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On the transitivity of the comonotonic and countermonotonic comparison of random variables

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Abstract

A recently proposed method for the pairwise comparison of arbitrary independent random variables results in a probabilistic relation. When restricted to discrete random variables uniformly distributed on finite multisets of numbers, this probabilistic relation expresses the winning probabilities between pairs of hypothetical dice that carry these numbers and exhibits a particular type of transitivity called dice-transitivity. In case these multisets have equal cardinality, two alternative methods for statistically comparing the ordered lists of the numbers on the faces of the dice have been studied recently: the comonotonic method based upon the comparison of the numbers of the same rank when the lists are in increasing order, and the countermonotonic method, also based upon the comparison of only numbers of the same rank but with the lists in opposite order. In terms of the discrete random variables associated to these lists, these methods each turn out to be related to a particular copula that joins the marginal cumulative distribution functions into a bivariate cumulative distribution function. The transitivity of the generated probabilistic relation has been completely characterized. In this paper, the list comparison methods are generalized for the purpose of comparing arbitrary random variables. The transitivity properties derived in the case of discrete uniform random variables are shown to be

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generic. Additionally, it is shown that for a collection of normal random variables, both comparison methods lead to a probabilistic relation that is at least moderately stochastic transitive.

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1. Introduction

Recently, we have established and analyzed a method for comparing a finite number of independent random variables (r.v.) $X_1, X_2, \ldots, X_m$ in a pairwise manner [5]. In particular, a so-called probabilistic relation Q is generated, which can be interpreted as a graded preference relation expressing intensities of preference [11]. For discrete r.v., these intensities of preference can be regarded as winning probabilities in a dice game, each r.v. being associated to a (possibly unfair) hypothetical dice with an arbitrary number of faces, each containing an arbitrary number of eyes [4].

For the sake of comparing discrete r.v. that are uniformly distributed on finite (multi)sets, we have recently incorporated into the original dice model some alternative statistical comparison strategies [6]. The main idea is to order the numbers on the faces of the dice (or, equivalently, the elements of the multisets) so as to associate to each r.v. a unique ordered list. In [6], we have established two extreme ways of comparing two such lists: either couples of elements of the same rank are compared when the lists are in the same order, which is characteristic for the so-called comonotonic comparison strategy, or couples of elements of the same rank are compared when the lists are in the opposite order, which is characteristic for the so-called countermonotonic comparison strategy.

These two extreme comparison strategies are unambiguously related to the particular way the bivariate c.d.f. of any couple of r.v. used to compute the pairwise winning probabilities and to generate the probabilistic relation, depends upon the marginal discrete uniform distributions, in other words, the comparison strategies are completely characterized by the copula which for the purpose of comparison artificially couples the marginal cumulative distributions into a bivariate c.d.f. More precisely, the comonotonic comparison strategy is related to the minimum operator $T_{\mathbf{M}}$ ($T_{\mathbf{M}}(x, y) = \min(x, y)$, also called the Fréchet–Hoeffding upper bound), whereas the countermonotonic comparison strategy is related to the Lukasiewicz copula $T_{\mathbf{L}}$ ($T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$, also called the Fréchet–Hoeffding lower bound). Note that these two copulas are also the two extreme copulas in between which all other copulas are situated. It should also be remarked that in the literature on copulas, these copulas are usually denoted W and M instead of their t-norm equivalents $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$. Let us recall that a binary operation $T : [0, 1]^2 \to [0, 1]$ is called a t-norm if it is increasing, associative, commutative and possesses 1 as neutral element [13].

The original dice model is also related to a specific copula. Indeed, the statistical comparison of two dice amounts to the elementwise comparison of their associated (ordered) lists so that each element of one list is compared to each element of the other (which, in fact, makes the ordering of the list numbers irrelevant). The copula that characterizes such a comparison strategy is the ordinary product copula $T_{\mathbf{P}}(x, y) = xy$. As a consequence, all bivariate c.d.f. are simply the product of two one-dimensional marginal c.d.f. and therefore the r.v. can be regarded as pairwisely independent. It should, however, be emphasized that the two alternative extreme comparison strategies and particularly the copulas that characterize these strategies, should not be regarded as a means of taking into account a possible pairwise dependence of the given r.v. The dependence structure of a random vector with m components being entirely captured by the m-dimensional joint c.d.f., it is very unlikely that all bivariate c.d.f. derived from it be expressible by means of a same copula. In fact, though there does exist a random vector with all pairs of its components coupled comonotonically, no random vector can be found such that all pairs of components are coupled countermonotonically. The existence and construction of a joint c.d.f., given all the bivariate distributions, in other words finding an m-copula that has prescribed marginal 2-copulas is a famous open problem in the theory of copulas, closely related to the so-called compatibility problem [15]. We circumvent this problem by interpreting the copula as an artificial device for comparison purposes not related to the possible dependence between the r.v.

For uniformly distributed discrete r.v. we have studied the type of transitivity exhibited by the generated probabilistic relation when the r.v. are coupled either by $T_{\mathbf{P}}$, $T_{\mathbf{M}}$ or $T_{\mathbf{L}}$ [4,6]. These types of transitivity, respectively, called dice-transitivity, Łukasiewicz-transitivity and partial stochastic transitivity [9], perfectly fit into the framework of cycle-transitivity introduced by the present authors [3]. Furthermore, we have proven in [5] that dice-transitivity is the genuine type of transitivity corresponding to the coupling by $T_{\mathbf{P}}$, in the sense that for arbitrary discrete or continuous r.v., the generated probabilistic relation is at least dice-transitive.

In the present paper, we demonstrate that, whatever the marginal c.d.f. of the r.v. be, Lukasiewicz-transitivity is the genuine type of transitivity of the probabilistic relation when the coupling is done by $T_{\mathbf{M}}$, whereas partial stochastic transitivity is the genuine type of transitivity when the coupling is done by $T_{\mathbf{L}}$. The outline of the paper is as follows. First, the general recipe for generating a probabilistic relation from a given collection of r.v. is briefly discussed. Then, two sets of formulae to compute the probabilistic relations $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$, respectively, obtained with the coupling by $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, are established: one set for the case of arbitrary discrete r.v., the other set for the case of arbitrary continuous r.v. Furthermore, the transitivity properties previously derived for discrete uniformly distributed r.v. are shown to hold for arbitrary r.v. Finally, for both extreme couplings the transitivity of the probabilistic relation generated by a collection of normal r.v. is analyzed.

2. A general method for comparing random variables

An immediate way of comparing two r.v. X_1 and X_2 is to consider the probability that the first one takes a greater value than the second one. Proceeding along this line of thought, a collection $\{X_1, X_2, \dots, X_m\}$ of r.v. generates a probabilistic relation, also called reciprocal relation or ipsodual relation, in the following way.

