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Hamiltonian knot projections and lengths of thick knots ☆

Yuanan Diao [a,*], Claus Ernst [b], Xingxing Yu [c]

[a] *Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA*
[b] *Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101, USA*
[c] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

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Abstract

For a knot or link K, $L(K)$ denotes the rope length of K and $Cr(K)$ denotes the crossing number of K. An important problem in geometric knot theory concerns the bound on $L(K)$ in terms of $Cr(K)$. It is well known that there exist positive constants c_1, c_2 such that for any knot or link K, $c_1 \cdot (Cr(K))^{3/4} \leqslant L(K) \leqslant c_2 \cdot (Cr(K))^2$. In this paper, we prove that there exists a constant $c > 0$ such that for any knot or link K, $L(K) \leqslant c \cdot (Cr(K))^{3/2}$. This is done through the study of regular projections of knots and links as 4-regular plane graphs. We show that for any knot or link K there exists a knot or link K' and a regular projection G of K' such that K' is of the same knot type as that of K, G has at most $4 \cdot Cr(K)$ crossings, and G is a Hamiltonian graph. We then use this result to develop an embedding algorithm. Using this algorithm, we are able to embed any knot or link K into the simple cubic lattice such that the length of the embedded knot is of order at most $\mathrm{O}((Cr(K))^{3/2})$. This result in turn establishes the above mentioned upper bound on $L(K)$ for smooth knots and links. Moreover, for many knots and links with special Hamiltonian projections, our embedding algorithm ensures that the bound on $L(K)$ can be of order $\mathrm{O}(Cr(K))$. The study of Hamilton cycles in a regular knot projection plays a very important role and many questions can be raised in this direction.
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* Corresponding author.
E-mail addresses: ydiao@uncc.edu (Y. Diao), claus.ernst@wku.edu (C. Ernst), yu@math.gatech.edu (X. Yu).

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1. Introduction

In this paper, we are interested in geometric properties of knots when they are considered as physical subjects, that is, when the knots are tied with ropes which have thickness and volumes. This is in sharp contrast with the traditional mathematical treatment of knots which views knots as volumeless simple closed curves in the 3-dimensional space $\mathbb{R}^3$. It is well known that knots play an important role in studying the behavior of various enzymes known as topoisomerases, see, for example, [14,16,17,27,31,32]. Since the (effective) diameter of DNA can be measured, it is reasonable to treat it as a rope with certain physical properties, see, for example, [26,25]. In many cases it is also important to recognize the geometric shapes and volumes of physical knots [19]. An essential issue here is to relate the length of a rope (with certain thickness) to those knots that can be tied with this rope. Such information plays an important role in studying the effect of topological entanglement in subjects such as circular DNA and long chain polymers, where knots occur and cannot be treated as volumeless curves.

There are different ways to define the thickness of a knot [7,11,21]. In this paper, we will be using the so-called *disk thickness* introduced in [21] and described as follows. Let K be a C^2 knot. A number $r > 0$ is said to be *nice* if for any distinct points x, y on K, we have $D(x,r) \cap D(y,r) = \emptyset$, where $D(x,r)$ and $D(y,r)$ are the discs of radius r centered at x and y which are normal to K. The *disk thickness* of K is defined to be $t(K) = \sup\{r\colon\ r \text{ is nice}\}$.

It is shown in [7] that the disk thickness definition can be extended to all $C^{1,1}$ curves. Therefore, we will restrict our discussions to such curves in this paper. However, the results obtained in this paper also hold for other thickness definitions with a suitable change in the constant coefficient.

1.1. Definition. For any given knot K, a *thick realization* K_0 of K is a knot of unit thickness which is of the same knot type as that of K. The *rope length* $L(K)$ of K is the infimum of the length of K_0 taken over all thick realizations of K. The existence of $L(K)$ is shown in [7].

In this paper, we are interested in finding lower and upper bounds on $L(K)$ in terms of $Cr(K)$, the minimum crossing number of K.

It is shown in [2,3] that there is a constant $a > 0$ such that for any knot K, $L(K) \geqslant a \cdot (Cr(K))^{3/4}$ (this is called the *three-fourth power law*). The constant a is estimated to be at least 1.105 by a result in [3]. This is improved to 2.135 in [24].

In [16], it is reported that a linear relation between $L(K)$ and $Cr(K)$ is observed. Consequently, one conjectures that the minimum rope length of any knot K is proportional to its crossing number. In other words, there exist constants $0 < a < b$ such that $a \leqslant L(K)/Cr(K) \leqslant b$ for any nontrivial knot K. Half of this conjecture is proven to be false since the three-fourth power law is also shown to be achievable for some knot families [6,9]. That is, there exists an infinite family $\{K_n\}$ of knots and a constant $a_0 > 0$ such that $Cr(K_n) \to \infty$ as $n \to \infty$ and $L(K_n) \leqslant a_0 \cdot (Cr(K_n))^{3/4}$. However, the other half of the conjecture is still open. That is, it is still not known if there exists an infinite family $\{K_n\}$ of knots such that $L(K_n)/Cr(K_n) \to \infty$ as $k \to \infty$. In fact, for a long time, it was not clear whether there exists an infinite family $\{K_n\}$ of prime knots such that $Cr(K_n) \to \infty$ as $n \to \infty$ and $L(K_n)$ is of order more than $\mathrm{O}((Cr(K_n))^{3/4})$. It is shown very recently in [13] that there indeed exists an infinite family $\{K_n\}$ of prime knots such that $Cr(K_n) \to \infty$ as $n \to \infty$ and $L(K_n) = \mathrm{O}(Cr(K_n))$. Let us restate the above unsolved conjecture as the following problem.

1.2. Problem. *Does there exist a constant $c > 0$ such that for any knot or link K, we have*

$$L(K) \leqslant c \cdot Cr(K)?$$

In the case that K is a link of two components (of unit thickness) with lengths L_1 and L_2, it is shown in [12] that $L_1 L_2^{1/3}$ and $L_1^{1/3} L_2$ are both bounded below by $a_1 \cdot |\ell(K)|$, where $a_1 > 0$ is a constant and $\ell(K)$ is the linking number between the two components of K. This result also holds when $\ell(K)$ is replaced by $Cr(K)$, see [10]. For a thick link K of m components, it is well known that the length of K is of the order at least m. So it is not difficult to construct links whose lengths grow linearly with their crossing numbers.

There is very little in the literature about the upper bounds on $L(K)$. It is known that there is a constant $a_2 > 0$ such that for any knot K, $L(K) \leqslant a_2 \cdot (Cr(K))^2$ (see [15,7]). The constant a_2 is estimated to be around 24 [7] and is improved to less than 3 in a recent report [5]. The main objective of this paper is to establish the following significant breakthrough on the upper bound of the rope length of knots.

1.3. Theorem. *There exists a constant $c > 0$ such that for any knot K,*

$$L(K) \leqslant c \cdot \bigl(Cr(K)\bigr)^{3/2}.$$

Due to difficulties of dealing with a thick smooth knot, a physical knot is often modelled by a polygon in the cubic lattice, called a *lattice knot*. (The cubic lattice consists of all points in $\mathbb{R}^3$ with integral coordinates and all unit line segments joining these points.) We will first prove Theorem 1.3 for lattice knots. Since knots realized in the cubic lattice can easily be modified into smooth $C^{1,1}$ knots of thickness $1/2$, Theorem 1.3 can then be extended to smooth knots as well.

To prove Theorem 1.3 for lattice knots, we will design an algorithm which embeds a knot into the cubic lattice. This algorithm will make use of a particular projection of a knot where the projection can be thought of as a 4-regular Hamiltonian graph (to be defined in Section 3).

This paper is organized as follows. In Section 2, we view regular projections of knots as graphs and study their connectivity. In Section 3, we show that every knot K has a regular projection which is a Hamiltonian graph with at most $4 \cdot Cr(K)$ vertices. In Section 4, such a regular projection is embedded into the cubic lattice using at most $c \cdot (Cr(K))^{3/2}$ unit line segments in the cubic lattice. Then in Section 5, the embedding produced in Section 4 is used to prove Theorem 1.3 for lattice knots and finally for smooth thick knots.

2. Regular projection graphs

In this section, we introduce some basic results in knot theory and graph theory. In addition, some new terms are defined as they will be needed in our discussions. Most terms used in this section are well-known definitions in knot theory and graph theory. The reader is referred to [1,30].

A *geometric realization* of a graph G is an embedding of G in $\mathbb{R}^2$ or $\mathbb{R}^3$ such that the edges of G are represented by simple arcs that do not intersect each other in their interior and the vertices of G are represented by the end points of these arcs. Of course, if two edges of G are adjacent, then the two corresponding arcs in the geometric realization of G will share a common vertex of the geometric realization. It is convenient to view the cubic lattice as the geometric realization in $\mathbb{R}^3$ of the infinite graph with vertex set $\{(x, y, z)\colon x, y, z \in \mathbb{N}\}$ and edge set $\{(x, y, z)(x', y', z')\colon x, y, z, x', y', z' \in \mathbb{N}, (x - x')^2 + (y - y')^2 + (z - z')^2 = 1\}$. Here, $\mathbb{N}$ is the set of all integers and the edges are represented by the unit line segments between their ends. We say that a graph G is *planar* if it has a geometric realization in a plane. Such a geometric realization is called a *plane graph*. Plane graphs are related to knots through regular projections of knots.

A common measure for the complexity of a knot or link K is its *crossing number*, which is the minimum number of crossings in all possible regular projections of knots having same knot type K. This is denoted by $Cr(K)$. Of course, by this definition, if K and K' are of the same knot type, then $Cr(K) = Cr(K')$. We say that $P(K)$ is a *minimum projection* of K if it is a regular projection with $Cr(K)$ crossings.

