A new discrete filled function algorithm for discrete global optimization

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Abstract
A definition of the discrete filled function is given in this paper. Based on the definition, a discrete filled function is proposed. Theoretical properties of the proposed discrete filled function are investigated, and an algorithm for discrete global optimization is developed from the new discrete filled function. The implementation of the algorithms on several test problems is reported with satisfactory numerical results.

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1. Introduction
Optimization of a general cost function over discrete variables arises frequently in various applications such as combinatorics, scheduling, design, and operate problems. Even though some discrete optimization problems have been studied since ancient times, the impact of discrete optimization have become influential only in last few decades due to advancement of computer technologies. Like the continuous global optimization problems, the existence of multiple local minima of a general nonconvex objective function makes discrete global optimization a great challenge. For continuous global optimization problems, many deterministic methods have been proposed to search for a globally optimal solution of a function of several variables. The filled function algorithm (see [5,6]) is an effective and practical method among determinate algorithms.

The primary filled function was proposed by Ge in paper [5]. The definition of the filled function is as follows:

**Definition 1.1.** Let \( x_1^* \) be a current minimizer of \( f(x) \). A function \( P(x) \) is called a filled function of \( f(x) \) at \( x_1^* \) if \( P(x) \) has the following properties:

1. \( x_1^* \) is a maximizer of \( P(x) \) and the whole basin \( B_1^* \) of \( f(x) \) at \( x_1^* \) becomes a part of a hill of \( P(x) \);
2. \( P(x) \) has no minimizers or saddle points in any higher basin of \( f(x) \) than \( B_1^* \);

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3. if $f(x)$ has a lower basin than $B_1^*$, then there is a point $x'$ in such a basin that minimizes $P(x)$ on the line through $x$ and $x_1^*$.

For the definitions of basin and hill, refer to paper [5].

Its main idea is as follows: a local minimizer of objective function is found by classic local search algorithm at first; a filled function is constructed over the feasible region; making use of the properties of the filled function, a better minimizer is found, or show that the local minimizer has been the global minimizer.

In the paper, we consider the follow discrete global optimization problem:

$$\min \{ f(x) : x \in X \subset \mathbb{Z}^n \},$$

(1)

where $f : \mathbb{Z}^n \rightarrow \mathbb{R}$, and $\mathbb{Z}^n$ is the set of integer points in $\mathbb{R}^n$, and $X$ is box, i.e., $X = \{ x \in \mathbb{Z}^n : a \leq x \leq b, a,b \in \mathbb{Z}^n \}$.

We try to solve discrete global optimization problem (1) by the same idea as that of solving continuous global optimization problems. A new definition of the discrete filled function is given at first, and a discrete filled function satisfying the definition is presented. The discrete filled function has some properties, such as, any point $x$ where the function value is greater than or equal to the current minimum of the cost function is not the discrete local minimizer of the discrete filled function; if the current discrete minimizer of cost function is not its global discrete minimizer, then there must exist a discrete filled function’s discrete minimizer where the cost function value is less its current discrete minimum, and so on. Second, an algorithm for discrete global optimization is developed from the discrete filled function. Adopting the concept of the discrete filled functions, a discrete global optimization problem can be solved via a two-phase cycle. In Phase 1, we first find a discrete local minimizer $x_1^*$ of the cost function $f(x)$ by the classic search algorithm. In Phase 2, a discrete filled function is constructed at $x_1^*$ and minimize the discrete filled function in order to identify a point $x'$ with $f(x') < f(x_1^*)$. If such a $x'$ is found, then we can use $x'$ as the initial point in Phase 1 again, and hence we can find a better minimizer $x_2^*$ of $f(x)$ with $f(x_2^*) < f(x_1^*)$. This process repeats until the time when minimizing a filled function does not yield a better solution. The current discrete local minimum will be taken as a discrete global minimizer of $f(x)$.