Definition 1. Given a collection $\{X_1, X_2, \ldots, X_m\}$ of random variables, the binary relation Q defined by

$$Q(X_i, X_j) = \text{Prob}\{X_i > X_j\} + \tfrac{1}{2}\,\text{Prob}\{X_i = X_j\} \tag{1}$$

is a probabilistic relation, i.e. for all i, j it holds that $Q(X_i, X_j) + Q(X_j, X_i) = 1$.

In general, probabilistic relations are not only a convenient tool for expressing the result of the pairwise comparison of a set of alternatives [1], but they also appear in various fields such as game theory [8], voting theory [12,16] and psychological studies on preference and discrimination in (individual or collective) decision making methods [7].

It is clear from the definition that in the case of discrete r.v. the relation Q can be immediately computed as

$$Q(X_i, X_j) = \sum_{k>l} p_{X_i,X_j}(k,l) + \frac{1}{2}\sum_{k} p_{X_i,X_j}(k,k), \tag{2}$$

with p_{X_i,X_j} the joint probability mass function (p.m.f.) of (X_i, X_j) which essentially depends upon the copula used to compare the discrete r.v.

In the case of a collection of continuous r.v., Q is computed as

$$Q(X_i, X_j) = \int_{x>y} dF_{X_i,X_j}(x,y) + \frac{1}{2}\int_{x=y} dF_{X_i,X_j}(x,y), \tag{3}$$

with the bivariate c.d.f. $F_{X_i,X_j}(x,y)$ as given in (4).

The cornerstone for computing this probabilistic relation Q is the knowledge of the bivariate cumulative distribution function (c.d.f.) F_{X_i,X_j} of all pairs X_i, X_j of r.v. Sklar's theorem [15,17] tells us that if a joint c.d.f. F_{X_i,X_j} has marginals F_{X_i} and F_{X_j}, then there exists a 2-copula (or simply a copula) C_{ij}, such that for all x, y:

$$F_{X_i,X_j}(x,y) = C_{ij}(F_{X_i}(x), F_{X_j}(y)). \tag{4}$$

Let us recall [15] that a copula is a binary operation $C : [0,1]^2 \to [0,1]$, that has neutral element 1 and absorbing element 0 and that satisfies the property of moderate growth: for any $(x_1, x_2, y_1, y_2) \in [0,1]^4$

$$(x_1 \leqslant x_2 \wedge y_1 \leqslant y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geqslant C(x_1, y_2) + C(x_2, y_1).$$

If X_i and X_j are continuous, then C in (4) is unique; otherwise, C is uniquely determined on $\mathrm{R}an(F_{X_i}) \times \mathrm{R}an(F_{X_j})$. Conversely, if C is a copula and F_{X_i} and F_{X_j} are c.d.f. then the function defined by (4) is a joint c.d.f. with marginals F_{X_i} and F_{X_j}.

If the copula C_{ij} is absolutely continuous, then the second integral in (3) vanishes and in the first integral $dF_{X_i,X_j}(x,y)$ can be written as $f_{X_i,X_j}(x,y)\,dx\,dy$ with f_{X_i,X_j} the joint probability density function (p.d.f.) of (X_i, X_j). This is, for instance, the case for the product $T_{\mathbf{P}}$ where, moreover, $f_{X_i,X_j}(x,y) = f_{X_i}(x) f_{X_j}(y)$. On the other hand, $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are examples of singular copulas for which the second part in (3) does not vanish.

3. Comparison of discrete random variables

We now turn to the case of a collection $\{X_1, X_2, \ldots, X_m\}$ of discrete r.v., each X_i distributed on a finite integer multiset A_i with elements $x_1^i \leqslant x_2^i \leqslant \cdots \leqslant x_{n_i}^i$ and associated marginal probability masses $p_1^i, p_2^i, \ldots, p_{n_i}^i$.

In the case of coupling by $T_{\mathbf{P}}$, the following formula for the associated probabilistic relation $Q^{\mathbf{P}}$ was derived in [5]:

$$Q^{\mathbf{P}}(X_i, X_j) = \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} p_k^i p_l^j \delta_{kl}^{\mathbf{P}}, \tag{5}$$

with

$$\delta_{kl}^{\mathbf{P}} = \begin{cases} 1 & \text{if } x_k^i > x_l^j, \\ 1/2 & \text{if } x_k^i = x_l^j, \\ 0 & \text{if } x_k^i < x_l^j. \end{cases} \tag{6}$$

In particular, if the r.v. are uniformly distributed on multisets A_i of the same cardinality n, formula (5) simplifies to

$$Q^{\mathbf{P}}(X_i, X_j) = \frac{1}{n^2} \sum_{k,l=1}^{n} \delta_{kl}^{\mathbf{P}},$$

showing that if the elements of the multisets A_i denote the number of eyes on the faces of hypothetical fair dice with n faces, then $Q^{\mathbf{P}}(X_i, X_j)$ corresponds to the winning probability of dice i w.r.t. dice j when both dice are independently thrown (assuming that a tie leads to a replay). This interpretation of the proposed comparison method leads to what we previously have called the dice model.

If the coupling is done by $T_{\mathbf{M}}$ the following result holds:

Proposition 2. *Consider a collection $\{X_1, X_2, \ldots, X_m\}$ of discrete random variables distributed on (not necessarily disjoint) finite integer multisets A_i with elements $x_1^i \leqslant x_2^i \leqslant \cdots \leqslant x_{n_i}^i$ and associated marginal probability masses $p_1^i, p_2^i, \ldots, p_{n_i}^i$ and assume that for comparison purposes these random variables are coupled by $T_{\mathbf{M}}$. For the computation of $Q^{\mathbf{M}}(X_i, X_j)$ the multisets A_i and A_j are first transformed into new multisets $\bar{A}_i$* and *$\bar{A}_j$ of the same cardinality n with elements $\bar{x}_1^i \leqslant \bar{x}_2^i \leqslant \cdots \leqslant \bar{x}_n^i$ and $\bar{x}_1^j \leqslant \bar{x}_2^j \leqslant \cdots \leqslant \bar{x}_n^j$, and such that the associated probability masses are pairwisely equal, i.e. for all $k = 1, 2, \ldots, n$, it holds that $\bar{p}_k^i = \bar{p}_k^j = \bar{p}_k^{ij}$. Then $Q^{\mathbf{M}}(X_i, X_j)$ is given by*

$$Q^{\mathbf{M}}(X_i, X_j) = \sum_{k=1}^{n} \bar{p}_k^{ij} \bar{\delta}_k^{\mathbf{M}} \tag{7}$$

with

$$\bar{\delta}_k^{\mathbf{M}} = \begin{cases} 1 & \textit{if } \bar{x}_k^i > \bar{x}_k^j, \\ 1/2 & \textit{if } \bar{x}_k^i = \bar{x}_k^j, \\ 0 & \textit{if } \bar{x}_k^i < \bar{x}_k^j. \end{cases} \tag{8}$$

Proof. The transformation of the multisets A_i and A_j into multisets $\bar{A}_i$ and $\bar{A}_j$ of the same cardinality n, such that for all k the probability masses of $\bar{x}_k^i$ and $\bar{x}_k^j$ are both equal to $\bar{p}_k$, should leave the marginal c.d.f. F_{X_i} and F_{X_j} and therefore also the bivariate c.d.f. F_{X_i,X_j} invariant. Such a transformation can be easily established by duplicating some elements of A_i and A_j and by carefully partitioning the associated probability masses over duplicated elements. This is illustrated after this proof.

