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Oscillation of Solutions for Second-Order Nonlinear Difference Equations with Nonlinear Neutral Term

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Abstract—Some Riccati type difference inequalities are given for the second-order nonlinear difference equations with nonlinear neutral term

$$\Delta(a_n \Delta(x_n + \varphi(n, x_{\tau_n}))) + q_n f(x_{g_n}) = 0$$

and using these inequalities, we obtain some oscillation criteria for the above equation.

Keywords—Nonlinear difference equations, Nonlinear neutral term, Oscillation.

1. INTRODUCTION

In this paper, we consider the second-order nonlinear difference equations with nonlinear neutral term

$$\Delta(a_n \Delta(x_n + \varphi(n, x_{\tau_n}))) + q_n f(x_{g_n}) = 0, \tag{1.1}$$

where $n \geq n_0$, n_0 is a positive integer, $\{\tau_n\}$ and $\{g_n\}$ are sequences of nondecreasing nonnegative integers with $\tau_n \leq n$, $g_n \leq n$, and $\lim_{n\to\infty} \tau_n = \infty$, $\lim_{n\to\infty} g_n = \infty$, $\{a_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers and $a_n > 0$, $\sum_{n=1}^{\infty} (1/a_n) = \infty$, $q_n \geq 0$, $q_n \not\equiv 0$, $0 \leq \varphi(n,u)/u \leq p_n < 1$ for $u \neq 0$, p_n is a sequence of positive real numbers, $f(u)/u \geq \varepsilon_0$, for $u \neq 0$, ε_0 is a positive real number. Δ is forward difference operator: $\Delta x_n = x_{n+1} - x_n$.

A solution of (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, otherwise, it is nonoscillatory. Equation (1.1) is called oscillatory if and only if all its solutions are oscillatory, otherwise, is nonoscillatory.

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Recently, there has been an increasing interest in the study of oscillation for solutions of second-order difference equations. The papers [1–6] discuss second-order self-conjugate difference equations, the papers [7,8] discuss second-order neutral difference equations, the paper [9] discusses second-order self-conjugate neutral difference equation. In this paper, we are concerned with the second-order nonlinear delay difference equations with nonlinear neutral term. Some Riccati type difference inequalities are established for these equations, and using these inequalities, we obtain some oscillation criteria. The results obtained here imply and extend those in [5,9].

2. RELATED LEMMAS

To obtain our main results, we need the following lemmas.

LEMMA 1. *(See [5].) Assume x_n is an eventually positive solution of (1.1), let $z_n = x_n + \varphi(n, x_{\tau_n})$. Then $\Delta(a_n \Delta z_n) \le 0$, $\Delta z_n > 0$, $z_n > 0$ eventually.*

LEMMA 2. *If equation (1.1) is nonoscillatory, for an arbitrary positive sequence $\{A_n\}$, then there exists $n_1 \ge n_0$, such that the Riccati difference inequalities*

$$\Delta u_n + Q_n + \frac{A_n u_{n+1}^2}{a_{g_n} A_{n+1}^2} \le 0, \qquad n \ge n_1, \tag{2.1}$$

have solution $\{u_n\}$, where

$$Q_n = A_n \left\{ \varepsilon_0 q_n (1 - p_{g_n}) + a_{g_n} \frac{(\Delta A_n)^2}{4A_n} + \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \right\}. \tag{2.2}$$

PROOF. Suppose $\{x_n\}$ is an eventually positive solution of (1.1), let $z_n = x_n + \varphi(n, x_{\tau_n})$, by Lemma 1, there exist $n_1 \ge n_0$, such that $\Delta(a_n \Delta z_n) \le 0$, $\Delta z_n > 0$, $z_n > 0$ for $n \ge n_1$. From (1.1), we have

