ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT

J. Math. Anal. Appl. 281 (2003) 657–662

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Weak shadowing for discrete dynamical systems on nonsmooth manifolds ☆

Marcin Mazur

Uniwersytet Jagielloński, Instytut Matematyki, Reymonta 4, 30-059 Kraków, Poland

Received 15 March 2002

Submitted by U. Kirchgraber

Abstract

In J. Math. Anal. Appl. 189 (1995) 409–423, Corless and Pilyugin proved that weak shadowing is a C^0 generic property in the space of discrete dynamical systems on a compact smooth manifold M. In our paper we give another proof of this theorem which does not assume that M has a differential structure. Moreover, our method also works for systems on some compact metric spaces that are not manifolds, such as a Hilbert cube (or generally, a countably infinite Cartesian product of manifolds with boundary) and a Cantor set.

Keywords: (Weak) shadowing; (Discrete) dynamical system; Pseudo-trajectory; Homogeneity; Generic property

1. Introduction

Let (M,d) denote a compact metric space and let $f:M\to M$ be a homeomorphism (a discrete dynamical system on M).

A sequence $\{y_n\}_{n\in\mathbb{Z}}\subset M$ is called a δ-pseudo-trajectory ($\delta>0$) of f if

$$d\big(f(x_n),x_{n+1}\big)\leqslant\delta \quad \text{for every } n\in\mathbb{Z}.$$

Note that 0-pseudo-trajectory of f is simply its "real" trajectory.

We say that f has the (weak) shadowing property if for every $\varepsilon>0$ there exists $\delta>0$ satisfying the following condition: given a δ-pseudo-trajectory $\mathrm{y}=\{y_n\}_{n\in\mathbb{Z}}$ we can find a corresponding trajectory $\mathrm{x}=\{x_n\}_{n\in\mathbb{Z}}$ which (weakly) ε-traces y, i.e.,

☆ Supported by KBN Grant 5P03A01620.

E-mail address: mazur@im.uj.edu.pl.

doi:10.1016/S0022-247X(03)00186-0

$d(x_n, y_n) \leqslant \varepsilon$ for every $n \in \mathbb{Z}$ (shadowing),

$\mathrm{y} \subset U_\varepsilon(\mathrm{x})$ (weak shadowing).

Here and subsequently $U_\varepsilon(S)$ denote the ε-neighborhood of the set $S \subset M$, i.e., the set of all $x \in M$ such that $\mathrm{dist}(x, S) \leqslant \varepsilon$.

The concept of shadowing was investigated by many authors (see, e.g., [5,10,18,20, 21]). In [6] Corless and Pilyugin proved that weak shadowing is a C^0 generic property for discrete dynamical systems of a compact smooth manifold M. Subsequently, Pilyugin and Plamenevskaya [22] improved this theorem showing C^0 genericity of the shadowing property. The other related results were obtained in [14,17,25,31].

Both of the proofs given in [6] and [22] required that M was a C^∞ smooth manifold (see Remarks 4 and 5). The aim of this paper is to show that for C^0 genericity of weak shadowing neither the differential structure nor even being a manifold is a crucial assumption on the space M, but a generalized version of a topological property called homogeneity (see [1,3,8,9,27] and references therein).

2. Results

At first we complete notation and, for the convenience of the reader, recall some known definitions.

We denote the set of all homeomorphisms of M by $\mathcal{H}(M)$. Introduce in $\mathcal{H}(M)$ the complete metric

$$\rho_0(f, g) := \max\Big\{\max_{x\in M} d\big(f(x), g(x)\big), \max_{x\in M} d\big(f^{-1}(x), g^{-1}(x)\big)\Big\},$$

which generates the C^0 topology.

A property $\mathcal{P}$ of elements of a topological space X is said to be generic if the set of all $x \in X$ satisfying $\mathcal{P}$ is residual, i.e., it includes a countable intersection of open and dense subsets of X.

We say that the space M is generalized homogeneous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\} \subset M$ is a pair of sets of mutually disjoint points satisfying $d(x_i, y_i) \leqslant \delta$, $i \in \{1, \dots, n\}$, then there exists $h \in \mathcal{H}(M)$ with $\rho_0(h, \mathrm{id}_M) \leqslant \varepsilon$ and $h(x_i) = y_i$, $i \in \{1, \dots, n\}$. We will call such δ an ε-modulus of homogeneity of M.

The theorem stated below is the main result of this paper.

Theorem 1. *If the space M is a generalized homogeneous and has no isolated points then the weak shadowing property is generic in $\mathcal{H}(M)$.*

As a corollary we also prove the following

Theorem 2. *If the space M is one of the following*:

(i) *a topological manifold with boundary* ($\dim(M) \geqslant 2$ *if* $\partial M \neq \emptyset$),

(ii) *a Cartesian product of a countably infinite number of manifolds with nonempty boundary,*
(iii) *a Cantor set,*

then weak shadowing is a generic property in $\mathcal{H}(M)$.