Assuming that this first transformation step has been carried out, there remains to prove that (7) holds for the coupling by $T_{\mathbf{M}}$. In general, the bivariate probability masses are computed as

$$\begin{aligned}\bar{p}_{X_i,X_j}(\bar{x}_k^i, \bar{x}_l^j) = {} & F_{X_i,X_j}(\bar{x}_k^i, \bar{x}_l^j) + F_{X_i,X_j}(\bar{x}_{k-1}^i, \bar{x}_{l-1}^j) \\ & -F_{X_i,X_j}(\bar{x}_{k-1}^i, \bar{x}_l^j) - F_{X_i,X_j}(\bar{x}_k^i, \bar{x}_{l-1}^j),\end{aligned} \tag{9}$$

where k and l run from 1 to n. Also, by convention, it holds for any i that $F_{X_i}(\bar{x}_k^i)$ is zero for all $k \leqslant 0$ and one for all $k \geqslant n$. Since

$$F_{X_i,X_j}(\bar{x}_k^i, \bar{x}_l^j) = \min(F_{X_i}(\bar{x}_k^i), F_{X_j}(\bar{x}_l^j)) = F_{X_i}(\bar{x}_{\min(k,l)}^i) = F_{X_j}(\bar{x}_{\min(k,l)}^j),$$

substitution in (9) leads to

$$\bar{p}_{X_i,X_j}(\bar{x}_k^i, \bar{x}_l^j) = \begin{cases} 0 & \text{if } k \neq l, \\ F_{X_i}(\bar{x}_k^i) - F_{X_i}(\bar{x}_{k-1}^i) = \bar{p}_k & \text{if } k = l. \end{cases}$$

Taking into account (2), formulae (7) and (8) follow. □

Let us illustrate the above procedure on an example.

Example 3. Suppose X_i is a discrete r.v. on the set $\{1, 3, 4\}$, i.e. $x_1^i = 1, x_2^i = 3$ and $x_3^i = 4$, with probabilities $p_1^i = 0.15$, $p_2^i = 0.40$ and $p_3^i = 0.45$, and X_j a discrete r.v. on the set $\{2, 3, 5\}$, i.e. $x_1^j = 2$, $x_2^j = 3$ and $x_3^j = 5$, with probabilities $p_1^j = 0.35$, $p_2^j = 0.35$ and $p_3^j = 0.30$. The step of duplicating elements and partitioning probabilities goes as follows. Since $p_1^j > p_1^i$ and $p_1^i + p_2^i > p_1^j$, the element x_1^j is duplicated and to the two new elements $\bar{x}_1^j$ and $\bar{x}_2^j$, both equal to 2, are assigned the probabilities $\bar{p}_1^j = 0.15$ and $\bar{p}_2^j = 0.20$, respectively. We also set $\bar{x}_1^i = 1$ and $\bar{p}_1^i = 0.15$. Proceeding in the same way until all probabilities are pairwisely the same, we obtain that the first set $\{1, 3, 4\}$ is finally transformed into the multiset $\{1, 3, 3, 4, 4\}$ and the second set $\{2, 3, 5\}$ into the multiset $\{2, 2, 3, 3, 5\}$, while for both multisets the associated probabilities are 0.15, 0.20, 0.20, 0.15, 0.30. Note that $n = 5$. Application of (7) now immediately results in $Q^{\mathbf{M}}(X_i, X_j) = 0.20+0.20/2+0.15 = 0.45$.

We next turn our attention to the pairwise coupling of discrete r.v. by $T_{\mathbf{L}}$.

Proposition 4. *Consider a collection* $\{X_1, X_2, \ldots, X_m\}$ *of discrete random variables distributed on (not necessarily disjoint) finite integer multisets* A_i *with elements* $x_1^i \leqslant x_2^i \leqslant \cdots \leqslant x_{n_i}^i$ *and associated marginal probability masses* $p_1^i, p_2^i, \ldots, p_{n_i}^i$ *and assume that for comparison purposes these random variables are coupled by* $T_{\mathbf{L}}$. *For the computation of* $Q^{\mathbf{L}}(X_i, X_j)$ *the multisets* A_i *and* A_j *are first transformed into new multisets* $\bar{A}_i$ *and* $\bar{A}_j$ *of the same cardinality n with elements* $\bar{x}_1^i \leqslant \bar{x}_2^i \leqslant \cdots \leqslant \bar{x}_n^i$ *and* $\bar{x}_1^j \leqslant \bar{x}_2^j \leqslant \cdots \leqslant \bar{x}_n^j$, *respectively, and such that for the associated probability masses* $\bar{p}_k^i$ *and* $\bar{p}_k^j$ *it holds that* $\bar{p}_k^i = \bar{p}_{n-k+1}^j = \bar{p}_k^{ij}$ *for all* $k = 1, 2, \ldots, n$. *Then* $Q^{\mathbf{L}}(X_i, X_j)$ *is given by*

$$Q^{\mathbf{L}}(X_i, X_j) = \sum_{k=1}^{n} \bar{p}_k^{ij} \bar{\delta}_k^{\mathbf{L}} \tag{10}$$

with

$$\bar{\delta}_k^{\mathbf{L}} = \begin{cases} 1 & \text{if } \bar{x}_k^i > \bar{x}_{n-k+1}^j, \\ 1/2 & \text{if } \bar{x}_k^i = \bar{x}_{n-k+1}^j, \\ 0 & \text{if } \bar{x}_k^i < \bar{x}_{n-k+1}^j. \end{cases} \tag{11}$$

Proof. The transformation of the given multisets A_i and A_j into equivalent multisets $\bar{A}_i$ and $\bar{A}_j$ is exactly the same as in the case of the coupling by $T_{\mathbf{M}}$, provided the elements of one of the two given sets, say A_j, are listed in reversed order before the transformation is carried out, and the increasing ordering is restored after the transformation is carried out. Next, the proof proceeds in the same way as for the coupling by $T_{\mathbf{M}}$ and one obtains

$$\bar{p}_{X_i,X_j}(\bar{x}_k^i, \bar{x}_l^j) = \begin{cases} 0 & \text{if } k \neq n+1-l, \\ F_{X_i}(\bar{x}_k^i) - F_{X_i}(\bar{x}_{k-1}^i) = \bar{p}_k & \text{if } k = n+1-l, \end{cases}$$

from which, again taking into account (2), formulae (10) and (11) immediately follow. □

Example 5. Let us illustrate the computation of $Q^{\mathbf{L}}(X_i, X_j)$ on the same example as before. Now, the first set $A_i = \{1, 3, 4\}$ is transformed into the multiset $\bar{A}_i = \{1, 3, 3, 4, 4\}$ with associated probabilities 0.15, 0.15, 0.25, 0.10, 0.35 while the second set $A_j = \{2, 3, 5\}$ is transformed into the equivalent multiset $\bar{A}_j = \{2, 3, 3, 5, 5\}$ with associated probabilities 0.35, 0.10, 0.25, 0.15, 0.15. Again $n = 5$ and application of (10) results in $Q^{\mathbf{L}}(X, Y) = 0.25/2 + 0.10 + 0.35 = 0.575$.