Let K be a knot or link and let $P(K)$ be a regular projection of K. If we treat the crossings in $P(K)$ as vertices and the arcs of $P(K)$ joining these crossings as edges, then $P(K)$ can be viewed as a 4-regular plane graph. Thus, from now on, we may view a regular projection $P(K)$ as a 4-regular plane graph G. To stress the fact that G arises from a regular projection of a knot or link K, we will call it an *RP-graph* of K. If G arises from a minimum projection of a knot or link K, we will then call it a *minimum RP-graph* of K. Note that any 4-regular plane graph is an RP-graph of some knot or link. Thus, a graph is an RP-graph if, and only if, it is a 4-regular plane graph.

If an RP-graph G of a knot (or link) K contains a loop edge e incident with a vertex w, then K can be isotoped to some knot (or link) K' through a Reidemeister move such that K' has an RP-graph G' which can be obtained from G by replacing e, w, and the other two edges incident with w by a single edge. See Fig. 1. For the definition and properties of Reidemeister moves, see [1] or [4]. It follows that we only need to consider RP-graphs without loop edges.

Therefore, for RP-graphs in the rest of this paper, we will assume that no loop edges are present. It follows that every vertex is incident with four distinct edges. In order to

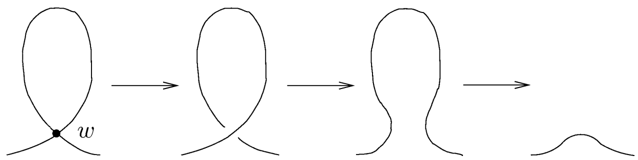

Fig. 1. Loop edges in RP-graphs are removable.

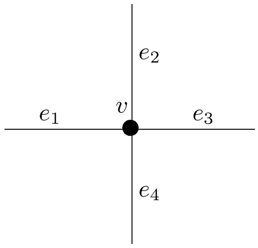

Fig. 2. Pairs of opposite edges: $\{e_1, e_3\}$, $\{e_2, e_4\}$.

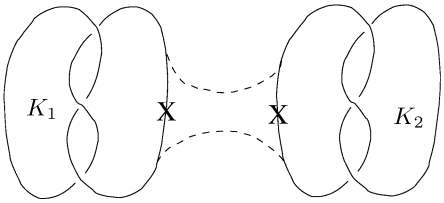

Fig. 3. The connected sum $K_1 \# K_2$ of K_1 and K_2.

recover a knot from an RP-graph, it is important to keep track of the over-strands and under-strands at crossings. Therefore, we introduce the concepts of adjacent and opposite edges in RP-graphs.

2.1. Definition. Let G be an RP-graph, let v be a vertex of G, and let e_1, e_2, e_3, e_4 be the edges of G incident with v. Suppose that e_1, e_2, e_3, and e_4 occur around v in this cyclic order. See Fig. 2. Then we say that e_i is *opposite* to e_j if $|j - i| = 2$, and e_i and e_j are *adjacent* otherwise.

Recall that a composite knot K can be constructed from two nontrivial knots K_1 and K_2 as shown in Fig. 3 by cutting the arcs marked with X and then adding the dashed arcs. We say that K is a *connected sum* of K_1 and K_2 in this case and also denote K by $K_1 \# K_2$. One can similarly define the connected sum of more than two knots.

The following theorems are classical results in knot theory [4] or [20].

2.2. Theorem. *Any nontrivial knot K can be decomposed as the connected sum of prime knots. That is, for any nontrivial knot K, there exist prime knots $K_1, K_2, \ldots, K_j$ $(j \geqslant 1)$ such that $K = K_1 \# K_2 \# \cdots \# K_j$.*

2.3. Theorem [18,23,28]. *For any knots K_1 and K_2, we have $Cr(K_1 \# K_2) \leqslant Cr(K_1) + Cr(K_2)$. If K_1 and K_2 are alternating knots, then we have $Cr(K_1 \# K_2) = Cr(K_1) + Cr(K_2)$.*

It is still an open problem whether $Cr(K_1 \# K_2) = Cr(K_1) + Cr(K_2)$ is true for any two knots K_1 and K_2, although it has recently been proven by one of the authors that this is also true for all torus knots [8]. Since we are not sure if $Cr(K_1 \# K_2) = Cr(K_1) + Cr(K_2)$

in general, the upper bounds on $L(K_1)$ and $L(K_2)$ (in terms of $Cr(K_1)$ and $Cr(K_2)$) do not automatically provide us an upper bound on $L(K_1 \# K_2)$ in terms of $Cr(K_1 \# K_2)$.

We devote the rest of this section to the study of connectivity of RP-graphs.

A graph G is said to be *connected* if for any $u, v \in V(G)$, there is a path in G from u to v. A *component* of G is a maximal subgraph of G that is connected. It can be shown that G is *connected* if, and only if, for any partition V_1 and V_2 of $V(G)$, G has an edge with one end in V_1 and the other in V_2.

For any $X \subset V(G)$, let $G - X$ denote the subgraph of G obtained from G by deleting vertices of G in X and edges of G with at least one end in X. Similarly, for any $Y \subset E(G)$, we use $G - Y$ to denote the subgraph of G obtained from G by deleting the edges in Y (but keeping all vertices of G). We say that G is *k-connected*, where k is a positive integer, if $|V(G)| \geqslant k + 1$ and, for any subset $X \subset V(G)$ with $|X| < k$, $G - X$ is connected. We say that G is *k-edge-connected* if, for any $Y \subset E(G)$ with $|Y| < k$, $G - Y$ is connected. The *connectivity* (respectively, *edge-connectivity*) of G is the largest integer k such that G is k-connected.

An easy observation is that an RP-graph is at most 4-connected and at most 4-edge-connected. For a minimum RP-graph of a knot, we can say more.

2.4. Lemma. *If G is a minimum RP-graph of a nontrivial knot or link K, then*

(a) *G is 2-connected,*
(b) *the edge connectivity of G is 2 or 4, and*
(c) *if K is a prime knot or link, then G is 4-edge-connected.*

Proof. Suppose G is not 2-connected. Then G has a single vertex v such that $G - \{v\}$ is disconnected. See Fig. 4. By twisting part of the corresponding projection $P(K)$ as shown in Fig. 4, we obtain a new projection of K with crossing number $Cr(K) - 1$, a contradiction. So (a) holds.

Assume that there is a set $Y \subset E(G)$ of size at most four such that $G - Y$ is not connected. Then there exists a simple closed curve α which intersects G exactly once at each edge in Y. Since there must be an even number of intersections between any two simple closed curves in general position in a plane, we have $|Y| = 2$ or 4. So (b) holds.

Now assume that G is not 4-edge-connected. Then G is 2-edge-connected. So there is a set $Y \subset E(G)$ such that $|Y| = 2$ and $G - Y$ is not connected. In fact, $G - Y$ has two components, say G_1 and G_2. Moreover, there is a topological 2-sphere S^2 intersecting K exactly twice such that the part of K corresponding to G_1 is inside S^2 and the part of K corresponding to G_2 is outside S^2. If one of these two parts of K corresponding to G_1 and G_2 forms a trivial knot with an arc on S^2 connecting the points in $K \cap S^2$, then G is not a

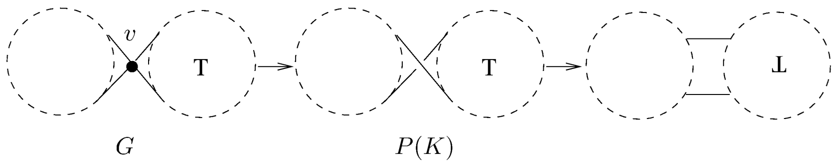

Fig. 4. 1-Connected RP-graph is not minimum.

minimum RP-graph. If both parts form nontrivial knots with an arc on S^2 connecting the points in $K \cap S^2$, then K is not a prime knot by definition. So we have (c). □

3. Hamiltonian knots and graphs

Let G be a graph. A *Hamilton cycle* in G is a cycle that contains all vertices of G. A graph with a Hamilton cycle is said to be *Hamiltonian*. In this section, we first study the following question: given a knot K, does K always have a minimum RP-graph which is Hamiltonian? After obtaining a negative answer to this question, we then ask a weaker question: given a knot K, does K have an RP-graph that is Hamiltonian? Furthermore, if K does have a Hamiltonian RP-graph, how many crossings does this Hamiltonian RP-graph have?

We show that any knot K has a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices. This result will be used in the subsequent sections to establish a thick realization of K with length at most $\mathrm{O}((Cr(K))^{3/2})$. For convenience, let us introduce the following concepts.

3.1. Definition. A knot K is said to be *Hamiltonian* if there exists some knot K' such that K' and K have the same knot type and K' has a Hamiltonian RP-graph (not necessarily minimum). A knot K is said to be *minimally Hamiltonian* if there exists some knot K' such that K' and K have the same knot type and K' has a Hamiltonian minimum RP-graph.

Almost all minimum RP-graphs of small prime knots as listed in the knot tables are Hamiltonian. Notice that the knot 9_{46} has a non-Hamiltonian minimum RP-graph as shown in the left portion of Fig. 5. However, it is minimally Hamiltonian since it does have a minimum RP-graph that is Hamiltonian as shown in the right portion of Fig. 5.

So is it true that all prime knots are minimally Hamiltonian? The following theorem gives a negative answer to this question.