$$\begin{aligned} \Delta(a_n \Delta z_n) + q_n f(x_{g_n}) &= 0, \\ \Delta(a_n \Delta z_n) + \varepsilon_0 q_n x_{g_n} &\le 0, \\ \Delta(a_n \Delta z_n) + \varepsilon_0 q_n (z_{g_n} - \varphi(n, x_{\tau_{g_n}})) &\le 0, \\ \Delta(a_n \Delta z_n) + \varepsilon_0 q_n (z_{g_n} - p_{g_n} z_{\tau_{g_n}}) &\le 0, \\ \Delta(a_n \Delta z_n) + \varepsilon_0 q_n (1 - p_{g_n}) z_{g_n}) &\le 0, \end{aligned} \tag{2.3}$$

and

$$\Delta \left(\frac{a_n \Delta z_n}{z_{g_n}} \right) = -\varepsilon_0 q_n (1 - p_{g_n}) - \frac{a_n \Delta z_n \Delta z_{g_n}}{z_{g_n} z_{g_{n+1}}} < 0, \tag{2.4}$$

hence, $\{a_n \Delta z_n\}, \{a_n \Delta z_n / z_{g_n}\}$ are all decreasing.

Set

$$u_n = A_n \left\{ \frac{a_n \Delta z_n}{z_{g_n}} - a_{g_{n-1}} \frac{\Delta A_{n-1}}{2An - 1} \right\}.$$

Then

$$\begin{aligned} \Delta u_n &= \frac{\Delta A_n}{A_{n+1}} u_{n+1} + A_n \frac{z_{g_{n+1}} \Delta(a_n \Delta z_n) - \Delta z_{g_n} a_{n+1} \Delta z_{n+1}}{z_{g_{n+1}} z_{g_n}} - A_n \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \\ &\le \frac{\Delta A_n}{A_{n+1}} u_{n+1} - A_n \varepsilon_0 q_n (1 - p_{g_n}) - \frac{A_n a_{n+1} \Delta z_{n+1} a_{g_n} \Delta z_{g_n}}{a_{g_n} z_{g_n} z_{g_{n+1}}} - A_n \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \\ &\le \frac{\Delta A_n}{A_{n+1}} u_{n+1} - A_n \varepsilon_0 q_n (1 - p_{g_n}) - \frac{A_n a_n \Delta z_n a_{n+1} \Delta z_{n+1}}{a_{g_n} z_{g_n} z_{g_{n+1}}} - A_n \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \\ &\le \frac{\Delta A_n}{A_{n+1}} u_{n+1} - A_n \varepsilon_0 q_n (1 - p_{g_n}) - \frac{A_n}{a_{g_n}} \left(\frac{u_{n+1}}{A_{n+1}} + a_{g_n} \frac{\Delta A_n}{2A_n} \right)^2 - A_n \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \\ &\le -A_n \left\{ \varepsilon_0 q_n (1 - p_{g_n}) + a_{g_n} \frac{(\Delta A_n)^2}{4A_n} + \Delta \left(a_{g_{n-1}} \frac{\Delta A_{n-1}}{2A_{n-1}} \right) \right\} - \frac{A_n u_{n+1}^2}{a_{g_n} A_{n+1}^2}. \end{aligned}$$

Hence,

$$\Delta u_n + Q_n + \frac{A_n u_{n+1}^2}{a_{g_n} A_{n+1}^2} \le 0, \qquad n \ge n_1.$$

The proof is complete.

LEMMA 3. *If equation (1.1) is nonoscillatory, and for any positive sequence $\{A_n\}$ with*

$$\sum_{s=n}^{\infty} \frac{A_s}{a_{g_s} A_{s+1}^2} = \infty, \tag{2.5}$$

such that

$$P_n = \sum_{s=n}^{\infty} Q_s > -\infty, \tag{2.6}$$

then there exists $n_1 \ge n_0$, such that the Riccati difference inequality

$$u_n \ge P_n + \sum_{s=n}^{\infty} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2}, \qquad n \ge n_1, \tag{2.7}$$

has a solution $\{u_n\}$ and

$$P_n = \sum_{s=n}^{\infty} Q_s < \infty, \qquad \sum_{s=n}^{\infty} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} < \infty. \tag{2.8}$$