If the discrete r.v. X_i and X_j are both uniformly distributed on finite multisets of the same cardinality n, then the first transformation step is superfluous and according to (7) and (10), for both extreme couplings the comparison of these r.v. amounts to the comparison of the lists of increasingly ordered numbers derived from the respective multisets. If the coupling is comonotonic, then the two lists are compared by comparing the elements of same rank; if the coupling is countermonotonic, then the lists are compared by comparing the elements of complementary rank, in other words, the elements to be compared have ranks whose sum is $n + 1$. This latter situation is clearly equivalent with the comparison of elements of the same rank if one list is increasingly and the other one decreasingly ordered.

4. Comparison of continuous random variables

The comparison of pairwise independent continuous r.v., i.e. of continuous r.v. that are coupled by $T_{\mathbf{P}}$ (the bivariate joint p.d.f. are factorizable as $f_{X_i,X_j} = f_{X_i} f_{X_j}$), yields a probabilistic relation denoted $Q^{\mathbf{P}}$. According to (3), $Q^{\mathbf{P}}$ can be computed as

$$Q^{\mathbf{P}}(X_i, X_j) = \int_{-\infty}^{+\infty} f_{X_i}(x) F_{X_j}(x)\, dx = E_{X_i}[F_{X_j}], \tag{12}$$

the last expression denoting the expected value w.r.t. X_i of the c.d.f. F_{X_j}.

Let us next compare continuous r.v. $X_1, X_2, \ldots, X_m$, each couple (X_i, X_j) being (artificially) coupled by $T_{\mathbf{M}}$, i.e. for all (i, j) we define a bivariate c.d.f. by:

$$F_{X_i,X_j}(x, y) = \min(F_{X_i}(x), F_{X_j}(y)). \tag{13}$$

Proposition 6. *Consider a collection $\{X_1, X_2, \ldots, X_m\}$ of continuous random variables with probability density function f_{X_i} and assume that for comparison purposes these random variables are pairwisely coupled by $T_{\mathbf{M}}$. Then the probabilistic relation $Q^{\mathbf{M}}$, defined by $Q^{\mathbf{M}}(X_i, X_j) = \text{Prob}\{X_i > X_j\} + 1/2\text{Prob}\{X_i = X_j\}$, can be computed as*

$$Q^{\mathbf{M}}(X_i, X_j) = \int_{x:F_{X_i}(x)<F_{X_j}(x)} f_{X_i}(x)\, dx + \frac{1}{2} \int_{x:F_{X_i}(x)=F_{X_j}(x)} f_{X_i}(x)\, dx. \tag{14}$$

Proof. Expression (13) for the bivariate c.d.f. of any couple (X_i, X_j) can be written as

$$F_{X_i,X_j}(x, y) = \begin{cases} F_{X_j}(y) & \text{if } y \leqslant F_{X_j}^{-1}(F_{X_i}(x)), \\ F_{X_i}(x) & \text{if } y \geqslant F_{X_j}^{-1}(F_{X_i}(x)), \end{cases}$$

where $F_{X_j}^{-1}$ denotes the pseudo-inverse of F_{X_j}. It follows that

$$\frac{\partial F_{X_i,X_j}(x, y)}{\partial x} = \begin{cases} 0 & \text{if } y < F_{X_j}^{-1}(F_{X_i}(x)), \\ f_{X_i}(x) & \text{if } y \geqslant F_{X_j}^{-1}(F_{X_i}(x)) \end{cases}$$

and

$$\frac{\partial^2 F_{X_i,X_j}(x, y)}{\partial x \partial y} = f_{X_i}(x)\delta(y - F_{X_j}^{-1}(F_{X_i}(x))), \tag{15}$$

where $\delta(\cdot)$ denotes the Dirac-delta functional. By substituting (15) into the first (double) integral on the r.h.s. of (3), the domain of integration in $\mathbb{R}^2$ is defined by $y = F_{X_j}^{-1}(F_{X_i}(x))$ and $x > y$, reducing the double integral to a single integral on the domain in $\mathbb{R}$ defined by the inequality $F_{X_i}(x) < F_{X_j}(x)$. Similarly, substituting (15) into the second (double) integral on the r.h.s. of (3), the domain of integration is now defined by the two equalities $y = F_{X_j}^{-1}(F_{X_i}(x))$ and $x = y$, reducing the double integral to a single integral on the domain in $\mathbb{R}$ defined by the equality $F_{X_i}(x) = F_{X_j}(x)$. This immediately leads to the result (14). $\square$

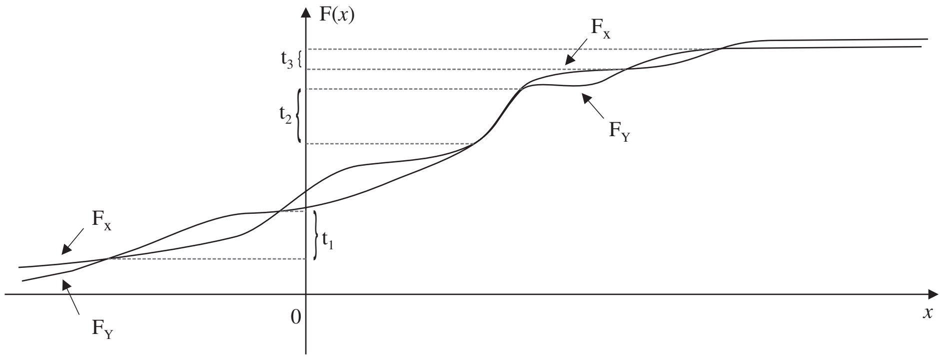

Fig. 1. Comparison of two continuous random variables coupled by $T_{\mathbf{M}}$.

A graphical interpretation of (14) is shown in Fig. 1. The two curves correspond to the marginal c.d.f. F_X and F_Y of two r.v. X and Y. According to (14), we have to distinguish three domains: the domain where F_X lies beneath F_Y, the domain where F_X lies above F_Y, and the domain where F_X and F_Y coincide. The value of $Q^{\mathbf{M}}(X, Y)$ is computed as the sum of the increment of F_X over the first domain and half of the increment of F_X (or F_Y) over the third domain. With the notations shown in Fig. 1, we obtain for the example: $Q^{\mathbf{M}}(X, Y) = t_1 + t_3 + \frac{1}{2}t_2$.