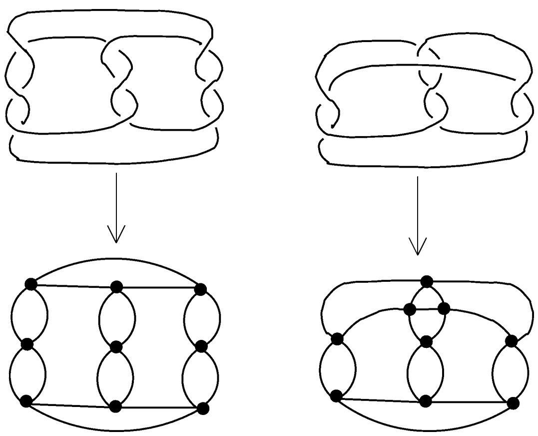

Fig. 5. Non-Hamiltonian and Hamiltonian minimum RP-graphs of 9_{46}.

3.2. Theorem. *Not all prime knots are minimally Hamiltonian.*

Proof. It suffices to show an example. We claim that 9_{35} is a prime knot such that none of its minimum RP-graphs is Hamiltonian.

Notice that 9_{35} is an alternating knot that can be obtained from the knot 9_{46} shown in the left portion of Fig. 5 by taking the mirror image of the three crossings on the left. So an RP-graph of 9_{35} is identical to the RP-graph of 9_{46} shown in the left portion of Fig. 5, which is not Hamiltonian.

Therefore, it suffices to show that any minimum projection of 9_{35} leads to an RP-graph that is isomorphic to the one shown. A *flype* is the modification of a knot diagram as shown in Fig. 6.

Clearly a flype does not alter the number of crossings in the knot diagram nor does it change the knot type. In [22] it is shown that for an alternating knot K, any minimum regular projection of K can be obtained from any given minimum regular projection of K through a finite sequence of flypes.

Note that 9_{35} is an alternating knot, and one can easily check that any flype on the minimum projection of 9_{35} shown in Fig. 5 produces a minimum RP-graph isomorphic to the original one. Therefore 9_{35} is not minimally Hamiltonian. $\square$

Note that the above proof cannot be applied to the knot 9_{46} since 9_{46} is nonalternating and the modification of its diagram shown in Fig. 5 is not a sequence of flypes.

Our next result shows that Hamiltonicity is preserved under connected sum.

3.3. Theorem. *Suppose for each $i \in \{1, \dots, j\}$, K_i is knot which admits a Hamiltonian RP-graph with n_i vertices. Then $K = K_1 \# K_2 \# \cdots \# K_j$ admits a Hamiltonian RP-graph with $n_1 + \cdots + n_j$ vertices.*

Proof. For $i = 1, 2$, let G_i be a Hamiltonian RP-graph of K_i contained in some plane P_i, and let H_i be a Hamilton cycle in G_i. Notice that G_i divides the plane P_i into closed regions. Choose a point O_i in one of these regions whose boundary contains an edge $u_i v_i$ of H_i, and choose a circle C_i contained in P_i centered at O_i that does not intersect G_i.

An inversion of the plane P_i about C_i maps G_i to a graph G_i' isomorphic to G_i, maps H_i to a Hamilton cycle H_i' in G_i', and maps $u_i v_i$ to an edge $u_i' v_i'$ of H_i'. This inversion in the plane P_i about C_i can be extended to an inversion in $\mathbb{R}^3$ about S_i^2, where S_i^2 is the 2-sphere with C_i as a great circle. We can assume that the knot K_i is very close to the plane P_i and, for a suitably chosen C_i, we can assume that S_i^2 does not intersect the knot K_i. Then, this inversion maps K_i to a knot K_i' that is of the same knot type as that of the mirror image of K_i. Furthermore, since K_i is close to P_i, we can assume that (after a small

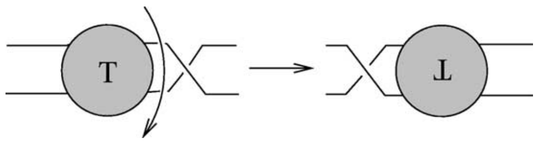

Fig. 6. A flype in a knot diagram.

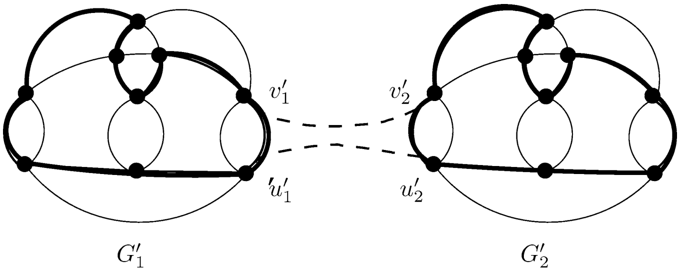

Fig. 7. Connecting two Hamilton cycles.

isotopy of K_i') G_i' is the RP-graph of K_i' in P_i. So the mirror image K_i'' of K_i' through the plane P_i is of the same knot type as that of K_i. Also, G_i' is the RP-graph of K_i'' in P_i. So G_i' is still an RP-graph of K_i. Notice that H_i' is a Hamilton cycle in G_i' and $u_i'v_i'$ is on the boundary of the unbounded region of $P_i - G_i'$. See Fig. 7 for an example.

We can now obtain a new graph G from the disjoint union of G_1' and G_2' (as illustrated in Fig. 7) by deleting edges $u_1'v_1'$ and $u_2'v_2'$ and by adding edges $u_1'u_2'$ and $v_1'v_2'$. We also obtain a Hamilton cycle H of G from the disjoint union of H_1' and H_2' by deleting edges $u_1'v_1'$ and $u_2'v_2'$ and by adding edges $u_1'u_2'$ and $v_1'v_2'$. Clearly, G is a Hamiltonian RP-graph of $K_1'' \# K_2''$ and $|V(G)| = |V(G_1')| + |V(G_2')| = n_1 + n_2$.

The same argument can be repeated for $K_1'' \# K_2''$ and K_3, and so on. So the statement of the theorem holds by induction on the number of summands. □

As mentioned before, our main objective in this section is to show that every knot or link is Hamiltonian. Notice that RP-graphs are plane graphs. Whitney [33] showed in 1931 that every 4-connected plane triangulation contains a Hamilton cycle. In 1956, Tutte [29] generalized Whitney's result to all plane graphs.

3.4. Theorem. *If G is a 4-connected plane graph, then G has a Hamilton cycle.*

Using Theorem 3.4, we are able to prove the following theorem.

3.5. Theorem. *If K is a prime knot or link, then K admits a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices.*

Proof. Let K be a prime knot (or link) and let G be a minimum RP-graph of K in a plane P. By Lemma 2.4, G is 2-connected and 4-edge-connected. The proof proceeds as follows. First, we construct a plane graph H from G such that H is 4-connected and thus has a Hamilton cycle C. We then use C and H to construct a knot K' such that K' and K have the same knot type and K' has a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices.

(I) The construction of a plane graph H from G.

For each vertex v of G, let D_v be a disk in P centered at v with a small radius such that D_v contains no other vertex of G and the boundary γ_v of D_v intersects G at exactly four points v_1, v_2, v_3 and v_4. Let $\Gamma_v = \{v, v_1, v_2, v_3, v_4\}$ and call it the *vertex cluster at v*

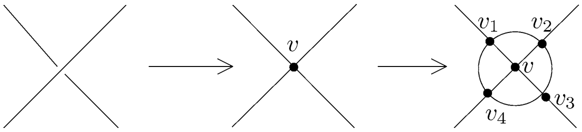

Fig. 8. The graph H around a vertex cluster Γ_v.

to stress the fact that the points v_1, v_2, v_3 and v_4 are created around v and will be vertices of H. These points in $\bigcup_{v\in V(G)}\Gamma_v$ divide each edge of G into 3 arcs and divide each γ_v into 4 arcs. Let H denote the graph whose vertex set is $\bigcup_{v\in V(G)}\Gamma_v$ and whose edge set consists of all these arcs in G and γ_v created by the points in $\bigcup_{v\in V(G)}\Gamma_v$. Then H is a 4-regular plane graph. A local picture of H around a vertex cluster Γ_v is shown in Fig. 8.

(II) The proof that H is 4-connected.

Assume to the contrary that H is not 4-connected. Then there exists a set $X\subset V(H)$ such that $|X|\leqslant 3$ and $H-X$ is not connected. Choose X such that $|X|$ is minimum. Hence, there exists a partition X_1 and X_2 of $V(H)-X$ such that H has no edge with one end in X_1 and the other in X_2. Since $|X|$ is minimum, each vertex in X is adjacent to a vertex in X_1 and also to a vertex in X_2.

(1) We claim that for any $v\in V(G)$, $v\notin X$.

Suppose (1) does not hold. Let $v\in V(G)$ such that $v\in X$. Then v is adjacent to a vertex in X_1 and a vertex in X_2. Since H has no edge from X_1 to X_2 and by the local structure of H on Γ_v, $|X|=3$ and $X\subset\Gamma_v$. For $i=1,2$, let $V_i=\{u\in V(G)\colon\ \Gamma_u\subset X_i\}$. Since H is 4-regular and $|X|=3$, $|X_i|\geqslant 2$ for $i=1,2$. Since $X\subset\Gamma_v$, $V_i\neq\emptyset$ for $i=1,2$. Clearly, V_1, V_2 form a partition of $V(G)-\{v\}$ such that G has no edge from V_1 to V_2. This means that $G-\{v\}$ is not connected, a contradiction (since G is 2-connected). So we have (1).

By (1) and by the local structure of H on Γ_v, we have

(2) For any $v\in V(G)$, either $\Gamma_v\subset X\cup X_1$ or $\Gamma_v\subset X\cup X_2$.