Statement (i) of Theorem 2 was announced in [19] and the proof appeared in the author's Ph.D. thesis [13]. However, the argumentation presented there based on Kuratowski's theorem [12] providing genericity of continuity points of a semi-continuous multivalued map (see also [23,25]). The author is indebted to the anonymous referee for indicating a possibility of applying another, simpler method. Actually, the proof presented here is based on the referee's suggestions.

3. Proofs

Proof of Theorem 1. Fix any $\varepsilon > 0$. Let $\mathrm{U} = \{U_i\}_{i=1}^k$ be a finite covering of M by open sets with diameters not greater than ε and let $K := \{1, 2, \ldots, k\}$. For $f \in \mathcal{H}(M)$ consider the family $\mathrm{I}_f \subset \mathcal{S}(K)$ consisting of these sets $L \subset K$ for which we can find a trajectory having a nonempty intersection with each of the sets U_i for $i \in L$ (here and subsequently $\mathcal{S}(T)$ denotes the family of all subsets of the set T). It is easily seen that for any $f \in \mathcal{H}(M)$ the following holds:

$$\text{There exists a neighborhood } \mathcal{W} \text{ of } f \text{ such that } \mathrm{I}_f \subset \mathrm{I}_g \text{ for } g \in \mathcal{W}. \tag{1}$$

Define the set $\mathcal{R}_{\mathrm{U}}$ as the collection of such $f \in \mathcal{H}(M)$ that $\mathrm{I}_f = \mathrm{I}_g$ for g sufficiently close to f. Obviously, it is an open subset of $\mathcal{H}(M)$. To prove that $\mathcal{R}_{\mathrm{U}}$ is dense in $\mathcal{H}(M)$, fix any open set $\mathcal{V} \subset \mathcal{H}(M)$. Observe that the set $\mathrm{I}_{\mathcal{V}} := \{\mathrm{I}_g \mid g \in \mathcal{V}\} \subset \mathcal{S}(\mathcal{S}(K))$ is a finite set, partially ordered by the relation of inclusion. Let I_f, corresponding to some $f \in \mathcal{V}$, be one of its maximal elements. Then, applying condition (1), we obtain a neighborhood $\mathcal{W} \subset \mathcal{V}$ of f such that $\mathrm{I}_f = \mathrm{I}_g$ for $g \in \mathcal{W}$. Thus, $f \in \mathcal{R}_{\mathrm{U}} \cap \mathcal{V}$, which completes the proof of density of the set $\mathcal{R}_{\mathrm{U}}$.

Take $f \in \mathcal{R}_{\mathrm{U}}$ and choose $\beta > 0$ such that $\mathrm{I}_f = \mathrm{I}_g$ for $g \in \mathcal{H}(M)$ with $\rho_0(f, g) \leqslant \beta$. Let $\gamma > 0$ be a β-modulus of homogeneity of M. Set $\delta := \gamma/2$. To make the proof complete it is sufficient to show that each δ-pseudo-trajectory of f has some weakly 3ε-tracing trajectory. Fix any δ-pseudo-trajectory $\mathrm{y} = \{y_n\}_{n\in\mathbb{Z}}$ and notice that it is contained in an ε-neighborhood of its "finite part," i.e., there exist $l, r \in \mathbb{Z}$ such that $\mathrm{y} \subset U_\varepsilon(\mathrm{y}_l^r)$, where $\mathrm{y}_l^r = \{y_n\}_{n=l}^r$. Since M has no isolated points we can easily find (see, for instance, the proof of Lemma 9 in [28][1]) a finite 2δ-pseudo-trajectory $\bar{\mathrm{y}}_l^r = \{\bar{y}_n\}_{n=l}^r$ such that $\mathrm{y}_l^r \subset U_\varepsilon(\bar{\mathrm{y}}_l^r)$ and $\bar{y}_i \neq \bar{y}_j$ for $i, j \in \{l, \ldots, r\}$, $i \neq j$. Let $h \in \mathcal{H}(M)$, $\rho_0(h, \mathrm{id}_M) \leqslant \beta$, be a homeomorphism connecting $f(\bar{y}_i)$ with $\bar{y}_{i+1}$ for $i \in \{1, \ldots, l-1\}$. Set $g := h \circ f$. The sequence

$$\bar{\mathrm{y}} = \left\{\ldots, g^{-2}(\bar{y}_l), g^{-1}(\bar{y}_l), \bar{y}_l, \bar{y}_{l+1}, \ldots, \bar{y}_r, g(\bar{y}_r), g^2(\bar{y}_r), \ldots\right\}$$