Let us next couple the continuous r.v. $X_1, X_2, \ldots, X_m$ by $T_{\mathbf{L}}$, i.e. for all (i, j) we define a bivariate c.d.f. by

$$F_{X_i,X_j}(x, y) = \max(F_{X_i}(x) + F_{X_j}(y) - 1, 0). \tag{16}$$

Proposition 7. *Consider a collection $\{X_1, X_2, \ldots, X_m\}$ of continuous random variables with probability density function f_{X_i} and assume that for comparison purposes these random variables are pairwisely coupled by $T_{\mathbf{L}}$. Then the probabilistic relation $Q^{\mathbf{L}}$, defined by $Q^{\mathbf{L}}(X_i, X_j) = \text{Prob}\{X_i > X_j\}$, can be computed as*:

$$Q^{\mathbf{L}}(X_i, X_j) = \int_{x:F_{X_i}(x)+F_{X_j}(x)\geqslant 1} f_{X_i}(x)\,dx, \tag{17}$$

or, *equivalently*

$$Q^{\mathbf{L}}(X_i, X_j) = F_{X_j}(u) \text{ with } u \text{ such that } F_{X_i}(u) + F_{X_j}(u) = 1. \tag{18}$$

Proof. Expression (16) of the assumed bivariate c.d.f. of any couple (X_i, X_j) can be written as

$$F_{X_i,X_j}(x, y) = \begin{cases} 0 & \text{if } y \leqslant F_{X_j}^{-1}(1 - F_{X_i}(x)), \\ F_{X_i}(x) + F_{X_j}(y) - 1 & \text{if } y \geqslant F_{X_j}^{-1}(1 - F_{X_i}(x)), \end{cases}$$

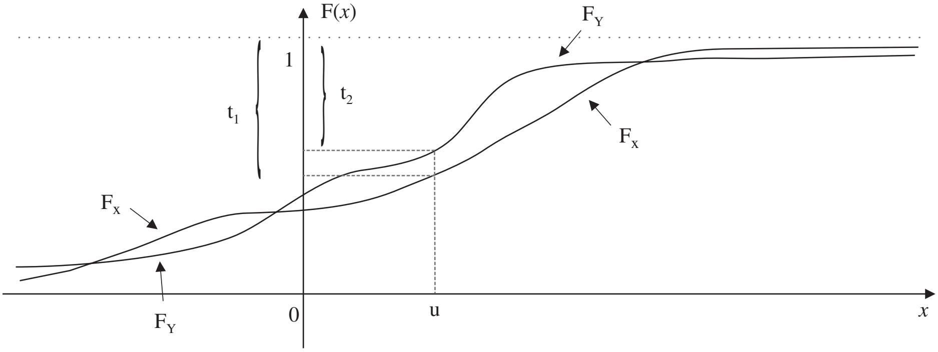

Fig. 2. Comparison of two continuous random variables coupled by $T_{\mathbf{L}}$.

from which it follows that

$$\frac{\partial F_{X_i,X_j}(x,y)}{\partial x} = \begin{cases} 0 & \text{if } y < F_{X_j}^{-1}(F_{X_i}(x)), \\ f_{X_i}(x) & \text{if } y \geqslant F_{X_j}^{-1}(1 - F_{X_i(}x)) \end{cases}$$

and

$$\frac{\partial^2 F_{X_i,X_j}(x,y)}{\partial x \partial y} = \delta(y - F_{X_j}^{-1}(1 - F_{X_i}(x))). \tag{19}$$

Substitution of (19) into (3) now leads to

$$Q^{\mathbf{L}}(X_i, X_j) = \int_{F_{X_i}(x)+F_{X_j}(x)>1} f_{X_i}(x)\,dx + \frac{1}{2}\int_{F_{X_i}(x)+F_{X_j}(x)=1} f_{X_i}(x)\,dx.$$

The last integral vanishes since F_{X_i} is necessarily constant on its integration domain, whence $f_{X_i} = 0$ on that domain. This proves (17), a formula in which it is now optional to add the equality sign in the definition of the integration domain. □

Note that u in (18) might not be unique, in which case any u fulfilling the right equality may be considered. This is illustrated in Fig. 2 for two r.v. X and Y. $Q^{\mathbf{L}}(X, Y)$ is simply the value of F_Y in u, since $Q^{\mathbf{L}}(X, Y) = F_Y(u) = t_1$ and $t_1 + t_2 = 1$.

5. Transitivity of the probabilistic relations $Q^{\mathbf{M}}$ and $Q^{\mathbf{L}}$

Let us briefly recall the concept of cycle-transitivity. In the framework of cycle-transitivity [3], for a probabilistic relation $Q = [q_{ij}]$, the quantities

$$\alpha_{ijk} = \min(q_{ij}, q_{jk}, q_{ki}),\ \beta_{ijk} = \mathrm{m}ed(q_{ij}, q_{jk}, q_{ki}),\ \gamma_{ijk} = \max(q_{ij}, q_{jk}, q_{ki}),$$

are defined for all (i, j, k). Obviously, $\alpha_{ijk} \leqslant \beta_{ijk} \leqslant \gamma_{ijk}$. Also, the notation $\Delta = \{(x, y, z) \in [0, 1]^3 \mid x \leqslant y \leqslant z\}$ is used.

Definition 8. A function $U : \Delta \to \mathbb{R}$ is called an upper bound function if it satisfies:
(i) $U(0, 0, 1) \geqslant 0$ and $U(0, 1, 1) \geqslant 1$;
(ii) for any $(\alpha, \beta, \gamma) \in \Delta$:

$$U(\alpha, \beta, \gamma) + U(1-\gamma, 1-\beta, 1-\alpha) \geqslant 1.$$

The original definition of cycle-transitivity given in [3] turns out to be equivalent to

Proposition 9. *A probabilistic relation $Q = [q_{ij}]$ is cycle-transitive w.r.t. an upper bound function U, if for all (i, j, k) it holds that*

$$\alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1 \leqslant U(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}). \tag{20}$$

Note that a value of $U(\alpha, \beta, \gamma)$ equal to 2 is used to express that for the given values there is no restriction at all (indeed, $\alpha + \beta + \gamma - 1$ is always bounded by 2).

Cycle-transitivity includes as special cases T-transitivity and all known types of g-stochastic transitivity. A $[0, 1]$-valued relation R on a set of alternatives A is called T-transitive [10] if for any $(a, b, c) \in A^3$ it holds that $T(R(a, b), R(b, c)) \leqslant R(a, c)$. The following proposition shows how T-transitivity fits into the framework of cycle-transitivity in case the t-norm T is 1-Lipschitz continuous (for short, 1-Lipschitz), which means that for all $(x, y, z) \in [0, 1]^3$ it holds that $|T(x, y) - T(x, z)| \leqslant |y - z|$ [3].