PROOF. By Lemma 2, for any positive sequence $\{A_n\}$, there exist $n_1 \ge n_0$ and $\{u_n\}$, such that $\Delta u_n + Q_n + (A_n u_{n+1}^2)/(a_{g_n} A_{n+1}^2) \le 0, n \ge n_1$. Claim $\sum_{s=n}^{\infty} (A_s u_{s+1}^2)/(a_{g_s} A_{s+1}^2)$ satisfying (2.8). If not, i.e.,

$$\sum_{s=n}^{\infty} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} = \infty. \tag{2.9}$$

In view of (2.6) and (2.9), there exits an $N_1 > n$ for fixed n, such that for $\xi \ge N_1$, we have

$$u_{\xi+1} \le u_n - \sum_{s=n}^{\xi} Q_s - \sum_{s=n}^{N_1-1} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} \le -1 - \sum_{s=N_1}^{\xi} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2},$$

which implies

$$\sum_{s=N_1}^{\xi} \frac{A_s u_{s+1}}{a_{g_s} A_{s+1}^2} \le - \sum_{s=N_1}^{\xi} \frac{A_s}{a_{g_s} A_{s+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s}{a_{g_s} A_{s+1}^2} \sum_{i=N_1}^{\xi} \frac{A_i u_{i+1}^2}{a_{g_{i+1}} A_{i+1}^2}.$$

Set

$$v_\xi = \sum_{s=N_1}^{\xi} \frac{A_s u_{s+1}}{a_{g_s} A_{s+1}^2}.$$

According to the discrete Cauchy-Schmartz inequality, we have

$$\sum_{i=N_1}^{s} \frac{A_i u_{i+1}^2}{a_{g_i} A_{i+1}^2} \ge v_s^2 \left(\sum_{i=N_1}^{s} \frac{A_i}{a_{g_i} A_{i+1}^2} \right)^{-1}$$

$$v_\xi \le - \sum_{i=N_1}^{\xi} \frac{A_i}{a_{g_i} A_{i+1}^2} - \sum_{s=N_1}^{\xi} \frac{A_s v_s^2}{a_{g_s} A_{s+1}^2} \left(\sum_{i=N_1}^{s} \frac{A_i}{a_{g_1} A_{i+1}^2} \right)^{-1} \equiv H_\xi,$$

then that $\sum_{s=N_1}^{\xi} (A_s)/(a_{g_s} A_{s+1}^2) \le |H_\xi| \le |v_\xi|, H_\xi < 0$, and $H_\xi \to -\infty$, as $\xi \to \infty$.

$$\Delta H_\xi = -\frac{A_{\xi+1}}{a_{g_{\xi+1}} A_{\xi+2}^2} - \frac{A_{\xi+1} v_{\xi+1}^2}{a_{g_{\xi+1}} A_{\xi+2}^2} \left(\sum_{s=N_1}^{\xi+1} \frac{A_s}{a_{g_s} A_{s+1}^2} \right)^{-1} < 0,$$

hence, is decreasing, which implies

$$\frac{\Delta H_\xi}{H_\xi H_{\xi+1}} \le \frac{\Delta H_\xi}{H_{\xi+1}^2} \le \frac{\Delta H_\xi}{v_{\xi+1}^2} \le -\frac{A_{\xi+1}}{a_{g_{\xi+1}} A_{\xi+2}^2} \left(\sum_{i=N_1}^{\xi+1} \frac{A_i}{a_{g_i} A_{i+1}^2} \right)^{-1}. \tag{2.10}$$

Summing of (2.10) from $N-1$ to ξ,

$$\frac{1}{H_{N-1}} - \frac{1}{H_\xi} \le -\sum_{s=N}^{\xi+1} \frac{A_s}{a_{g_s} A_{s+1}^2} \left(\sum_{i=N_1}^{s} \frac{A_i}{a_{g_i} A_{i+1}^2} \right)^{-1} \le -\sum_{s=N}^{\xi+1} \frac{A_s}{a_{g_s} A_{s+1}^2} \left(\sum_{i=N_1}^{\xi+1} \frac{A_i}{a_{g_i} A_{i+1}^2} \right)^{-1} \equiv B_\xi.$$