For $i=1,2$, let $U_i=\{u\in V(G)\colon\ \Gamma_u\subset X\cup X_i\}$. Then U_1 and U_2 form a partition of $V(G)$. We claim that

(3) G has at most $|X|$ edges from U_1 to U_2.

For any edge e of G connecting U_1 to U_2, it contains two vertices of $V(H)\backslash V(G)$ by the construction of H. At least one of them is in X, since otherwise there is any edge in $E(H)$ connecting X_1 to X_2. This shows (3).

Since $|X|\leqslant 3$, (3) contradicts Theorem 2.4 which asserts that G is 4-edge-connected. Thus X cannot disconnect H, and so, H is 4-connected.

(III) The construction of a knot K' such that K' and K have the same knot type and K' has a Hamiltonian RP-graph with at most $4\cdot Cr(K)$ vertices.

By Theorem 3.4, H has a Hamilton cycle, say C. Since a Hamilton cycle passes each vertex of H exactly once, for any vertex cluster Γ_v either (a) C enters and leaves Γ_v exactly once or (b) C enters and leaves Γ_v exactly twice. We will make changes to the projection of K corresponding to G by applying a sequence of Reidemeister moves according to (a) or (b). The result will be a knot K' isotopic to K such that K' has a Hamiltonian RP-graph with at most $4\cdot Cr(K)$ vertices.

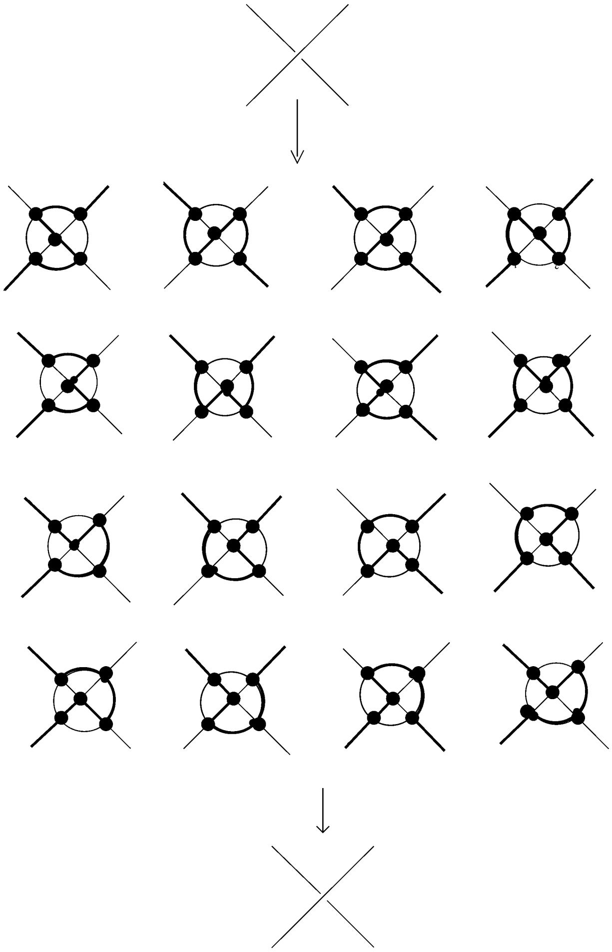

Fig. 9. No change to a crossing at v when Γ_v is of type (a).

The middle portion of Fig. 9 shows all possibilities when Γ_v is of type (a), where the thickened edges are in C. For each vertex cluster Γ_v of type (a), we leave unchanged the crossing at v in the projection of K corresponding to G. See bottom portion of Fig. 9.

The top portion of Fig. 10 shows all possibilities when $\Gamma(v)$ is of type (b), where the thickened edges are in C. For each vertex cluster Γ_v of type (b), we first make the changes locally as shown in the middle section of Fig. 10. Notice that these changes do not affect the Hamilton cycle C, so C is still a Hamilton cycle in the resulting new graph H'. We then modify K to a new knot K' through some suitable Reidemeister moves as shown in the bottom portion of Fig. 11 such that the RP-graph of K' is H'.

Therefore we have constructed a knot K' such that: (1) K' is obtained from K by a sequence of Reidemeister moves (and so, K' is isotopic to K); (2) K' has a projection H' with at most $4 \cdot Cr(K)$ crossings; and (3) C is a Hamilton cycle in H'. $\square$

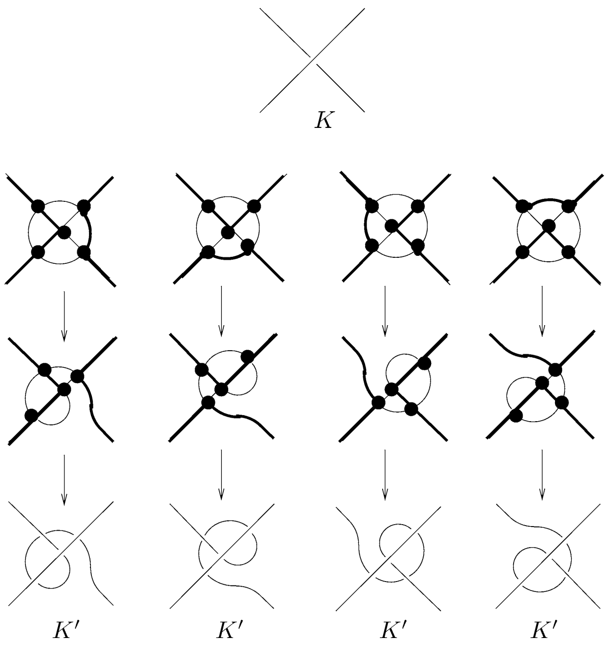

Fig. 10. The changes to a crossing at v when Γ_v is of type (b).

The following theorem is the main result of this section.

3.6. Theorem. *Every knot or link K admits a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices.*

Proof. Suppose Theorem 3.6 is false. Let K be a counter-example with the smallest minimum crossing number. That is, Theorem 3.6 holds for any knot with crossing number less than $Cr(K)$. Let G be a minimum projection of K. By Lemma 2.4, G has edge-connectivity either 2 or 4. In fact, G has edge-connectivity 2, for otherwise, the same proof of Theorem 3.5 shows that K has a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices, contradicting the assumption that K is a counter-example to Theorem 3.6.

So let $u_1'u_2', v_1'v_2'$ be the edges of G such that $G - \{u_1'u_2', v_1'v_2'\}$ has two components G_1' and G_2', $u_1', v_1' \in V(G_1')$, and $u_2', v_2' \in V(G_2')$. See Fig. 7 for an example. Therefore, there exists a topological 2-sphere S^2 intersecting K exactly twice such that the part of K corresponding to G_1' is inside S^2 and the part of K corresponding to G_2' is outside S^2. For $i = 1, 2$, let G_i be the graph obtained from G_i' by adding the edge $u_i'v_i'$. Then G_1 (respectively, G_2) is an RP-graph of the knot K_1 (respectively, K_2) formed by the part of K inside (respectively, outside) S^2 and an arc on S^2 connecting the points in $K \cap S^2$.

Because $Cr(K_1) + Cr(K_2) \geqslant Cr(K) = |V(G_1)| + |V(G_2)|$ and $|V(G_i)| \geqslant Cr(K_i)$ for $i = 1, 2$, we have $Cr(K_i) = |V(G_i)|$ for $i = 1, 2$. So for $i = 1, 2$, G_i is a minimum RP-graph of K_i. Since $Cr(K_1) < Cr(K)$ and $Cr(K_2) < Cr(K)$, Theorem 3.6 holds for K_1 and K_2 by the choice of K. That is, K_i has a Hamiltonian RP-graph with at most $4 \cdots Cr(K_i)$

vertices. So by Theorem 3.3 and the fact that $Cr(K) = Cr(K_1) + Cr(K_2)$, $K = K_1 \# K_2$ also has a Hamiltonian RP-graph with at most $4 \cdot Cr(K)$ vertices, a contradiction. $\square$

4. The embedding of RP-graphs

In this section, we will show a way to embed a Hamiltonian RP-graph of a knot K into the cubic lattice, and such an embedding will then be modified in the next section to give a thick realization of K.

Recall that the cubic lattice is the infinite graph in $\mathbb{R}^3$ whose vertices are points with integer coordinates and whose edges are unit length line segments connecting these points. Hence, there are three types of edges in the cubic lattice: those parallel to the x-axis, those parallel to the y-axis, and those parallel to the z-axis. An edge of the cubic lattice that is parallel to the x-axis is called an *x-step*. If an oriented x-step increases (respectively, decreases) the x-coordinate, then it is called an *x^+-step* (respectively, *x^--step*). The terms *y-step*, *y^+-step*, *y^--step*, *z-step*, *z^+-step*, and *z^--step* are similarly defined. For the sake of clarity, the vertices and edges of the cubic lattice are called *lattice vertices* and *lattice edges*, respectively. A *lattice path* is a simple curve in the cubic lattice between two distinct lattice vertices, and the ends of curve are called the *ends* of the lattice path. A *lattice graph* is a graph whose vertices are lattice vertices and whose edges are lattice paths that are pairwise disjoint except possibly at their ends.

4.1. Definition. We say that a graph G in $\mathbb{R}^3$ can be *embedded* into the cubic lattice if there is an ambient isotopy $H : I \times \mathbb{R}^3 \to I \times \mathbb{R}^3$ such that $H(0, x)$ is the identity map and $H(1, x)$ maps G onto a lattice graph F. The *length* of the lattice graph F, denoted by $L(F)$, is the total number of lattice edges in F.