The paper is organized as follows: to simplify the discussion in this paper, we first recall some definitions in discrete analysis and discrete optimization in Section 2. In Section 3, a definition of discrete filled function is given, a discrete filled function is presented, and we investigate its properties. An algorithm is developed from the discrete filled function are proposed in Section 4. Numerical experiments are presented in Section 5. In the last section, we give the conclusion.

2. Preliminary

To simplify the discussion in this paper, we recall some definitions in discrete analysis and discrete optimization.

**Definition 2.1.** A sequence $\{x_i\}_{i=-1}^n$ is called a discrete path in $X$ between two distinct point $x^*$ and $x^{**}$ in $X$ if $x^{-1}=x^*$, $x^u=x^{**}$, $x^i \in X$, for all $i$; $x^i \neq x^j$, for $i \neq j$; and $\|x^0-x^*\| = \|x^{i+1}-x^i\| = \|x^{**}-x^{u-1}\| = 1$, for all $i$. If such a discrete path exists, then $x^*$ and $x^{**}$ are said to be pathwise connected in $X$. Furthermore, if every two distinct points in $X$ are pathwise connected in $X$, then $X$ is called a pathwise connected set.

**Definition 2.2.** The set of all axial directions in $\mathbb{Z}^n$ is defined by $D = \{ \pm e_i : i = 1, 2, \ldots, n \}$, where $e_i$ is the $i$th unit vector (the $n$-dimensional vector with the $i$th component equal to one and all other components equal to zero).

**Definition 2.3.** The set of all feasible directions at $x \in X$ is defined by $D_x = \{ d \in D : x + d \in X \}$, where $D$ is the set of axial directions.

**Definition 2.4.** For any $x \in \mathbb{Z}^n$, the discrete neighborhood of $x$ is defined by $N(x) = \{ x, x \pm e_i : i = 1, 2, \ldots, n \}$.

**Definition 2.5.** The discrete interior of $X$ is defined by $\text{int} X = \{ x \in X : N(x) \subset X \}$. While, the discrete boundary of $X$ is denoted by $\partial X = X \setminus \text{int} X$.

Note that, if $X$ contains less than or equal to $2n$ points, then $\text{int} X = \emptyset$ and $\partial X = X$. 
**Definition 2.6.** A point \( x^* \in X \) is called a **discrete local minimizer** of \( f \) over \( X \) if \( f(x^*) \leq f(x) \), for all \( x \in X \cap N(x^*) \). If, in addition, \( f(x^*) < f(x) \), for all \( x \in X \cap N(x^*) \setminus \{x^*\} \), then \( x^* \) is called a **strict discrete local minimizer** of \( f \) over \( X \).

**Definition 2.7.** A point \( x^* \in X \) is called a **discrete global minimizer** of \( f \) over \( X \) if \( f(x^*) \leq f(x) \), for all \( x \in X \). If, in addition, \( f(x^*) < f(x) \), for all \( x \in X \), then \( x^* \) is called a **strict discrete global minimizer** of \( f \) over \( X \).

**Definition 2.8.** For any \( x \in X \), \( d \in D \) is said to be a **discrete descent direction** of \( f \) at \( x \) over \( X \) if \( x + d \in X \) and \( f(x + d) < f(x) \); beside, \( d^* \in D \) is called a **discrete steepest descent direction** of \( f \) at \( x \) over \( X \) if \( f(x + d^*) \leq f(x + d) \), for all \( d \in D^* \), where \( D^* \) is the set of all descent direction of \( f \) at \( x \) over \( X \).

In the following, we present a discrete steepest descent method for finding a local minimizer of \( f \) over \( X \) from a given initial point \( x \in X \).

**Algorithm 2.1 (Discrete steepest descent method).**

1. Start from the initial point \( x \in X \).
2. If \( x \) is a local minimizer of \( f \) over \( X \), then stop. Otherwise, a discrete steepest descent direction \( d^* \) of \( f \) at \( x \) over \( X \) can be found.
3. Let \( x = x + \lambda d^* \), where \( \lambda \in \mathbb{Z}_+ \) is the step length such that \( f \) has maximum reduction in direction \( d^* \), and go to Step 2.