Proposition 10. *Let T be a 1-Lipschitz t-norm. A probabilistic relation is T-transitive if and only if it is cycle-transitive w.r.t. the upper bound function U_T defined by*

$$U_T(\alpha, \beta, \gamma) = \alpha + \beta - T(\alpha, \beta). \tag{21}$$

Note that 1-Lipschitz t-norms can also be regarded as associative and commutative copulas. The special t-norms $T_{\mathbf{P}}$, $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are examples of 1-Lipschitz t-norms. By means of (21) we immediately find that $T_{\mathbf{M}}$-transitivity, $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity are equivalent with cycle-transitivity w.r.t. the upper bound functions $U_{\mathbf{M}}(\alpha, \beta, \gamma) = \beta$, $U_{\mathbf{P}}(\alpha, \beta, \gamma) = \alpha + \beta - \alpha\beta$ and $U_{\mathbf{L}}(\alpha, \beta, \gamma) = \min(\alpha + \beta, 1)$, respectively. For the case of $T_{\mathbf{L}}$-transitivity, an equivalent upper bound function is given by $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$.

In the literature one finds various types of stochastic transitivity [1,14]. They can, however, be regarded as special cases of a generic type of stochastic transitivity, which we have called g-stochastic transitivity. Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping. A probabilistic relation Q on A is called g-stochastic transitive if for any $(a, b, c) \in A^3$ it holds that

$$(Q(a, b) \geqslant 1/2 \wedge Q(b, c) \geqslant 1/2) \Rightarrow Q(a, c) \geqslant g(Q(a, b), Q(b, c)).$$

In [3], we have proven the following proposition.

Proposition 11. *Let g be a commutative, increasing $[1/2, 1]^2 \to [1/2, 1]$ mapping such that $g(1/2, x) \leqslant x$ for any $x \in [1/2, 1]$. A probabilistic relation Q is g-stochastic transitive* *if and only if it is cycle-transitive w.r.t. the upper bound function U_g defined by*

$$U_g(\alpha, \beta, \gamma) = \begin{cases} \beta + \gamma - g(\beta, \gamma) & \text{if } \beta \geqslant 1/2 \wedge \alpha < 1/2, \\ 1/2 & \text{if } \alpha \geqslant 1/2, \\ 2 & \text{if } \beta < 1/2. \end{cases} \tag{22}$$

We obtain as special cases (only mentioning the function g):

(i) strong stochastic transitivity: $g_{\text{ss}}(\beta, \gamma) = \max(\beta, \gamma) = \gamma$;
(ii) moderate stochastic transitivity: $g_{\text{ms}}(\beta, \gamma) = \min(\beta, \gamma) = \beta$;
(iii) weak stochastic transitivity: $g_{\text{ws}}(\beta, \gamma) = 1/2$.

The transitivity exhibited by a probabilistic relation $Q^{\mathbf{P}}$ generated by a collection of arbitrary independent discrete or continuous r.v., is called dice-transitivity [4,5]. It has the particularity that it can neither be classified as a type of T-transitivity, nor as a type of g-stochastic transitivity, but nicely fits into the framework of cycle-transitivity [2]. More precisely, a dice-transitive probabilistic relation is cycle-transitive w.r.t. the upper bound function U_{D} defined by

$$U_{\text{D}}(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma. \tag{23}$$

Dice-transitivity can be situated between $T_{\mathbf{P}}$-transitivity and $T_{\mathbf{L}}$-transitivity, and also between moderate stochastic transitivity and $T_{\mathbf{L}}$-transitivity.

In [6], it has been proven that for discrete r.v. uniformly distributed on finite multisets of the same cardinality, the probabilistic relation $Q^{\mathbf{M}}$ is $T_{\mathbf{L}}$-transitive, that is, cycle-transitive w.r.t. the upper bound function $U'_{\mathbf{L}}(\alpha, \beta, \gamma) = 1$. Also, any three-dimensional $T_{\mathbf{L}}$-transitive probabilistic relation with rational elements can be generated by the application of the comonotonic comparison strategy to a collection of three ordered lists of the same length. This also proves that there does not exist a stronger type of transitivity than $T_{\mathbf{L}}$-transitivity that holds for all probabilistic relations $Q^{\mathbf{M}}$. The following proposition complements these results.

Proposition 12. *The probabilistic relation $Q^{\mathbf{M}}$ generated by a collection of random variables that are pairwisely coupled by the copula $T_{\mathbf{M}}$ is $T_{\mathbf{L}}$-transitive.*

Proof. One possible proof is completely analogous to the one given in [5] which allowed to conclude that the transitivity exhibited by the probabilistic relation $Q^{\mathbf{P}}$ generated by discrete uniformly distributed r.v., remains unchanged for collections of arbitrary r.v. The main steps are the following. Since the set of rationals is dense in the set of reals, any discrete distribution can be approximated with arbitrary precision by a discrete distribution with rational probability masses. The latter can be regarded as a uniform distribution on a multiset. Also, any continuous distribution can be approximated with arbitrary precision by a discrete distribution. If the precision of the approximations is sufficiently high, the transitivity of the generated probabilistic relation remains unaltered. □

We now turn to the case of a collection of random variables pairwisely coupled by the copula $T_{\mathbf{L}}$. In [6], it has been proven that for discrete r.v. uniformly distributed on finite multisets of the same cardinality, the probabilistic relation $Q^{\mathbf{L}}$ is partially stochastic transitive, that is, cycle-transitive w.r.t. the upper bound function $U_{\mathrm{B}}(\alpha, \beta, \gamma) = \gamma$. Also, any three-dimensional partially stochastic transitive probabilistic relation with rational elements can be generated by the application of the countermonotonic comparison strategy to a collection of three ordered lists of the same length, from which it follows that there does not exist a stronger type of transitivity than partial stochastic transitivity that holds for all probabilistic relations $Q^{\mathbf{L}}$. Note that partial stochastic transitivity should be situated between $T_{\mathbf{M}}$-transitivity and dice-transitivity and also between $T_{\mathbf{M}}$-transitivity and moderate stochastic transitivity. In fact, it can be regarded as a variant of moderate stochastic transitivity, since a probabilistic relation Q on A is called partially stochastic transitive if for any $(a, b, c) \in A^3$ it holds that

$$\left(Q(a,b) > \frac{1}{2} \ \wedge \ Q(b,c) > \frac{1}{2}\right) \Rightarrow Q(a,c) \geqslant \min(Q(a,b), Q(b,c)).$$

Proposition 13. *The probabilistic relation $Q^{\mathbf{L}}$ generated by a collection of random variables that are pairwisely coupled by the copula $T_{\mathbf{L}}$ is partially stochastic transitive.*

Proof. The same reasoning as for r.v. that are pairwisely coupled by $T_{\mathbf{P}}$ or $T_{\mathbf{M}}$ leads to the required result. For the case of a collection of continuous r.v., it is nonetheless possible to give an elegant direct proof of the stated transitivity property.