Throughout the rest of this section, we fix the following notation.

4.2. Notation. Let G be a Hamiltonian RP-graph of a knot K in a plane $z = 0$ and let C be a Hamilton cycle in G. Let $n = |V(G)|$. Let $v_1, \ldots, v_n$ denote the vertices of G which occur on C in the cyclic order listed. Let $k = \lceil \sqrt{n} \rceil$. For any point p in $\mathbb{R}^3$, we use $y(p)$ to denote the y-coordinate of p. Observe that as a simple closed curve in the plane $z = 0$, C divides the plane $z = 0$ into two closed regions, one bounded and one unbounded. The edges in the set $E(G) - E(C)$ are then divided into two groups: those in the bounded region, called *B-edges*, and those in the unbounded region, called *U-edges*.

Before we describe our algorithm, let us first look at an example.

4.3. Example. An embedding of 9_{32}.

The top portion of Fig. 11 shows a minimum RP-graph of 9_{32} and a Hamilton cycle represented by the thickened curve. The bottom portion of Fig. 11 shows an embedding of the given RP-graph in the cubic lattice where again the embedding of the Hamilton cycle is marked as the thickened lattice polygon.

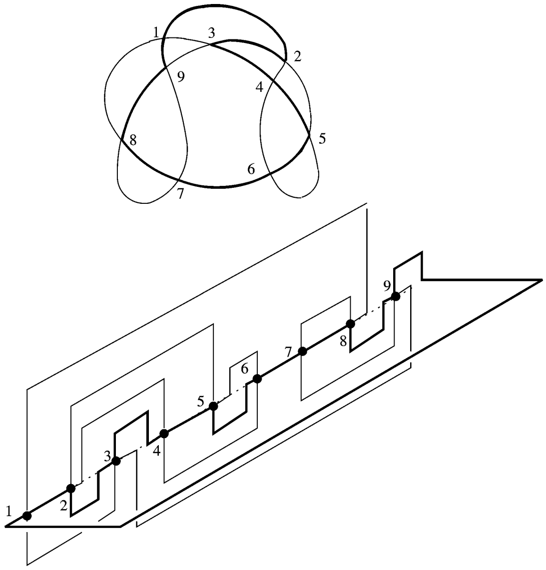

Fig. 11. A Hamiltonian RP-graph of 9_{32} and its realization in the cubic lattice.

The thickened Hamilton cycle allows us to embed the RP-graph into the cubic lattice in a systematic way. First, we embed the vertices along the y-axis following the order inherited from the Hamilton cycle so that any two consecutive vertices are three y-steps apart. The edges of the Hamilton cycle are then embedded in the plane $x = 0$ (with the exception of one edge) and between the planes $z = 1$ and $z = -1$. The remaining edges of G are then embedded according to their relations with the Hamilton cycle. The U-edges are embedded in the plane $x = 0$ above y-axis, and the B-edges are embedded in the plane $x = 0$ below y-axis. Fig. 11 outlines this idea. Note that the lattice graph shown in Fig. 11 is essentially contained in a single square lattice.

The above example gives an intuitive idea how an algorithm may be designed to embed an RP-graph into the cubic lattice using $\mathrm{O}((Cr(K))^2)$ lattice edges. Since we aim to achieve an upper bound better than $\mathrm{O}((Cr(K))^2)$, we leave out a detailed description of such an algorithm. An interested reader can easily work this out.

To achieve the upper bound $\mathrm{O}((Cr(K))^{3/2})$, we need to embed a Hamiltonian RP-graph into the cubic lattice in a more compact and subtle way. In particular, we need to construct an embedding that fills up a part of the lattice by making use of all 6 directions available in the cubic lattice. Similar to the embedding scheme sketched in Example 4.3, we will embed G into the cubic lattice in four steps. First, we embed the vertices of G into the plane $z = 0$. We then embed the edges of C between the planes $z = 1$ and $z = -1$. Next we embed the U-edges in the half-space $z \geqslant 0$. Finally, the B-edges are embedded in a similar way into the half-space $z \leqslant 0$.

4.4. *The embedding of $V(G)$ in the plane $z = 0$.*

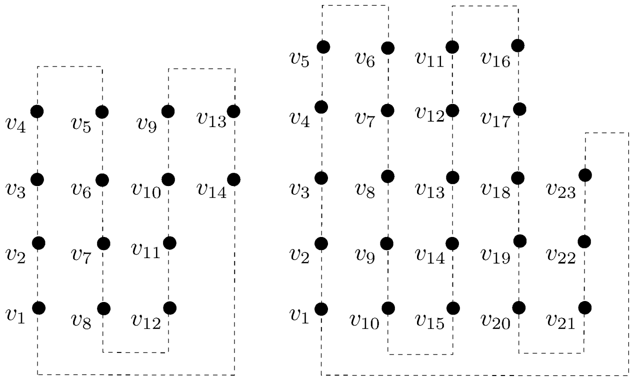

Fig. 12. The embedding of vertices of C for $n = 14$ and $n = 23$.

We will embed $V(G)$ into the set of points $\{(3i, 3j, 0)\colon\ 0 \leqslant i < k, 1 \leqslant j \leqslant k\}$. Recall the notations defined in Notation 4.2.

Intuitively, we embed $v_1, \dots, v_k$ in the order listed on the y-axis by starting with v_1 at $(0, 3, 0)$, increasing the y-coordinate by 3 at a time, ending with v_k at $(0, 3k, 0)$. We then embed $v_{k+1}, \dots, v_{2k}$ on the line determined by $x = 3$ and $z = 0$, by starting with v_{k+1} at $(3, 3k, 0)$, decreasing the y-coordinate by 3 at a time, ending with v_{2k} at $(3, 3, 0)$. We then continue in this way by embedding $v_{ik+1}, \dots, v_{(i+1)k}$ on the line determined by $x = 3i$ and $z = 0$ in the order from $(3i, 3, 0)$ to $(3i, 3k, 0)$ when i is even or from $(3i, 3k, 0)$ to $(3i, 3, 0)$ when i is odd, with y-coordinates differing by 3 at a time. Fig. 12(a) and (b) illustrate the embedding with $n = 14$ and $n = 23$, respectively.

To precisely state this embedding, let $p_{i,j} = (3i, 3j, 0)$ for $0 \leqslant i < k$ and $1 \leqslant j \leqslant k$. For each $1 \leqslant \ell \leqslant n$, there exist unique integers i, j such that $0 \leqslant i \leqslant k - 1$, $1 \leqslant j \leqslant k$, and $\ell = ik + j$. If i is even then embed v_ℓ to $p_{i,j}$, otherwise embed v_ℓ to $p_{i,k-j+1}$.

One may view the points $p_{i,1}, p_{i,2}, \dots, p_{i,k}$ as in *column* i, and there are k such columns. However, it is possible that column $k - 1$ contains no vertices of G at all (for example, when $n = 12$ or $n = 20$). This completes the description of the embedding of $V(G)$.

For convenience, we will not distinguish between v_ℓ and $p_{i,j}$ if v_ℓ is embedded to $p_{i,j}$, and we sometimes write $v_\ell = p_{i,j}$. Also we let $v_0 = v_n$ so that $v_{\ell-1}v_\ell$ makes sense for all $\ell = 1, \dots, n$.

4.5. *The embedding of C as a lattice polygon between the planes $z = -1$ and $z = 1$.*

The embedding of edges of C depends on the edges in $E(G) - E(C)$. Let v_ℓ be a vertex of G. Since G is 4-regular and is free of loop edges, there are exactly two distinct edges of G incident with v_ℓ which are not on C, and these two edges can be both U-edges, both B-edges, or one U-edge and one B-edge. We say that v_ℓ is of *type* (a) if one of these two edges is a U-edge and the other one is a B-edge. We say that v_ℓ is of *type* (b) if these two edges are both U-edges. Finally, we say that v_ℓ is of *type* (c) if these two edges are both B-edges.

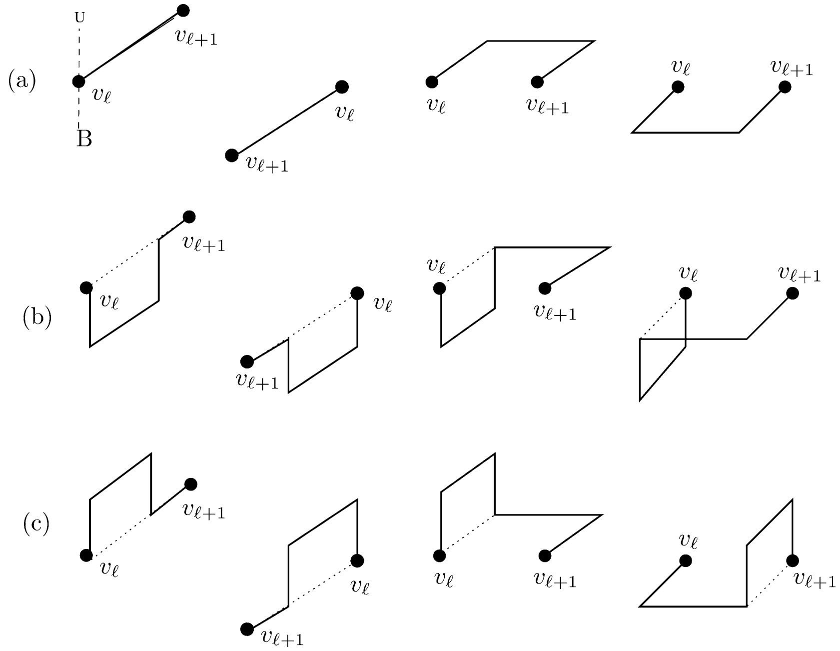

Fig. 13. The local embedding of C.