**Algorithm 2.2 (Modified discrete descent method).**

1. Start from the initial point \( x \in X \).
2. If \( x \) is a local minimizer of \( f \) over \( X \), then stop. Otherwise, let
   \[
   d^* = \arg\max \{ f(x + d_l) : d_l \in D_x, f(x + d_l) < f(x) \},
   \]
   where \( D_x \) denotes the set of feasible directions at \( x \).
3. Let \( x = x + d^* \), and go to Step 2.

Obviously, by Algorithms 2.1 and 2.2, we can only find a discrete local minimizer.

Finally, for the discrete global optimization problem (1), we make the following assumptions in this paper:

**Assumption 1.** \( X \) is a pathwise connected set.

**Assumption 2.** \( X \subset \mathbb{Z}^n \) is a bounded set which contains more than one point. This implies that there exists a constant \( K > 0 \) such that
\[
1 \leq K = \max_{x, y \in X} \|x - y\| \leq \infty,
\]
where \( \| \cdot \| \) is the usual Euclidean norm.

**Assumption 3.** \( f : \bigcup_{x \in X} N(x) \to \mathbb{R} \) satisfies the following Lipschitz condition for every \( x, y \in \bigcup_{x \in X} N(x) \):
\[
|f(x) - f(y)| \leq L\|x - y\|.
\]
where \( 0 < L < \infty \) is a constant, \( N(x) \) is the discrete neighborhood of \( x \).

Furthermore, when \( f \) is coercive, i.e., \( f \to \infty \) as \( \|x\| \to \infty \), there always exists a box which contains all discrete global minimizer of \( f \), and Assumption 1–3 hold on the box set. Thus, the unconstrained discrete global optimization problem, \( \min \{ f(x) : x \in \mathbb{Z}^n \} \) can be reduced into an equivalent problem formulation in (1).
3. The discrete filled function

In this subsection, we give a definition of discrete filled function at first. Further, a discrete filled function is presented, and we investigate some properties of the discrete filled function.

**Definition 3.1.** $P(x, x^*_1)$ is called a discrete filled function of $f(x)$ at a discrete local minimizer $x^*_1$ if $P(x, x^*_1)$ has the following properties:

1. $x^*_1$ is a strict discrete local maximizer of $P(x, x^*_1)$ over $X$.
2. $P(x, x^*_1)$ has no discrete local minimizers in the region $S_1 = \{x \mid f(x) \geq f(x^*_1), x \in X / x^*_1 \}$.
3. If $x^*_1$ is not a discrete global minimizer of $f(x)$, then $P(x, x^*_1)$ does have a discrete minimizer in the region $S_2 = \{x \mid f(x) < f(x^*_1), x \in X \}$.

These properties of the discrete filled function ensure that when the discrete steepest descent method is employed to minimize the constructed filled function, the sequence of iteration point will not terminate at any point at which the function value is larger than $f(x^*_1)$; if $x^*_1$ is not a global minimizer, then there must exist a discrete minimizer of the discrete filled function at which the cost function value is less than $f(x^*_1)$, that is, any local minimizer of $P(x, x^*_1)$ must belong to the set $S_2 = \{x \in X : f(x) < f(x^*_1) \}$. So that the current discrete local minimizer of the objective function is escaped by minimizing the discrete filled function and a better minimizer can be found by minimizing the cost function starting from the discrete minimizer of the discrete filled function.

Now, we give a discrete filled function for problem (1) at a local minimizer $x^*_1$ as follows:

$$F(x, x^*_1, q, r) = \frac{1}{q + \|x - x^*_1\|} \varphi_q(\max\{f(x) - f(x^*_1) + r, 0\}),$$  

(2)

where

$$\varphi_q(t) = \begin{cases} \exp\left(-\frac{t}{q}\right) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

(3)

$q > 0$ and $r$ satisfies

$$0 < r < \max_{x^*_1, x^*_1 \in L(P), f(x^*_1) < f(x^*_1)} (f(x^*_1) - f(x^*_1)),$$

where $L(P)$ stand for the set of discrete local minimizers of $f(x)$.