Indeed, consider three r.v. X, Y, Z with c.d.f. F_X, F_Y, F_Z, respectively. Denote by u a value such that $F_X(u) + F_Y(u) = 1$, by v a value such that $F_Y(v) + F_Z(v) = 1$ and by w a value such that $F_Z(w) + F_X(w) = 1$. From the graphical representation of $Q^{\mathbf{L}}$ as illustrated in Fig. 1 we know that $Q^{\mathbf{L}}(X,Y) = F_Y(u)$, $Q^{\mathbf{L}}(Y,Z) = F_Z(v)$, and $Q^{\mathbf{L}}(Z,X) = F_X(w)$. It follows that

$$\begin{aligned} &Q^{\mathbf{L}}(X,Y) + Q^{\mathbf{L}}(Y,Z) + Q^{\mathbf{L}}(Z,X) - 1 \\ &\quad = F_Y(u) + F_Z(v) + F_X(w) - 1 \\ &\quad = F_Y(u) + F_Z(v) - F_Z(w) \\ &\quad = F_Z(v) + F_X(w) - F_X(u) \\ &\quad = F_X(w) + F_Y(u) - F_Y(v). \end{aligned} \tag{24}$$

Whatever the ordering of u, v, w is, at least one of the expressions $F_Z(v) - F_Z(w)$, $F_X(w) - F_X(u)$, and $F_Y(u) - F_Y(v)$ is non-positive. In the last three expressions of (24) the term that comes with the minimum of $F_Z(v) - F_Z(w)$, $F_X(w) - F_X(u)$, and $F_Y(u) - F_Y(v)$ is the maximum of $F_Y(u)$, $F_Z(v)$, and $F_X(w)$. Since the minimum is non-positive, it follows that

$$Q^{\mathbf{L}}(X,Y) + Q^{\mathbf{L}}(Y,Z) + Q^{\mathbf{L}}(Z,X) - 1 \leqslant \max(F_Y(u), F_Z(v), F_X(w)),$$

which is equivalent to saying that $Q^{\mathbf{L}}$ is cycle-transitive w.r.t. U_{B}. □

6. Normal random variables

Let us consider a collection of normally distributed r.v. $X_i \stackrel{\mathrm{d}}{=} N(\mu_i, \sigma_i^2)$. Previously, we have shown that if the r.v. are pairwisely independent, in other words, the comparison of pairs of r.v. is based upon their coupling by $T_{\mathbf{P}}$, the probabilistic relation $Q^{\mathbf{P}}$ generated by the collection of r.v. can be computed as

$$Q^{\mathbf{P}}(X_i, X_j) = \Phi\left(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right), \tag{25}$$

where Φ denotes the c.d.f. of the standard normal distribution $N(0, 1)$. Moreover, we have proven that the probabilistic relation $Q^{\mathbf{P}}$ is moderately stochastic transitive [5]. In this section, we want to investigate the transitivity of the probabilistic relations generated by the same collection of normal r.v. but pairwisely coupled by either $T_{\mathbf{M}}$ or $T_{\mathbf{L}}$.

Proposition 14. *Let $X_i \stackrel{\mathrm{d}}{=} N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, m$, be normally distributed. If the random variables X_i, X_j, with $F_{X_i} \neq F_{X_j}$, are coupled by $T_{\mathbf{M}}$, the probabilistic relation $Q^{\mathbf{M}}$ can be computed as*

$$Q^{\mathbf{M}}(X_i, X_j) = \Phi\left(\frac{\mu_i - \mu_j}{|\sigma_i - \sigma_j|}\right), \tag{26}$$

whereas, if they are coupled by $T_{\mathbf{L}}$, the probabilistic relation $Q^{\mathbf{L}}$ can be computed as:

$$Q^{\mathbf{L}}(X_i, X_j) = \Phi\left(\frac{\mu_i - \mu_j}{\sigma_i + \sigma_j}\right). \tag{27}$$

If $F_{X_i} = F_{X_j}$ then it holds that $Q^{\mathbf{M}}(X_i, X_j) = Q^{\mathbf{L}}(X_i, X_j) = 1/2$.

Proof. Let us first consider the coupling by $T_{\mathbf{M}}$. We can make use of the graphical interpretation of $Q^{\mathbf{M}}$ (cf. Fig. 1). If $\sigma_i \neq \sigma_j$, the c.d.f. $F_{X_i}(x) = \Phi((x - \mu_i)/\sigma_i)$ of X_i has exactly one point in common with the c.d.f. $F_{X_j}(x) = \Phi((x - \mu_j)/\sigma_j)$ of X_j, namely the point with abscis $x_0 = (\mu_i\sigma_j - \mu_j\sigma_i)/(\sigma_j - \sigma_i)$, where both c.d.f. attain the value $\Phi((\mu_i - \mu_j)/(\sigma_j - \sigma_i))$. We have to compute the growth of F_{X_i} on the interval where F_{X_i} lies below F_{X_j}. If $\sigma_i > \sigma_j$, this interval extends from x_0 to $+\infty$ and

$$Q^{\mathbf{M}}(X_i, X_j) = 1 - \Phi\left(\frac{\mu_i - \mu_j}{\sigma_j - \sigma_i}\right) = \Phi\left(\frac{\mu_i - \mu_j}{\sigma_i - \sigma_j}\right).$$

If $\sigma_i < \sigma_j$, this interval extends from $-\infty$ to x_0 and

$$Q^{\mathbf{M}}(X_i, X_j) = \Phi\left(\frac{\mu_i - \mu_j}{\sigma_j - \sigma_i}\right).$$

Both cases can be combined into (26). Considering the appropriate limits, the latter formula is also valid when $\sigma_i = \sigma_j$ and $\mu_i \neq \mu_j$, as it yields $Q^{\mathbf{M}}(X_i, X_j) = \Phi(-\infty) = 0$ if $\mu_i < \mu_j$

and $Q^{\mathbf{M}}(X_i, X_j) = \Phi(+\infty) = 1$ if $\mu_i > \mu_j$. This is in agreement with the fact that in the former case F_{X_i} lies entirely above and in the latter case entirely below F_{X_j}. Finally, if $\mu_i = \mu_j$ and $\sigma_i = \sigma_j$ we have that $F_{X_i} = F_{X_j}$ and therefore $Q^{\mathbf{M}}(X_i, X_j) = 1/2$.

If the pairwise coupling is done by $T_{\mathbf{L}}$, we have to find a point u such that $F_{X_i}(u) + F_{X_j}(u) = 1$. The unique solution of the equation $\Phi((u-\mu_i)/\sigma_i) + \Phi((u-\mu_j)/\sigma_j) = 1$ being $u = (\mu_i\sigma_j + \mu_j\sigma_i)/(\sigma_i + \sigma_j)$, $Q^L(X_i, X_j)$, which according to the graphical interpretation of $Q^{\mathbf{L}}$ (see Fig. 2) is the growth of F_{X_j} on the interval $[-\infty, u]$, is readily seen to be given by (27). □

The transitivity of the probabilistic relation Q generated by arbitrary r.v. depends on the copula that pairwisely couples the r.v. In the remaining part of this section, we prove that in the case of normal r.v. pairwisely coupled by either $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$, the generated probabilistic relation is at least moderately stochastic transitive.