[1] In fact, in the cited lemma the space was assumed to be a manifold, but for the proof it was essential that it had no isolated points.

is a trajectory of g, so the set $L := \{i \in K \mid \bar{\mathrm{y}} \cap U_i \neq \emptyset\}$ belongs to I_g. As $\rho_0(f,g) \leqslant \beta$ we have $\mathrm{I}_f = \mathrm{I}_g$ and hence there exists a trajectory x of f having a nonempty intersection with each of the sets U_i, $i \in L$. From this we obtain $\bar{\mathrm{y}} \subset U_\varepsilon(\mathrm{x})$ and, in consequence, we conclude that the trajectory x weakly 3ε-traces the δ-pseudo-trajectory y, which completes the proof. □

Proof of Theorem 2. (i) Any manifold of the dimension at most 3 admits a C^∞ differential structure, which is compatible with a given topology (see [15,26,29]). So, in this case the conclusion is an immediate consequence of the mentioned results of [6,22]. If $\dim(M) \geqslant 2$ then $M \setminus \partial M$ is generalized homogeneous (see [1–4]). It is easily seen that then the conclusion can be obtained by a slight modification of the proof of Theorem 1.

(ii) In this case M is strongly homogeneous (see [8,30]), i.e., any bijective map between finite sets can be extended to a homeomorphism of M. So, Ungar's version [27] of the well-known theorem due to Effros [7] can be applied for the transformation group $(\mathcal{H}(M), F^n(M))$, where $F^n(M) := \{(x_1,\ldots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$, to show that M also satisfies a generalized homogeneity property. Theorem 1 completes the proof.

(iii) A Cantor set is generalized homogeneous (see [1,3]) and does not contain any isolated point. The conclusion follows immediately from Theorem 1. □

4. Remarks

Remark 3. Let us note that the absence of isolated points is not an essential assumption for Theorem 1 to hold. Indeed, since M satisfies the generalized homogeneity property the set $\mathrm{IP}(M)$ of points isolated in M, which is invariant for every homeomorphism of M, cannot cumulate in any point of M and, therefore, it may contain only a finite number of elements. So, in the proof of the theorem we can ignore δ-pseudo-trajectories that meet this set, taking notice of the fact that, when δ is sufficiently small, they are "real" trajectories contained in the set $\mathrm{IP}(M)$.

Remark 4. The argumentation presented in [6] employs a differential structure of a manifold M. For example, the proof of Lemma 2.1 stated there is based on the solution of a system of differential equations. On the other hand, the authors of [6] outline another possible argumentation making use of the theorem due to Shub and Smale [24] and Nitecki and Shub [16], which, in particular, says that any compact (boundaryless) C^∞ smooth manifold of the dimension at least 2 is a generalized homogeneous. So, since the latter also holds for a topological manifold, it is, in fact, a way to obtain the conclusion without a smoothness assumption. However, both of the mentioned methods depend, via Takens' result [25], on Kuratowski's theorem [12], and ours does not.

Remark 5. As we have already noted, in [22] C^0 genericity of the shadowing property was proved for homeomorphisms of a C^∞ smooth manifold M. The only place where the proof uses the smoothness assumption is the construction of a handle decomposition of M, based on a smooth triangulation which in this case always exists. However, although the existence of a handle decomposition does not require a differential structure if $\dim(M) \geqslant 6$

(see [11,22]), for the other dimensions this assumption cannot be omitted (note that it is essential if $\dim(M) = 4$ or 5).

Remark 6. The following problems seem to remain unsolved:

(1) Is the homogeneity of the compact metric space M a sufficient assumption to obtain genericity of the (nonweak) shadowing property in $\mathcal{H}(M)$?
(2) Is the homogeneity of the compact metric space M a necessary assumption to obtain genericity of the (weak) shadowing property in $\mathcal{H}(M)$?

In the proof of Theorem 1 we look at a trajectory of a dynamical system as at a finite collection of sets (from a given covering of the space M) that "it meets on its way." On the one hand, such a representation enable us to find easily the generic set of weakly shadowing systems, but on the other, results in loss of information about location of particular points of the trajectory that is crucial for the nonweak case. So, let us notice that applying this argument to solve the first of the above problems may have little or no effect.