Next we will show that the function $F(x, x^*_1, q, r)$ is a discrete filled function satisfying Definition 3.1 under certain conditions on the parameters $q$ and $r$.

**Theorem 3.1.** Suppose that $X$ holds Assumption 2. Further suppose that $x^*_1$ is a discrete local minimizer of $f(x)$. For any $r > 0$, when $q > 0$ is satisfactorily small, $x^*_1$ is a strict discrete local maximizer of $F(x, x^*_1, q, r)$.

**Proof.** Since $x^*_1$ is a discrete local minimizer of $f(x)$, for any $x \in N(x^*_1)$, $f(x) \geq f(x^*_1)$ and $\|x - x^*_1\| = 1$. Hence, we have

$$F(x, x^*_1, q, r) = \frac{1}{q + 1} \exp\left(-\frac{q}{f(x) - f(x^*_1) + r}\right)$$

and

$$F(x^*_1, x^*_1, q, r) = \frac{1}{q} \exp\left(-\frac{q}{r}\right).$$
Thus, when $0 < q \leq r/2$, \[
\frac{q(f(x) - f(x^*_i))}{r(f(x) - f(x^*_i) + r)} < 1 \quad \text{and} \quad e^y < \frac{1}{1 - y}
\] for all $y < 1$ and $y \neq 0$, for $0 < q \leq \min\{1, r/2\}$, we have

\[
\frac{F(x, x^*, q, r)}{F(x^*_1, x^*_1, q, r)} < 1
\]

for any $x \in N(x^*_1)$, $x^*_1$ is a strict discrete local maximizer of $F(x, x^*_1, q, r)$. \quad \square

**Lemma 3.1.** For every $x, x^* \in X$, if there exists $i \in \{1, 2, \ldots, n\}$ such that $x \pm e_i \in X$, then there exists $d \in D$ such that

$$
\|x + d - x^*\| > \|x - x^*\|.
$$

**Proof.** If there is an $i \in \{1, 2, \ldots, n\}$ such that $x_i \geq x^*_i$, then $d = e_i$. On the other hand, if there is an $i \in \{1, 2, \ldots, n\}$ such that $x_i \leq x^*_i$, then $d = -e_i$. This completes the proof. \quad \square

**Lemma 3.2.** Suppose that set $X$ holds Assumptions 1, 2. Given that $x_1, x_2, x^* \in X$. If $\|x_2 - x^*\| > \|x_1 - x^*\|$, then

$$
\frac{\|x_1 - x^*\|}{\|x_2 - x^*\|} < 1 - \frac{1}{2K^2}.
$$

**Proof.** We first consider for every $x \in Z^n$, then $\|x\|^2 \in Z$, and hence $\|x_2 - x^*\|^2 - \|x_1 - x^*\|^2 \in Z$, Since $\|x_2 - x^*\| > \|x_1 - x^*\|$, we have $\|x_2 - x^*\|^2 - \|x_1 - x^*\|^2 > 1$.

Moreover, by Assumption 2, we have $\|x_1 - x^*\| < \|x_2 - x^*\| \leq K$ and thus $0 < \|x_2 - x^*\| + \|x_1 - x^*\| < 2K$. Therefore,

$$
\|x_2 - x^*\| - \|x_1 - x^*\| \geq \frac{1}{\|x_2 - x^*\| + \|x_1 - x^*\|} > \frac{1}{2K},
$$

and

$$
\frac{\|x_2 - x^*\| - \|x_1 - x^*\|}{\|x_2 - x^*\|} > \frac{1/(2K)}{K} = \frac{1}{2K^2} \quad \square
$$
Theorem 3.2. Suppose that Assumptions 1–3 are satisfied. If \( x_1^* \) is a discrete local minimizer of \( f(x) \), then the function \( F(x, x_1^*, q, r) \) has no discrete local minimizers in the region \( S_1 = \{ x \mid f(x) \geq f(x_1^*), x \in X \setminus \{ x_1^* \} \} \) when \( r > 0 \) and \( q > 0 \) are satisfactorily small.

