated geometries, *Acta Math.*, *148*, 1982, pp. 47-157.

[W] R. M. W. WOOD, Framing the exceptional Lie group G_2, *Topology*, *15*, 1976, pp. 303-320.

[YI] K. YANO and S. ISHIHARA, Invariant submanifolds of an almost contact manifold, *Kodai Math. Sem. Rep.*, *21*, 1969, pp. 350-364.

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