In view of (2.5), $B_\xi \to -1$, as $\xi \to \infty$. Hence,

$$\frac{1}{H_{N-1}} \le -\frac{1}{2},$$

which contradicts that $H_n \to -\infty$, as $n \to \infty$. Hence, (2.8) holds. Therefore,

$$u_n \ge \lim_{\xi\to\infty} \operatorname{Sup} u_\xi + P_n + \sum_{s=n}^{\infty} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2}.$$

Claim $\lim_{\xi\to\infty} \operatorname{Sup} u_\xi \ge 0$. If not, $\lim_{\xi\to\infty} \operatorname{Sup} u_\xi < 0$, then there exists an $l > 0$ and an $N_2 \ge N_1$, such that $u_\xi \le -l$, as $\xi \ge N_2$, which implies

$$\sum_{s=n}^{\infty} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} \ge l^2 \sum_{s=n}^{\infty} \frac{A_s}{a_{g_s} A_{s+1}^2} \to \infty,$$

which contradicts (2.8), hence, $\lim_{\xi\to\infty} \operatorname{Sup} u_\xi \ge 0$. Thus, the proof is complete.

LEMMA 4. *In the assumptions of Lemma 3, further assume $P_n \ge 0$ eventually.*
Then there exist an $n_2 \ge n_1$ and a sequence $\{v_n\}$, such that

$$v_n \ge P_n^{(1)} + \sum_{s=n}^{\infty} \frac{A_s v_{s+1}^2}{a_{g_s} A_{s+1}^2} R_{s,n}^{(1)}, \qquad \text{as } n \ge n_2, \tag{2.11}$$

where

$$P_n^{(1)} = \sum_{s=n}^{\infty} \frac{A_s P_{s+1}^2}{a_{g_s} A_{s+1}^2} R_{s,n}^{(1)}, \qquad R_{s,n}^{(1)} = \prod_{i=n}^{s-1} \left(1 + \frac{2A_i P_{i+1}}{a_{g_i} A_{i+1}^2} \right), \qquad \prod_{n}^{n-1} = 1.$$

PROOF. By Lemma 3, there exists an n_1, such that (2.7) and (2.8) hold.
Define $v_n = \sum_{s=n}^{\infty} (A_s u_{s+1}^2)/(a_{g_s} A_{s+1}^2)$, then $\Delta v_n = -(A_n u_{n+1}^2)/(a_{g_n} A_{n+1}^2)$,

$$\Delta v_s R_{s,n}^{(1)} = -\frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} R_{s,n}^{(1)}. \tag{2.12}$$

Summing of (2.12) from n to $N-1$, we have

$$\sum_{s=n}^{N-1} \Delta v_s R_{s,n}^{(1)} = -\sum_{s=n}^{N-1} \frac{A_s u_{s+1}^2}{a_{g_s} A_{s+1}^2} R_{s,n}^{(1)},$$

$$v_N R^{(1)}_{N-1,n} - v_n - \sum_{s=n}^{N-1} 2v_{s+1} \frac{A_s P_{s+1}}{a_{g_s} A^2_{s+1}} R^{(1)}_{s,n} = - \sum_{s=n}^{N-1} \frac{A_s u^2_{s+1}}{a_{g_s} A^2_{s+1}} R^{(1)}_{s,n},$$

$$v_n \geq \sum_{s=n}^{N-1} \frac{A_s}{a_{g_s} A^2_{s+1}} R^{(1)}_{s,n} \left(u^2_{s+1} - 2v_{s+1} P_{s+1}\right).$$

From Lemma 3, $u_{s+1} \geq P_{s+1} + v_{s+1} \geq 0$, hence, $u^2_{s+1} \geq P^2_{s+1} + v^2_{s+1} + 2P_{s+1} v_{s+1}$,

$$v_n \geq \sum_{s=n}^{N-1} \frac{A_s}{a_{g_s} A^2_{s+1}} R^{(1)}_{s,n} \left(P^2_{s+1} + v^2_{s+1}\right),$$

$$v_n \geq \sum_{s=n}^{\infty} \frac{A_s}{a_{g_s} A^2_{s+1}} R^{(1)}_{s,n} \left(P^2_{s+1} + v^2_{s+1}\right).$$

This completes our proof.