Proof. Let \( \tilde{X} = \bigcup_{x \in X} N(x) \), obviously, \( \tilde{X} \) holds Assumptions 1–3, and we have \( S_1 \subseteq X \subseteq \text{int} \tilde{X} \). For every \( x \in S_1 \), by Lemma 3.1, there must exists a direction \( d \in D \) such that \( x + d \in \tilde{X} \) and

\[
\|x + d - x_1^*\| > \|x - x_1^*\|.
\]

Consider the following two cases:

1. \( f(x + d) \geq f(x_1^*) \):

Since \( f(x + d) > f(x_1^*) \), we have

\[
\frac{F(x + d, x_1^*, q, r)}{F(x, x_1^*, q, r)} = \frac{q + \|x - x_1^*\|}{q + \|x + d - x_1^*\|} \exp \left( \frac{q}{f(x) - f(x_1^*) + r} - \frac{q}{f(x + d) - f(x_1^*) + r} \right) 
\]

\[
= \frac{q + \|x - x_1^*\|}{q + \|x + d - x_1^*\|} \exp \left( \frac{q f(x)}{q (f(x) - f(x_1^*) + r)(f(x + d) - f(x_1^*) + r)} \right) 
\]

\[
\leq \frac{q + \|x - x_1^*\|}{q + \|x + d - x_1^*\|} \exp \left( \frac{q L}{r^2} \right).
\]

By Lemma 3.2, we have

\[
\frac{q + \|x - x_1^*\|}{q + \|x + d - x_1^*\|} < 1 - \frac{1}{2K^2} + \frac{q}{\|x + d - x_1^*\|}.
\]

If \( 0 < q \leq \frac{1}{4k^2} \), then

\[
\frac{q + \|x - x_1^*\|}{q + \|x + d - x_1^*\|} < 1 - \frac{1}{2K^2} + \frac{1}{4K^2} = \frac{4K^2 - 1}{4K^2}
\]

and

\[
\frac{F(x + d, x_1^*, q, r)}{F(x, x_1^*, q, r)} < \frac{4K^2 - 1}{4K^2} \exp \left( \frac{q L}{r^2} \right).
\]

Hence, when

\[
0 < q < \min \left\{ 1, \frac{r^2}{L} \ln \frac{4K^2}{4K^2 - 1} \right\},
\]

we have

\[
\frac{F(x + d, x_1^*, q, r)}{F(x, x_1^*, q, r)} < 1.
\]

2. \( f(x + d) < f(x_1^*) \) :
Theorem 3.3. If \( \text{Assumption 3} \) is satisfied, then there exists a discrete minimizer \( x_1^* \) of \( F(x, x_1^*, q, r) \) in the region \( S_2 = \{ x | f(x) < f(x_1^*), x \in X \} \).

Proof. Since \( x_1^* \) is not a discrete global minimizer and \( F(x, x_1^*, q, r) \geq 0 \), there exist a point \( \bar{x}_1^* \in S_2 \) and \( r > 0 \) such that \( f(\bar{x}_1^*) < f(x_1^*) - r \). Hence, \( F(\bar{x}_1^*, x_1^*, q, r) = 0 \), it implies that \( \bar{x}_1^* \in S_2 \) is a discrete minimizer of \( F(x, x_1^*, q, r) \).

Theorems 3.1–3.3 show that the function \( F(x, x_1^*, q, r) \) at point \( x_1^* \) is a discrete filled function satisfying Definition 3.1 with satisfactorily small \( q \) and \( r \). The following theorems further show that the proposed filled function has some good properties which classical functions have.

Theorem 3.4. Suppose that Assumption 3 is satisfied. If \( x_1, x_2 \in X \) and satisfy the following conditions:

1. \( f(x_1) \geq f(x_2) \) and \( f(x_2) \geq f(x_1^*) \),
2. \( \|x_2 - x_1^*\| > \|x_1 - x_1^*\| \).