References

[1] E. Akin, The General Topology of Dynamical Systems, in: Grad. Stud. Math., Vol. 1, American Mathematical Society, Providence, 1993.
[2] E. Akin, Stretching the Oxtoby–Ulam theorem, Colloq. Math. 84/85 (2000) 83–94.
[3] E. Akin, M. Hurley, J.A. Kennedy, Dynamics of topologically generic homeomorphisms, Mem. Amer. Math. Soc., to appear.
[4] N. Aoki, K. Hiraide, Topological Theory of Dynamical Systems. Recent Advances, in: North-Holland Math. Library, Vol. 52, North-Holland, Amsterdam, 1994.
[5] R. Bowen, ω-limit sets for axiom A diffeomorphisms, J. Differential Equations 18 (1975) 333–339.
[6] R.M. Corless, S.Yu Pilyugin, Approximate and real trajectories for generic dynamical systems, J. Math. Anal. Appl. 189 (1995) 409–423.
[7] E.G. Effros, Transformation groups and C^*-algebras, Ann. of Math. (2) 81 (1965) 38–55.
[8] M.K. Fort Jr., Homogeneity of infinite products of manifolds with boundary, Pacific J. Math. 12 (1962) 879–884.
[9] J. Kennedy, Some facts about homogeneity properties, Colloq. Math. 59 (1990) 103–116.
[10] P.E. Kloeden, J. Ombach, Hyperbolic homeomorphisms are bishadowing, Ann. Polon. Math. 65 (1997) 171–177.
[11] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings and Triangulations, in: Ann. of Math. Stud., Vol. 88, Princeton Univ. Press, Princeton, NJ, 1977.
[12] C. Kuratowski, Topologie, Vol. II (in French), in: Monogr. Mat., Vol. 21, PWN, Warszawa, 1961.
[13] M. Mazur, C^0 generic properties and asymptotic behavior of discrete dynamical systems, Ph.D. thesis, Uniwersytet Jagielloński, Kraków, 2001 (in Polish).
[14] M. Mazur, Tolerance stability conjecture revisited, Topology Appl., to appear.
[15] E.E. Moise, Affine structures in 3-manifolds, V. The triangulation theorem and Hauptvermutung, Ann. of Math. (2) 56 (1952) 96–114.
[16] Z. Nitecki, M. Shub, Filtrations, decompositions, and explosions, Amer. J. Math. 97 (1975) 1029–1047.
[17] K. Odani, Generic homeomorphisms have the pseudo-orbit tracing property, Proc. Amer. Math. Soc. 110 (1990) 281–284.
[18] J. Ombach, Shadowing, expansiveness and hyperbolic homeomorphisms, J. Austral. Math. Soc. Ser. A 61 (1996) 57–72.

[19] J. Ombach, M. Mazur, Shadowing and likes as C^0 generic properties, in: Proc. of the 3rd Polish Symposium on Nonlinear Analysis, in: Lecture Notes in Nonlinear Anal., Vol. 3, 2002, pp. 159–168.

[20] K. Palmer, Shadowing in Dynamical Systems, Kluwer Academic, Dordrecht, 2000.

[21] S.Yu. Pilyugin, Shadowing in Dynamical Systems, in: Lecture Notes in Math., Vol. 1706, Springer, Berlin, 1999.

[22] S.Yu. Pilyugin, O.B. Plamenevskaya, Shadowing is generic, Topology Appl. 97 (1999) 253–266.

[23] C. Pugh, An improved closing lemma and a general density theorem, Amer. J. Math. 89 (1967) 1010–1021.

[24] M. Shub, S. Smale, Beyond hyperbolicity, Ann. of Math. (2) 96 (1972) 587–591.

[25] F. Takens, On Zeeman's tolerance stability conjecture, in: Proc. Nuffic Summer School on Manifolds, in: Lecture Notes in Math., Vol. 197, Springer, Berlin, 1971, pp. 209–219.

[26] W.P. Thurston, Three-Dimensional Geometry and Topology, Vol. I, in: Princeton Math. Ser., Vol. 35, Princeton Univ. Press, Princeton, NJ, 1997.

[27] G.S. Ungar, On all kinds of homogeneity, Trans. Amer. Math. Soc. 264 (1981) 205–215.

[28] P. Walters, On the pseudo orbit tracing property and its relationship to stability, in: The Structure of Attractors in Dynamical Systems, in: Lecture Notes in Math., Vol. 668, Springer, Berlin, 1978, pp. 231–244.

[29] J.H.C. Whitehead, Manifolds with transverse fields in Euclidean space, Ann. of Math. (2) 73 (1961) 154–212.

[30] Z. Yang, Homogeneity of infinite products of manifolds with boundary, in: Papers on General Topology and Applications (Brookville, NY, 1990), in: Ann. New York Acad. Sci., Vol. 659, New York Acad. Sciences, New York, 1992, pp. 194–208.

[31] K. Yano, Generic homeomorphisms of S^1 have the pseudo-orbit tracing property, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987) 51–55.