Proposition 15. *The probabilistic relation generated by a collection of normally distributed random variables, pairwisely coupled by the same copula belonging to the set* $\{T_{\mathbf{M}}, T_{\mathbf{P}}, T_{\mathbf{L}}\}$, *is moderately stochastic transitive.*

Proof. The proof that $Q^{\mathbf{P}}$ is moderately stochastic transitive when the r.v. are coupled by $T_{\mathbf{P}}$ has been previously given in [5]. Here we focus on the proof for the two other copulas and start with the comonotonic case of $Q^{\mathbf{M}}$. We introduce first some notations.

Let $X_i \stackrel{\mathrm{d}}{=} N(\mu_i, \sigma_i^2)$, $X_j \stackrel{\mathrm{d}}{=} N(\mu_j, \sigma_j^2)$ and $X_k \stackrel{\mathrm{d}}{=} N(\mu_k, \sigma_k^2)$ be three normal r.v. with different variances and pairwisely coupled by $T_{\mathbf{M}}$. Let α, β and γ denote, as usual, the minimum, the median and the maximum of the three values $Q^{\mathbf{M}}(X_i, X_j)$, $Q^{\mathbf{M}}(X_j, X_k)$ and $Q^{\mathbf{M}}(X_k, X_i)$. Without loss of generality, we can assume that the labels i, j, k are attributed such that $\beta \geqslant 1/2$. Let us finally introduce the notations $u_\alpha = \Phi^{-1}(\alpha)$, $u_\beta = \Phi^{-1}(\beta)$ and $u_\gamma = \Phi^{-1}(\gamma)$. Clearly $u_\alpha \leqslant u_\beta \leqslant u_\gamma$ and $u_\beta \geqslant 0$, whereas it follows from (26) that $(u_\alpha, u_\beta, u_\gamma)$ is a permutation of

$$\left(\frac{\mu_i - \mu_j}{|\sigma_i - \sigma_j|}, \frac{\mu_j - \mu_k}{|\sigma_j - \sigma_k|}, \frac{\mu_k - \mu_i}{|\sigma_k - \sigma_i|}\right).$$

Let $(\phi_\alpha, \phi_\beta, \phi_\gamma)$ denote the corresponding permutation of $(|\sigma_i - \sigma_j|, |\sigma_j - \sigma_k|, |\sigma_k - \sigma_i|)$. It follows that the equality

$$\phi_\alpha u_\alpha + \phi_\beta u_\beta + \phi_\gamma u_\gamma = 0, \tag{28}$$

should hold for any $\phi_\alpha, \phi_\beta, \phi_\gamma$, which from their definition are strictly positive and satisfy

$$\max(\phi_\alpha, \phi_\beta, \phi_\gamma) = \mathrm{med}(\phi_\alpha, \phi_\beta, \phi_\gamma) + \min(\phi_\alpha, \phi_\beta, \phi_\gamma).$$

It follows that $u_\alpha \leqslant 0$. Note that if $\alpha = 1/2$, then $u_\alpha = 0$, and on account of (28) it must then hold that $u_\beta = u_\gamma = 0$, or, equivalently, $\beta = \gamma = 1/2$. Let us consider the case of $\alpha < 1/2$, or $u_\alpha < 0$.

(i) If $\max(\phi_\alpha, \phi_\beta, \phi_\gamma) = \phi_\alpha$, then (28) can be reduced to the equality

$$\phi_\beta(u_\alpha + u_\beta) + \phi_\gamma(u_\alpha + u_\gamma) = 0,$$

which is only satisfied if $u_\alpha + u_\beta \leqslant 0$ and $u_\alpha + u_\gamma \geqslant 0$. These two conditions are equivalent to $\alpha \leqslant 1 - \beta$ and $\alpha \geqslant 1 - \gamma$, or $1 - \alpha \geqslant \beta = \min(\beta, \gamma)$ and $\gamma \geqslant 1 - \alpha = \max(1 - \alpha, 1 - \beta)$.

(ii) If $\max(\phi_\alpha, \phi_\beta, \phi_\gamma) = \phi_\beta$, then (28) leads to the condition

$$\phi_\alpha(u_\alpha + u_\beta) + \phi_\gamma(u_\beta + u_\gamma) = 0,$$

which can only be satisfied if $u_\alpha + u_\beta \leqslant 0$ and $u_\beta + u_\gamma \geqslant 0$, or, equivalently, $\alpha \leqslant 1 - \beta$ and $\beta \geqslant 1 - \gamma$. In this case, it must therefore hold that $1 - \alpha \geqslant \beta = \min(\beta, \gamma)$, whereas the second condition is trivially fulfilled.

(iii) Finally, if $\max(\phi_\alpha, \phi_\beta, \phi_\gamma) = \phi_\gamma$, then (28) leads to the condition

$$\phi_\alpha(u_\alpha + u_\gamma) + \phi_\beta(u_\beta + u_\gamma) = 0,$$

which can only be satisfied if $u_\alpha + u_\gamma \leqslant 0$ and $u_\beta + u_\gamma \geqslant 0$, or, equivalently, $\alpha \leqslant 1 - \gamma$ and $\beta \geqslant 1 - \gamma$. Since the second condition is always fulfilled, it must hold that $1 - \alpha \geqslant \gamma = \max(\beta, \gamma)$.

In conclusion, the least restrictive condition encountered when $\alpha < 1/2$ is $1 - \alpha \geqslant \min(\beta, \gamma)$. Together with the fact that $\alpha = 1/2$ implies $\beta = \gamma = 1/2$ proves that $Q^{\mathbf{M}}$ is moderately stochastic transitive when the variances of the normal r.v. are mutually different. It can be verified that if two variances are equal, or if all variances are equal, then the transitivity of $Q^{\mathbf{M}}$ is not weaker than moderate stochastic transitivity.

We next turn to the case of the pairwise coupling by $T_{\mathbf{L}}$. Since for arbitrary r.v. $Q^{\mathbf{L}}$ is partially stochastic transitive, i.e. cycle-transitive w.r.t the upper bound function $U_{\mathrm{B}}(\alpha, \beta, \gamma) = \gamma$, this type of transitivity certainly applies to normal r.v. as well. However, $\alpha = 1/2$ implies $\beta = \gamma = 1/2$, therefore $Q^{\mathbf{L}}$ should be at least moderately stochastic transitive. □

7. Conclusions

In this paper, we have extended our method for the pairwise comparison of a collection of independent r.v. to a collection of r.v. that are artificially and pairwisely coupled by means of a same copula. In particular, the coupling by one of the extreme copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ has been considered in detail.

As for the original method, the present extension with one of the two extreme copulas generates a probabilistic relation of which the transitivity can be cast into the framework of cycle-transitivity. In future work, we will try to demonstrate similar results for other copulas and we will investigate whether the generated probabilistic relation can serve as a graded alternative to the concept of (first-order) stochastic dominance, popular in financial mathematics and welfare modelling.

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