Then, when \( r > 0 \) and \( q > 0 \) are satisfactorily small, \( F(x_2, x_1^*, q, r) < F(x_1, x_1^*, q, r) \).

Proof. Consider the following two cases:

1. If \( f(x_1^*) \leq f(x_2) \leq f(x_1) \), then it is obvious that the result follows.
2. If \( f(x_1^*) \leq f(x_1) < f(x_2) \), we will show the result also holds.

When \( f(x_1^*) \leq f(x_1) < f(x_2) \), we have

\[
\frac{F(x_2, x_1^*, q, r)}{F(x_1, x_1^*, q, r)} = \frac{q + \|x_1 - x_1^*\|}{q + \|x_2 - x_1^*\|} \exp\left(\frac{q}{f(x_1) - f(x_1^*) + r} - \frac{q}{f(x_2) - f(x_1^*) + r}\right) \\
\leq \frac{q + \|x_1 - x_1^*\|}{q + \|x_2 - x_1^*\|} \exp\left(\frac{q}{r} - \frac{q}{L\|x_2 - x_1^*\| + r}\right)
\]

and

\[
\lim_{q \to 0} \frac{q + \|x_1 - x_1^*\|}{q + \|x_2 - x_1^*\|} \exp\left(\frac{q}{r} - \frac{q}{L\|x_2 - x_1^*\| + r}\right) = \frac{\|x_1 - x_1^*\|}{\|x_2 - x_1^*\|} < 1.
\]

Then, there must exist a constant \( q_0 > 0 \) such that \( F(x_2, x^*, q, r) < F(x_1, x^*, q, r) \) while \( q < q_0 \), and \( q_0 \) is not related to these function values at \( x_1 \) and \( x_2 \).
Theorem 3.5. If \( x_1, x_2 \in X \) and satisfy the following conditions:

1. \( \|x_2 - x^*\| > \|x_1 - x^*\| \),
2. \( f(x_1) \geq f(x_1^*) > f(x_2), \) and \( f(x_2) - f(x_1^*) + r > 0 \).

Then, we have \( F(x_2, x_1^*, r, q) < F(x_1, x_1^*, r, q) \).

Proof. By Conditions 1 and 2, we have

\[
\frac{1}{q + \|x_2 - x_1^*\|} < \frac{1}{q + \|x_1 - x_1^*\|}
\]

and

\[
0 < f(x_2) - f(x_1^*) + r < f(x_1) - f(x_1^*) + r.
\]

Hence

\[ F(x_2, x_1^*, r, q) < F(x_1, x_1^*, r, q). \]

Now we make some remarks. Firstly, in the phase of minimizing the new discrete filled function, Theorems 3.4 and 3.5 guarantee that the current discrete local minimizer \( x_1^* \) of the objective function is escaped and the minimum of the new discrete filled function will be always achieved at a point where the objective function value is less than the current discrete minimum. Secondly, the parameters \( q \) and \( r \) are easier to be appropriately chosen. In the next section, a new discrete filled function algorithm is given.

4. A new discrete filled function algorithm

Based on the theoretical results in the previous section, a global optimization algorithm over \( X \) is proposed as follows:

Algorithm.

Initialization:

1. Choose any \( x_0 \in X \) as an initial point.
2. Let \( \varepsilon = 10^{-5} \) and \( q_0 = 0.01 \).
3. Let \( D_0 = \{ \pm e_i : i = 1, 2, \ldots, n \} \).

Main Program:

1. Starting from initial point \( x_0 \), minimize \( f(x) (x \in X) \) by the discrete steepest descent method (see Algorithm 2.1), we can obtain the discrete local \( x_1^* \).
   