LEMMA 5. *In the assumptions of Lemma 4, then for every positive integer m there exist an $N \geq n_1$ and a sequence for $\{w_n\}$, such that*

$$w_n \geq P^{(m)}_n = \sum_{s=n}^{\infty} \frac{A_s w^2_{s+1}}{a_{g_s} A^2_{s+1}} R^{(m)}_{s,n}, \qquad \text{as } n \geq N, \tag{2.13}$$

where

$$P^{(m)}_n = \sum_{s=n}^{\infty} \frac{A_s \left(P^{(m-1)}_{s+1}\right)^2}{a_{g_s} A^2_{s+1}} R^{(m)}_{s,n}, \qquad R^{(m)}_{s,n} = \prod_{i=n}^{s-1} \left(1 + \frac{2A_i P^{(m-1)}_{i+1}}{a_{g_i} A^2_{i+1}}\right).$$

The proof is similar to Lemma 4.

LEMMA 6. *In the assumptions of Lemma 4, then for every positive integer m,*

$$\lim_{n\to\infty} \operatorname{Sup} P^{(m)}_n \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_s P^{(m-1)}_{s+1}}{a_{g_s} A^2_{s+1}}\right) < \infty. \tag{2.14}$$

PROOF. By Lemma 5, there exists a sequence $\{w_n\}$ for $m = 1, 2, \dots$, such that

$$w_n \geq P^{(m)}_n + \sum_{s=n}^{\infty} \frac{A_s w^2_{s+1}}{a_{g_s} A^2_{s+1}} R^{(m)}_{s,n}.$$

Set $u_n = \sum_{s=n}^{\infty} (A_s w^2_{s+1})/(a_{g_s} A^2_{s+1}) R^{(m)}_{s,n}$, then $w_n \geq P^{(m)}_n + u_n$,

$$-\Delta u_n = \frac{A_n w^2_{n+1}}{a_{g_n} A^2_{n+1}} \geq \frac{4A_n P^{(m)}_{n+1} u_{n+1}}{a_{g_n} A^2_{n+1}},$$

$$u_n - u_{n+1} \geq \frac{4A_n P^{(m)}_{n+1} u_{n+1}}{a_{g_n} A^2_{n+1}},$$

hence,

$$u_{n+1} \leq u_n \left(1 + \frac{4A_n P^{(m)}_{n+1}}{a_{g_n} A^2_{n+1}}\right)^{-1} \leq u_{n_0} \prod_{s=n_0}^{n} \left(1 + \frac{4A_s P^{(m)}_{s+1}}{a_{g_s} A^2_{s+1}}\right)^{-1}.$$

From the proof of Lemma 5, $u_n \geq P^{(m+1)}_n$. Hence,

$$P^{(m+1)}_n \leq u_{n_0} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_s P^{(m)}_{s+1}}{a_{g_s} A^2_{s+1}}\right)^{-1},$$

$$P_n^{(m)} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_s P_{s+1}^{(m-1)}}{a_{g_s} A_{s+1}^2}\right)^{-1} \leq u_{n_0}.$$

Therefore, for every positive integer m,

$$\lim_{n\to\infty} \operatorname{Sup} P_n^{(m)} \prod_{s=n_0}^{n-1} \left(1 + \frac{4A_s P_{s+1}^{(m-1)}}{a_{g_s} A_{s+1}^2}\right) < \infty.$$

This completes our proof.

3. MAIN RESULTS

Using Lemmas 3–6 above, we can easily obtain the following Theorems 1–4, respectively.

THEOREM 1. *Assume for (1.1), there exists a positive sequence* $\{A_n\}$, *such that*

$$\sum_{s=n}^{\infty} \frac{A_s}{a_{g_s} A_{s+1}^2} = \infty \tag{3.1}$$

and

$$\sum_{s=n}^{\infty} Q_s = \infty. \tag{3.2}$$

Then equation (1.1) is oscillatory.