We now embed C onto a lattice polygon between the planes $z=-1$ and $z=1$. As we will see later that a reason to embed the edges between the planes $z=1$ and $z=-1$ is to make room for embedding the U-edges in the half space $z \geqslant 0$ and the B-edges in the half space $z \leqslant 0$. The lattice paths corresponding to $v_{ik}v_{ik+1}$, $i=1,\dots,k-1$, and $v_n v_1$ will require x-steps, while the lattice paths corresponding to the other edges will not use x-steps.

First, we embed $v_{\ell-1}v_\ell$ for all $\ell \in \{1,\dots,n-1\}$. Let $\ell = ik+j$, where $0 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant k$.

Assume that v_ℓ is of type (a). See Fig. 13(a) for an illustration of the embedding process described below. If $j \neq k$ and i is even, then we embed the edge $v_\ell v_{\ell+1}$ onto the lattice path from v_ℓ to $v_{\ell+1}$ obtained by taking three y^+-steps. If $j \neq k$ and i is odd, then we embed the edge $v_\ell v_{\ell+1}$ onto the lattice path from v_ℓ to $v_{\ell+1}$ obtained by taking three y^--steps. If $j=k$ and i is even, then we embed the edge $v_\ell v_{\ell+1}$ onto the lattice path from v_ℓ to $v_{\ell+1}$ obtained by taking two y^+-step, then three x^+-steps, and then two y^--steps. If $j=k$ and i is odd, then we embed the edge $v_\ell v_{\ell+1}$ onto the lattice path from v_ℓ to $v_{\ell+1}$ obtained by taking two y^--steps, then three x^+-steps, and then two y^+-step.

If v_ℓ is of type (b) or of type (c), the embedding process is illustrated in Fig. 13(b) and (c) and one can write down the corresponding lattice paths in a similar way to the above. The details are left to the reader.

It remains to embed the edge $v_n v_1$ into the cubic lattice. See Fig. 12 for an example when v_n is of type (a). When v_n is of type (b) or (c), make adjustments as in Fig. 13(b) and (c). By the definition of k, we have $n = ik+j$ for some $k-2 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant k$.

First, assume i is odd. Then $v_n = (3i, 3k - 3j + 3, 0)$. If v_n is of type (a), we embed the edge $v_n v_1$ onto the lattice path from v_n to v_1 obtained by taking $3k - 3j + 2$ y^--steps, $3i$ x^--steps, and two y^+-steps. If v_n is of type (b) (respectively, (c)), then we embed the edge $v_n v_1$ onto a lattice path from v_n to v_1 obtained by taking a z^--step (respectively, z^+-step), two y^--steps, a z^+-step (respectively, z^--step), $3k - 3j$ y^--steps, $3i$ x^--steps, and two y^+-steps. Now assume that i is even. Then $v_n = (3i, 3j, 0)$. If v_n is of type (a), then we embed the edge $v_n v_1$ onto the lattice path from v_n to v_1 obtained by taking two y^+-steps, two x^+-steps, $3j + 4$ y^--steps, $3i + 2$ x^--steps, and two y^+-steps. If v_n is of type (b) (respectively, (c)), then we embed the edge $v_n v_1$ onto a lattice path from v_n to v_1 obtained by taking a z^+ (respectively, z^--step), two y^+-steps, a z^- (respectively, z^+-step), two x^+-steps, $3j + 4$ y^--steps, $3i + 2$ x^--steps, and two y^+-steps. This concludes the embedding of C.

Remark. It is clear from the embedding scheme in 4.5 that the embedding of C is a lattice polygon between the planes $z = -1$ and $z = 1$. It is also easy to see that the process in 4.5 can be realized by an ambient isotopy that keeps the edges in $E(G) - E(C)$ on the plane $z = 0$, and maps C to the lattice polygon described above. Note that the lattice path from v_n to v_1 uses at most $6k + 10$ lattice edges, the lattice path from v_{ik} to v_{ik+1} use either seven or nine lattice edges, and the lattice path connecting other consecutive vertices of C uses either three or five lattice edges. Thus the total length of the embedding for C is bounded above by $6k + 10 + 9(k - 1) + 5(n - k) < 5n + 11k$.

Let F denote the lattice polygon constructed so far (which is the embedding of C). Then F is a lattice graph with $V(F) = V(G)$. We will update F by adding lattice paths to F as we embed U-edges and B-edges of G. During this process we will keep the symbol F for the up-to-date embedding until F becomes an embedding of G.

4.6. *Preparations for the embedding of U-edges.*

We want to embed the U-edges of G into the half space $z \geqslant 0$. More specifically, we need to connect the ends of every U-edge using a lattice path in the half space $z \geqslant 0$ such that all lattice paths corresponding to U-edges or edges of C are disjoint (except at their ends) and can be isotoped back to G in the half-space $z \geqslant 0$.

The embedding of a U-edge is based partially on which columns it connects. Let e be a U-edge which has one end in column i and the other in column j, and let $J(e) = |i - j|$. We call $J(e)$ the *jump number* of e. Clearly $0 \leqslant J(e) \leqslant k - 1$. Furthermore, we say that e jumps *from column i to column j* or *from column j to column i*. If $i < j$ then we say that e *starts in column i* and *ends in column j*. If $J(e) = 0$ then both ends of e are in the same column and the embedding of e needs not use any x-steps. If $J(e) > 0$, then x-steps are needed to embed e.

We will embed the U-edges in three stages: the U-edges with jump number 0 will be embedded first, then the U-edges with jump number 1, and the U-edges with jump number at least 2 will be embedded last.

The edges of G will be embedded so that at each vertex of G, edges which are opposite in G will use “opposite” lattice edges at that vertex. To keep track of opposite edges, we define a region R_e for each U-edge e. Let $e = v_r v_s$ be a U-edge of G, and assume that $r < s$. Let C_e denote the simple closed curve in the plane $z = 0$ corresponding to the union of e and the path $v_r v_{r+1} \dots v_s$ on C. Then C_e divides the plane P into two closed regions with boundary C_e, and we use R_e to denote the region that does not contain $C - C_e$.

Let v_ℓ be a vertex in column i and let v_ℓ^* be the lattice vertex such that $v_\ell v_\ell^*$ is the lattice edge corresponding to the y^+-step (respectively, y^--step) from v_ℓ when i is even (respectively, odd).

We are now ready to embed the U-edges with jump number 0.

4.7. *The embedding of U-edges with jump number* 0 *between the planes* $z = 0$ *and* $z = k + 1$.

To each U-edge of jump number 0, we associate a number, called the *level* of that edge, defined recursively. A U-edge e with jump number $J(e) = 0$ is called an *edge of level* 1 if there are no other U-edges of G inside R_e. Suppose that we have defined edges of level i for some integer $i \geqslant 1$. We say that a U-edge e with $J(e) = 0$ is an edge of *level* $i + 1$ if all U-edges of G contained in R_e (which are necessarily of jump number 0) are of level at most i and at least one of them is of level i. The level of an edge e is denoted by $t(e)$. Fig. 14 shows examples of edges of levels 1, 2 and 3.

We can now describe the embedding scheme of U-edges with jump number 0. Let $e = v_r v_s$ be a U-edge with $J(e) = 0$ and of level $t(e)$, that is, v_r and v_s are both in column i for some $0 \leqslant i \leqslant k - 1$.

If e is adjacent to both edges $v_{r-1}v_r$ and $v_{s-1}v_s$ in G, then we embed e onto the lattice path from v_r to v_s obtained by taking the following steps:

(1) $t(e) + 1$ z^+-steps from v_r,
(2) the minimum number of y-steps to reach the point with y-coordinate equal to $y(v_s)$, and
(3) $t(e) + 1$ z^--steps to v_s.

This process is illustrated by the edges e_1, e_3, and e_4 in Fig. 15.

If e is adjacent to $v_{r-1}v_r$ and opposite to $v_{s-1}v_s$ in G, then we embed e onto the lattice path from v_r to v_s obtained by taking the following steps: (1), (2), and (3) above with v_s^* replacing v_s, and (4) a y-step to reach v_s. This is illustrated by e_5 in Fig. 15.

If e is opposite to $v_{r-1}v_r$ and adjacent to $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps: (0) a y-step from v_r to v_r^*, (1), (2), and (3) above with v_r^* replacing v_r. This is illustrated by e_2 in Fig. 15.

If e is opposite to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps: (0) a y-step from v_r to

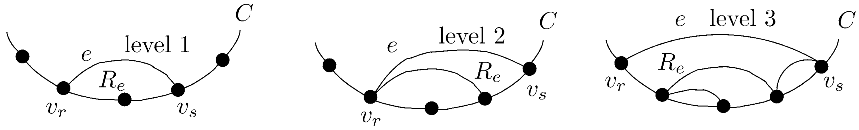

Fig. 14. Examples of edges of different levels.