   Let \( r = 1, q = q_0 \) and \( D = D_0 \).
2. Construct the discrete filled function

\[
F(x, x_1^*, q, r) = \frac{1}{q + \|x - x^*\|} \varphi_q(\max\{f(x) - f(x_1^*) + r, 0\}).
\]

3. If \( r \leq \varepsilon \), then terminate the iteration, the \( x_1^* \) is the global minimizer of \( f(x) \), otherwise, the next step
4. If \( D \neq \emptyset \), then goto 6, otherwise the next step.
5. If \( q < \varepsilon \times 10^{-2} \), then let \( r = r/10, q = q_0/10 \) and \( D = D_0 \), goto 2 otherwise let \( q = q/10 \), goto 2.
6. Take a direction \( d \in D \), and \( D \leftarrow D \setminus \{d\} \), turn to Inner Loop.

Inner Loop:

1. \( k = 0 \).
2. Let \( y_k = x_1^* + d \).
3. Minimize $F(x, x^*, q, r)$, starting from the point $y_k$, by implementing the modified discrete descent method (see Algorithm 2.2). $y_{k+1}$ denotes the next iterative point.

4. If $y_{k+1} \notin X$, then return Main Program 4, otherwise next step.

5. If $f(y_{k+1}) < f(x^*)$ then let $x_0 = y_{k+1}$ and return Main Program 1, otherwise let $k = k + 1$ and goto 3.

The idea and mechanism of algorithm are explained as follows.

There are two phrases in the algorithm. One is that of minimizing the original function $f$, the other is that of minimizing the new discrete filled function $F(x, x^*, q, r)$ in the inner loop. We let $r = 1$ and $q = 0.01$ in the initialization, afterwards, they are gradually reduced via the two-phase cycle until they are less than sufficiently small positive scales. If the parameters $r$ is sufficiently small, and we cannot find the point $x$ with $f(x) < f(x^*)$ yet, then we believe that there does not exist a better local minimizer of $f(x)$. The algorithm is terminated.

In the process of minimizing the discrete filled function $F(x, x^*, q, r)$, we choose every point in the discrete neighborhood of $x^*$ as initial point and adopt Algorithm 2.2 to minimize the discrete filled function. When we adopt the discrete steepest descent method to minimize the discrete filled function, the iterative points always quickly run to the boundary of $X$ due to properties of the discrete filled function. It is disadvantageous for us to find the discrete local minimizer of the discrete filled function. But Algorithm 2.2 is good choice.

Of course, it is the best that we can find the discrete minimizer of $F(x, x^*, q, r)$. But, when the point $y_k$ with $f(y_k) < f(x^*)$ is found, we can return Main Program to minimize the cost function $f(x)$ (see Inner Loop 5). Therefore it is not necessary that we find the discrete local minimizer of $F(x, x^*, q, r)$.

5. Numerical experiment

The algorithm in Fortran 90 is successfully used to find the global minimizers of some test problems. Through out the tests, we use the modified discrete descent method as shown in Algorithm 2.2 to perform local searches, in the initialization of the algorithm we let $q = 0.01$ and $r = 1$. In the following part, several test problem are given and results of the algorithm in solving these problem are reported.

The main iterative results are summarized in tables for each function. The symbols used are shown as follows:

- $x^0_k$ or $y^0_k$: The $k$th initial point.
- $k$: The iteration number in finding the $k$th local minimizer.
- $x^*_k$ or $y^*_k$: The $k$th local minimizer.
- $f(x^*_k)$ or $f(y^*_k)$: The function value of the $k$th local minimizer.
- time: The CPU time in seconds for the algorithm to stop.

**Problem 1.**

$$\min \quad f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10.1[(x_2 - 1)^2 + (x_4 - 1)^2] + 19.8(x_2 - 1)(x_4 - 1),$$

s.t. \quad -10 \leq x_i \leq 10, \quad x_i \text{ is integer}, \quad i = 1, 2, 3, 4.