THEOREM 2. *Assume for (1.1), there exists a positive sequence* $\{A_n\}$, *such that* $P_n \geq 0$ *eventually,* $\sum_{s=n}^{\infty} (A_s)/(a_{g_s} A_{s+1}^2) = \infty$,

$$P_n \sum_{s=n}^{\infty} Q_s < \infty, \tag{3.3}$$

but

$$\sum_{s=n}^{\infty} \frac{A_s P_{s+1}^2}{a_{g_s} A_{s+1}^2} \prod_{i=n}^{s-1} \left(1 + \frac{2A_i P_{i+1}}{a_{g_i} A_{i+1}^2}\right) = \infty. \tag{3.4}$$

Then equation (1.1) is oscillatory.

EXAMPLE 1. Consider the following second-order neutral delay difference equation:

$$\Delta\left(\frac{1}{(n+2)^2} \Delta\left(x_n + \frac{n^2+n-1}{(n+1)(n+2)} x_{n-1}\right)\right) + \frac{x_{n-1}}{n(n+1)} = 0. \tag{3.5}$$

Let $A_n \equiv 1$, then

$$Q_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}, \qquad P_n = \sum_{s=n}^{\infty} Q_s = \frac{1}{n^2} < \infty, \qquad \sum_{n}^{\infty} \frac{1}{a_{n-1}} = \infty,$$

$$\sum_{s=n}^{\infty} \frac{A_s P_{s+1}^2}{a_{g_s} A_{s+1}^2} \prod_{i=n}^{s-1} \left(1 + \frac{2A_i P_{i+1}}{a_{g_i} A_{i+1}^2}\right) = \sum_{s=n}^{\infty} \frac{3^{s-n}}{(s+1)^2} = \infty.$$

By Theorem 2, equation (3.5) is oscillatory.

THEOREM 3. *Assume for (1.1), there exists a positive sequence* $\{A_n\}$, *such that* $P_n \geq 0$ *eventually,* $\sum_{s=n}^{\infty} (A_s)/(a_{g_s} A_{s+1}^2) = \infty$, *and for some positive integer* k *have* $P_n^{(j)} < \infty$, $j = 0, 1, 2, \dots,$ $k-1$, *but*

$$P_n^{(k)} = \infty. \tag{3.6}$$

Then equation (1.1) is oscillatory.

THEOREM 4. *Assume for (1.1), there exists a positive sequence* $\{A_n\}$, *such that* $P_n \ge 0$ *eventually,* $\sum_{s=n}^{\infty}(A_s)/(a_{g_s}A_{s+1}^2) = \infty$, *there exists a positive integer* m_0, *such that*

$$\lim_{n\to\infty} \operatorname{Sup} P_n^{(m_0)} \prod_{s=n_0}^{n-1}\left(1+\frac{4A_sP_{s+1}^{(m_0-1)}}{a_{g_s}A_{s+1}^2}\right) = \infty. \tag{3.7}$$

Then equation (1.1) is oscillatory.

EXAMPLE 2. Consider the following second-order neutral delay difference equation:

$$\Delta\left(\frac{1}{(n+1)(n+2)}\Delta\left(x_n+\frac{1}{n+1}x_{n-1}\right)\right)+\frac{\lambda x_{n-1}}{(n-1)^2(n+1)(n+2)} = 0, \tag{3.8}$$

where λ is a nonnegative number.

Let $A_n \equiv 1$, then

$$\begin{aligned}
Q_n &= \lambda\left(1-\frac{1}{n}\right)\frac{1}{(n-1)^2(n+1)(n+2)} \\
&= \frac{\lambda}{(n-1)n(n+1)(n+2)}, \\
P_n &= \lambda\sum_{s=n}^{\infty}\frac{1}{(s-1)s(s+1)(s+2)} = \frac{\lambda}{3(n-1)n(n+1)}, \\
P_n^{(1)} &= \frac{\lambda^2}{9}\sum_{s=n}^{\infty}\frac{1}{s(s+1)(s+2)^2}\prod_{i=n}^{s-1}\left(1+\frac{2\lambda}{3(s+2)}\right) \\
&\ge \frac{\lambda^2}{9}\sum_{s=n}^{\infty}\frac{1}{s(s+1)(s+2)(s+3)} \\
&= \frac{\lambda^2}{27n(n+1)(n+2)}, \\
S_n &= P_n^{(1)}\prod_{s=n_0}^{n-1}\left(1+\frac{4A_sP_{s+1}}{a_{g_s}A_{s+1}^2}\right) \\
&\ge \frac{\lambda^2}{27n(n+1)(n+2)}\prod_{s=n_0}^{n-1}\left(1+\frac{4\lambda}{3(s+2)}\right) \\
&= \frac{\lambda^2}{27n(n+1)(n+2)}\prod_{s=n_0+2}^{n+1}\left(1+\frac{4\lambda}{3s}\right) \\
&\ge \frac{\alpha\lambda^2(n+1)^{(4\lambda)/3}}{27n(n+1)(n+2)}, \qquad \text{(see [10])},
\end{aligned}$$