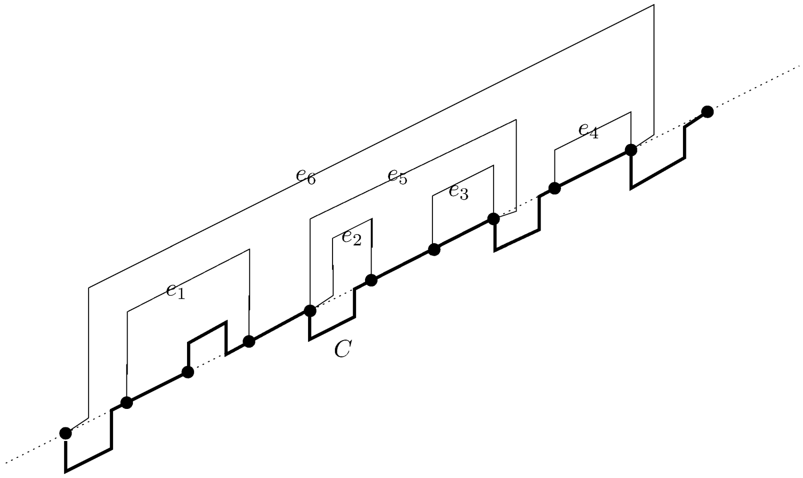

Fig. 15. The embedding of jump 0 edges.

v_r^*, (1), (2), (3) above with v_r^*, v_s^* replacing v_r, v_s, respectively, and (4) a y-step to reach v_s. This is illustrated by e_6 in Fig. 15.

Next, we observe that the jump 0 edges are embedded between the plane $z = 0$ and $z = k + 1$. This is because $t(e) \leqslant k$ for all jump 0 edges. To see this, note that in each column there are at most k U-edges with jump number 0 (since each column has k vertices, each vertex in a column is incident to at most 2 U-edges, and each U-edge with jump number 0 connects to exactly two vertices in the same column). It follows that each such lattice path onto which a jump 0 U-edge is embedded contains at most $2(k+1) + (3k-1) = 5k+1$ lattice edges. This completes the embedding of U-edges with jump number 0.

Remark. At this stage, F consists of the embedding of C and the embedding of all U-edges with jump number 0. Next we show that F is a lattice graph with $V(F) = V(G)$. To see that we need to show that all lattice paths used in 4.5 and 4.7 are disjoint except possibly at their ends. This is easy to see if one lattice path represents an edge on C and the other represents a jump 0 U-edge. Now suppose e_1 and e_2 are U-edges with jump number 0 such that R_{e_2} contains R_{e_1}. Then $t(e_2) > t(e_1)$ by planarity, and hence, the embedding of e_2 is on top of the embedding of e_1. Moreover, if $e_1 = v_r v_s$ and $e_2 = v_{r'} v_{s'}$ such that $r < s$, $r' < s'$, R_{e_1} does not contain R_{e_2}, and R_{e_2} does not contain R_{e_1}, then either $s \leqslant r'$ or $s' \leqslant r$ (by planarity). In other words, the lattice paths constructed in 4.7 are disjoint except possibly at their ends. Furthermore, this embedding algorithm allows us to isotope F to the plane $z = 0$ one lattice path at a time (starting with the edges with level one, then the edges with level two and so on).

To describe the embedding of U-edges with jump number at least 1, we need additional notation. For each column j, let E_j^+ (respectively, E_j^-) denote the set of U-edges starting (respectively, ending) in column j. Note that these sets may be empty. For any edge $e \in E_j^+$

(respectively, $e \in E_j^-$), we define $t_j^+(e)$ (respectively, $t_j^-(e)$) to be the number of edges in E_j^+ (respectively, E_j^-) which are contained in R_e.

The following observations will be useful. First, $|E_j^+| \leqslant 2k$ and $|E_j^-| \leqslant 2k$, and hence, $t_j^+(e) \leqslant 2k$ and $t_j^-(e) \leqslant 2k$. Secondly, if e jumps from column j to column $j+1$, then $t_j^+(e) = t_{j+1}^-(e)$. Finally, for any two distinct edges $e_1, e_2 \in E_j^+$ (respectively, $e_1, e_2 \in E_j^-$), either R_{e_1} contains R_{e_2} or R_{e_2} contains R_{e_1}, and hence, $t_j^+(e_1) \neq t_j^+(e_2)$ (respectively, $t_j^-(e_1) \neq t_j^-(e_2)$). The second and third observations make it possible to embed all edges with jump number at least 1 with two embedding schemes (and also, to isotope the lattice paths back onto the edges of G in the plane $z = 0$).

4.8. *The embedding of U-edges with jump number* 1 *between the planes* $z = 0$ *and* $z = 2k$.

Let $e = v_r v_s$ be a U-edge with v_r in column j and v_s in column $j+1$. That is, $e \in E_j^+ \cap E_{j+1}^-$, and hence, $t_j^+(e) = t_{j+1}^-(e)$. First, assume that e is adjacent to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G. Then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed:

(1) $t_j^+(e)$ z^+-steps starting from v_r,
(2) one x^+-steps,
(3) minimum number of y-steps to reach a point with y-coordinate equal to $y(v_s)$,
(4) two x^+-step,
(5) $t_j^+(e) = t_{j+1}^-(e)$ z^--steps to v_s.

For an illustration of this embedding, see Fig. 16(a).

If e is adjacent to $v_{r-1}v_r$ and opposite to $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (1)–(5) as above with v_s^* replacing v_s, and (6) one y-step from v_s^* to v_s. For an illustration, see Fig. 16(b).

If e is opposite to $v_{r-1}v_r$ and adjacent to $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (0) one y-step from v_r to v_r^*, and (1)–(5) as above with v_r^* replacing v_r. For an illustration, see Fig. 16(c).

If e is opposite to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (0) one y-step from v_r to v_r^*, (1)–(5) as above with v_r^*, v_s^* replacing v_r, v_s, respectively, and (6) a y-step from v_s^* to v_s. For an illustration, see Fig. 16(d).

Since $t_j^+(e) \leqslant 2k$, the embedding is between the planes $z = 0$ and $z = 2k$. By a simple counting, we see that the lattice path representing e has at most 3 x-steps, at most $4k$ z-steps, and at most $3k - 1$ y-steps. Hence the lattice path representing e uses at most $7k + 2$ lattice edges. This completes the description of the embedding of jump 1 U-edges.

Remark. The up-to-date F consists of the embedding of C and the embedding of all U-edges with jump number 0 or 1. Next we show that F is a lattice graph with vertex set

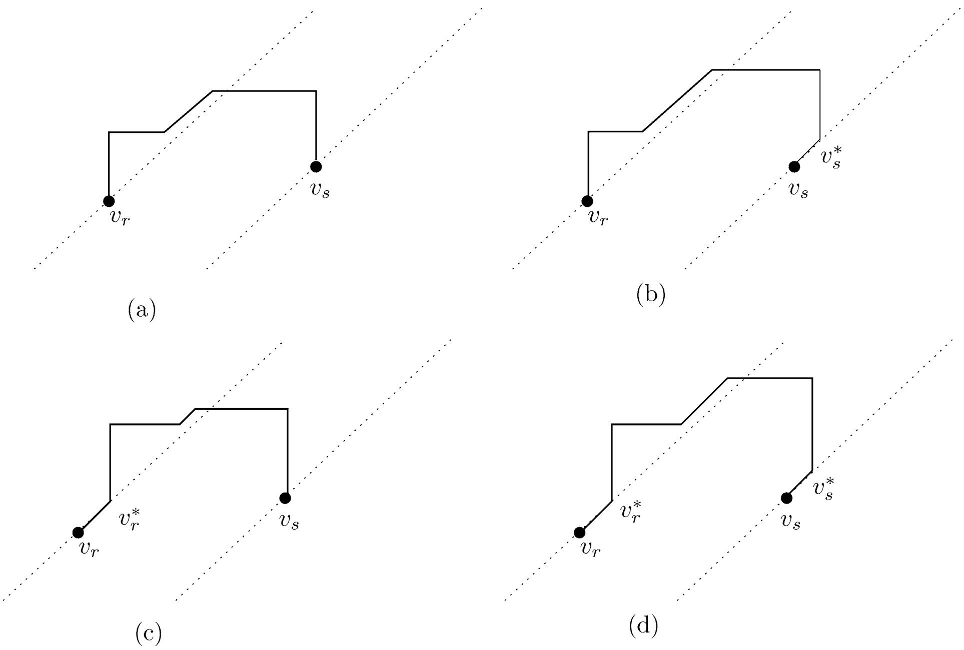

Fig. 16. The embedding of edges with jump number 1.

$V(F) = V(G)$. That is, for any two edges e_1 and e_2 which are on C or with jump number 0 or 1, the corresponding lattice paths P_1 and P_2 onto which they are embedded are disjoint except at their ends. By the remark following 4.7, we may assume that $e_1 = v_r v_s$ has jump number 1 with v_r in column j and v_s in column $j+1$, as shown in Fig. 16. If e_2 is on C, then it is easy to see that P_1 and P_2 are disjoint except possibly at their ends. So e_2 is a U-edge. If the ends of e_2 are not in column j or column $j+1$, then obviously P_1 and P_2 are disjoint. If e_2 has jump number 0 and its ends are both in column j or both in column $j+1$, then again it is easy to see that P_1 and P_2 are disjoint except possibly at their ends. So we may assume that e_2 has jump number 1 and has an end in column j or $j+1$. Then either (i) R_{e_1} contains R_{e_2} or (ii) $R(e_2)$ contains $R(e_1)$ or (iii) the interiors of R_{e_1} and R_{e_2} are disjoint. In case (i) P_2 also jumps from column j to column $j+1$. Moreover $t_j^+(e_1) = t_{j+1}^-(e_1) > t_j^+(e_2) = t_{j+1}^-(e_2)$ guarantees that P_1 and P_2 are disjoint except possibly at their ends. A similar argument works when (ii) occurs. In case (iii), P_2 connects column $j-1$ to column j or connects column $j+1$ to column $j+2$. In neither case there is an intersection of P_1 and P_2 (except possibly at their ends).