This problem is a discrete counterpart of the problem 38 in [8]. It is a box constrained/unconstrained nonlinear integer programming problem. It has $21^4 \approx 1.94 \times 10^7$ feasible points where 41 of them are discrete local minimizers but only one of those discrete local minimizers is the discrete global minimum solution: $x^*_\text{global} = (1, 1, 1, 1)$ with $f(x^*_\text{global}) = 0$. We used five initial points in our experiment: $(9, 6, 5, 6), (10, 10, 10, 10), (-10, -10, -10, -10), (-10, 10, -10, 10), (10, -10, -10, 10)$. For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. Let $x^0_1 = (9, 6, 5, 6)$, a summary of the computational results are displayed in Tables 1 and 3.

**Problem 2.**

$$\min \quad f(x) = g(x)h(x),$$

s.t. \quad x_i = 0.001y_i, \quad -2000 < y_i < 2000, \quad y_i \text{ is integer}, \quad i = 1, 2,
We used four initial points in our experiment:

<table>
<thead>
<tr>
<th>k</th>
<th>$x_k^0$</th>
<th>$f(x_k^0)$</th>
<th>$x_k^*$</th>
<th>$f(x_k^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(9, 6, 5, 6)</td>
<td>596070.0</td>
<td>(2, 4, 2, 3)</td>
<td>342.1000</td>
</tr>
<tr>
<td>2</td>
<td>(1, 1, 2, 3)</td>
<td>131.4000</td>
<td>(1, 1, 2, 4)</td>
<td>91.90000</td>
</tr>
<tr>
<td>3</td>
<td>(1, 1, 0, 1)</td>
<td>91.00000</td>
<td>(1, 1, 1, 1)</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Time = 0.1301872 s.

Problem 3 (Beale’s function).

$$g(x) = 1 + (x_1 + x_2 + 1)^2(19 − 14x_1 + 3x_1^2 − 14x_2 + 6x_1x_2 + 3x_2),$$
$$h(x) = 30 + (2x_1 − 3x_2)^2(18 − 32x_1 + 12x_1^2 + 48x_2 − 36x_1x_2 + 27x_2^2).$$

This problem is a discrete counterpart of the Goldstein and Price’s function in [10]. It is a box constrained/unconstrained nonlinear integer programming problem. It has 40012 ≈ 1.60 × 10^7 feasible points. More precisely, it has 207 and 2 discrete local minimizers in the interior and the boundary of box $−2.00 ≤ x_i ≤ 2.00$, $i = 1, 2$, respectively. Nevertheless, it has only one discrete global minimum solution: $x_{global}^* = (0.000, −1.000)$ with $f(x_{global}^*) = 3$. We used five initial points in our experiment: (2000, 2000), (−2000, −2000), (1196, 1156), (−2000, 2000), (2000, −2000).

For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum. The summary of the computational results are displayed in Tables 2 and 3.

Problem 4 (Powell’s singular function).

$$g(x) = 1 + (x_1 + 10x_2)^2 + 5(x_3 − x_4)^2 + (x_2 − 2x_3)^4 + 10(x_1 − x_4)^4,$$
$$h(x) = 0.001y_i, \quad 10^{-4} ≤ y_i ≤ 10^4, \quad y_i \text{ is integer}, \quad i = 1, 2, 3, 4.$$

It is a box constrained/unconstrained nonlinear integer programming problem. It has 20 001^4 ≈ 1.60 × 10^{17} feasible points and many local minimizers, but it has only one global minimum solution: $x_{global}^* = (0, 0, 0, 0)$ with $f(x_{global}^*) = 0$.

We used four initial points in our experiment: (1000, −1000, −1000, 1000), (1000, −1000, −1000, 1000), (−1000, 1000, 1000), (−1000, −1000, −1000), (1000, . . . , 1000). For every experiment, the proposed algorithm succeeded in identifying the discrete global minimum.

Let $y_1^0 = (1000, −1000, −1000, 1000)$, the summary of the computational results are displayed in Table 3.
A new definition of the discrete filled functions is given in this paper. It is not the extended definition of the classical filled functions for discrete global optimization. Based on a function satisfying the definition, an algorithm for discrete global optimization has been developed. The computational results show that this algorithm is quite efficient and reliable. So it may become a new and practical discrete filled function algorithm for discrete global optimization.

References

Further reading