where α is some positive number.

When $\lambda > 9/4, S_n \to \infty$, as $n \to \infty$. By Theorem 4, equation (3.8) is oscillatory.

THEOREM 5. *Assume for (1.1), there exists a positive sequence* $\{A_n\}$, *such that* $P_n \ge 0$ *eventually,* $\sum_{s=n}^{\infty}(A_s)/(a_{g_s}A_{s+1}^2) = \infty$, *there exists a positive integer* m_0, *such that*

$$\lim_{n\to\infty}\sum_{s=n_0}^{n}\prod_{i=n_0}^{s-1}\left(1+\frac{4A_iP_{i+1}^{(m_0-1)}}{a_{g_i}A_{i+1}^2}\right)^{-1} < \infty, \tag{3.9}$$

but

$$\lim_{n\to\infty}\sum_{s=n_0}^{n}P_s^{(m_0)} = \infty. \tag{3.10}$$

Then equation (1.1) is oscillatory.

PROOF. By the proof of Lemma 6, we have

$$\sum_{s=n_0}^{n} P_s^{(m_0)} \leq u_{n_0} \sum_{s=n_0}^{n} \prod_{i=n_0}^{s-1} \left(1 + \frac{4A_i P_{i+1}^{(m_0-1)}}{a_{g_i} A_{i+1}^2}\right)^{-1},$$

which contradicts (3.9) and (3.10).

REMARK. Theorems 1–3 all include and extend those in [5,9]. In fact, when $p_n \equiv 0$, letting $A_n \equiv 1$ can obtain their corresponding results.

REFERENCES

1. S.Z. Chen and L. Erbe, Oscillation results for second-order scalar and matrix difference equations, *Computers Math. Applic.* **28** (1–3), 55–70, (1994).
2. P.J.Y. Wong and R.P. Agarwal, Oscillation theorems and existence of positive monotone solutions for second-order nonlinear difference equations, *Mathl. Comput. Modelling* **21** (3), 63–84, (1995).
3. Z. Szafranski and B. Szmanda, Oscillation theorems for some nonlinear difference equations, *Appl. Math. Comput.* **83**, 43–52, (1997).
4. Z. Zhang and Q. Li, Oscillation theorems for second-order advanced functional difference equation, *Computers Math. Applic.* **36** (6), 11–18, (1998).
5. Z. Zhang and J. Zhang, Oscillation criteria for second-order functional difference equations with "summation small" coefficient, *Computers Math. Applic.* **38** (1), 25–31, (1999).
6. Z. Zhang and Y. Yu, Oscillation of solutions for a class second nonlinear difference equations, *Journal of Mathematical Research and Exposition* **19** (4), (1999).
7. E. Thandapani and P. Sanadarani, On the asymptotic and oscillatory behavior of second-order nonlinear neutral difference equations, *Indian J. Pure. Appl. Math.* **26** (19), (1995).
8. E. Thandapani and R. Arul, Oscillation properties of second-order nonlinear neutral delay difference equations, *Indian J. Pure. Appl. Math.* **28** (12), (1997).
9. R.P. Agarwal, M.M.S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second-order neutral delay difference equations II, *Appl. Math. Lett.* **10** (2), 103–109, (1997).
10. J. Dteudorme, *Foundations of Modern Analysis, Vol. 1*, Academic Press, (1969); *Vol. 2*, Academic Press, (1972).