When embedding U-edges with jump number at least 1, we need to use x-steps to construct a lattice path connecting these columns. The following concept is useful for making sure that these x-steps do not cause intersections among these lattice paths. To each U-edge e of G, we assign an integer $Y(e)$ between 0 and $4k-1$, called the *entrance index* of e such that $Y(e_1) \neq Y(e_2)$ if e_1 and e_2 start in the same column or end in the same column. We point out here that most x-steps that we will use in the embedding of e will be in the plane $y = Y(e)$.

The following lemma assures the existence of such a function Y.

4.9. Lemma. *There exists a function Y from the set of all U-edges of G to the set $W = \{0, 1, 2, \ldots, 4k - 1\}$ such that $Y(e_1) \neq Y(e_2)$ for any distinct U-edges e_1 and e_2 which start or end in the same column.*

Proof. We define Y in a doubly recursive way.

First, we define Y for the U-edges connecting column 0 and column 1. Let $e = v_q v_r$ where v_q is in column 0 and v_r is in column 1. Then we define $Y(e) = y(v_q)$ if e is adjacent to $v_{q-1}v_q$ in G, and let $Y(e) = y(v_q^*)$ if e is opposite to $v_{q-1}v_q$ in G. (Recall that $y(v_q)$ is the y-coordinate of v_q.) It is easy to see that for distinct U-edges e_1 and e_2 jumping from column 0 to column 1, $Y(e_1) \neq Y(e_2)$.

Assume that for some $j \in \{1, \ldots, k-2\}$, Y has been defined for all U-edges connecting two distinct columns lower than $j+1$ such that $Y(e_1) \neq Y(e_2)$ for any distinct U-edges e_1 and e_2 which start or end in the same column. We need to extend the definition of Y to all edges ending in column $j+1$ so that the condition in the lemma still holds.

This is done recursively. For each $m \in \{0, \ldots, j\}$, let J_m be the set of entrance indices already used by edges that start in column m.

For edges starting in column 1 and ending in column $j+1$, assign different Y values to them from the set $W \backslash J_0$ (in a rather arbitrary way). Since J_0 has at most $2k$ elements and W has $4k$ elements, this can be done without a problem. Call the collection of these newly assigned indices $I_{0,j+1}$.

Now assume that for some $i \in \{0, \ldots, j-1\}$, Y has been defined for all U-edges starting in columns lower than $i+1$ and ending in column $j+1$ such that $Y(e_1) \neq Y(e_2)$ for any two distinct U-edges e_1 and e_2 that connect distinct columns lower than $j+1$ or that connect some column lower than $i+1$ to column $j+1$. For $s = 1, \ldots, i$, let $I_{s,j+1}$ denote the set of indices assigned to edges connecting column s and column $j+1$. Since there are at most $2k$ edges of G jumping to column $j+1$, $\sum_{s=1}^{i} |I_{s,j+1}| \leqslant 2k$.

Next we define Y for edges connecting column $i+1$ to column $j+1$. Note that $|J_{i+1}|$ is the number of U-edges that start in column $i+1$ and end in some column lower than $j+1$. Since there are at most $2k$ edges jumping from column $i+1$, there are at most $2k - |J_{i+1}|$ edges connecting column $i+1$ to column $j+1$. On the other hand, the indices available are in $W - (\bigcup_{s=1}^{i} I_{s,j+1})$ whose size is $|W| - \sum_{s=1}^{i} |I_{s,j+1}| \geqslant 2k \geqslant 2k - |J_{i+1}|$. Therefore, we can define Y for U-edges connecting column $i+1$ to column $j+1$ by arbitrarily assigning distinct numbers in $W - (\bigcup_{s=1}^{i} I_{s,j+1})$ to them.

Continuing this process in the order $i = 0, \ldots, j-1$, we can define Y for all edges ending in column $j+1$.

Now continuing the whole process in the order $j = 1, \ldots, k-1$, we can define Y for all U-edges. $\square$

4.10. *The embedding of U-edges with jump number at least* 2 *between the planes* $z = 0$ *and* $z = 3k - 1$.

Let $e = v_r v_s$ be a U-edge with v_r in column i and v_s in column j such that $j \geqslant i + 2$. First, assume that e is adjacent to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G. Then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed:

(1) $t_i^+(e)$ z^+-steps starting from v_r,
(2) one x^+-steps,
(3) minimum number of y-steps to reach a point with y-coordinate equal to $Y(e)$,
(4) one x^+-step,
(5) $2k + J(e) - t_i^+(e)$ z^+-steps,
(6) $3J(e) - 4$ x^+-steps,
(7) $2k + J(e) - t_j^-(e)$ z^--steps,
(8) one x^+-step,
(9) minimum number of y-steps to reach a point with y-coordinate equal to $y(v_s)$,
(10) one x^+-step, and
(11) $t_j^-(e)$ z^--steps to v_s.

This embedding process is illustrated in Fig. 17(a). Note that the steps (4)–(8) are carried out in the plane $y = Y(e)$.

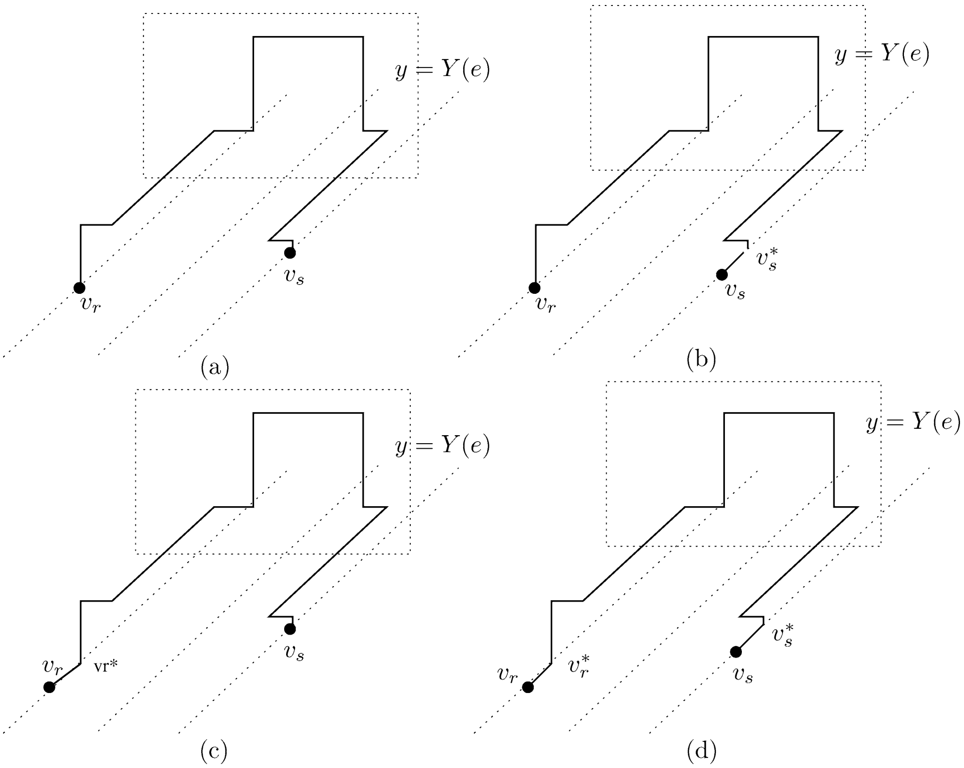

Fig. 17. The embedding of edges with jump number $\geqslant 2$.

If e is adjacent to $v_{r-1}v_r$ and opposite to $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (1)–(11) as above with v_s^* replacing v_s, and (12) one y-step from v_s^* to v_s. See Fig. 17(b).

If e is opposite to $v_{r-1}v_r$ and adjacent to $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (0) one y-step from v_r to v_r^*, and (1)–(11) as above with v_r^* replacing v_r. See Fig. 17(c).

If e is opposite to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G, then we embed the edge e onto the lattice path from v_r to v_s obtained by taking the following steps in the order listed: (0) a y-step from v_r to v_r^*, (1)–(11) as above with v_r^*, v_s^* replacing v_r, v_s, respectively, and (12) a y-step from v_s^* to v_s. See Fig. 17(d).

Since $t_j^+(e) \leqslant 2k$ and $t_j^-(e) \leqslant 2k$ and because $J(e) \leqslant k-1$, the embedding is between the planes $z=0$ and $z=3k-1$. Moreover, edges from column i to column j are embedded between the planes $x=3i$ and $x=3j$. Also note that the lattice path representing e uses at most

- 6 lattice edges for steps (0), (2), (4), (8), (10) and (12) of the construction,
- $3k-1$ lattice edges for steps (1) and (5) of the construction,
- $3k-1$ lattice edges for steps (7) and (11) of the construction,
- $8k$ lattice edges for steps (3) and (9) of the construction,
- $3k-7$ lattice edges for step (6) of the construction.

Thus the lattice path representing e contains at most $17k-3$ lattice edges. This completes the embedding of U-edges.

Remark. The up-to-date F consists of the embedding of C and all lattice paths representing all U-edges. We claim that F is a lattice graph with vertex set $V(F)=V(G)$. We need to show that any two lattice paths representing distinct edges of G are disjoint except possibly at their ends. This is done in the following lemma.

4.11. Lemma. *Let e_1 and e_2 be distinct edges which are not B-edges, and let P_1, P_2 be the lattice paths onto which e_1, e_2 are embedded respectively as constructed above. Then P_1 and P_2 do not intersect except possibly at their ends.*

Proof. By the remark following 4.8, we may assume that e_1 is a U-edge with $J(e_1) \geqslant 2$. Let $e_1 = v_r v_s$ and $e_2 = v_{r'}v_{s'}$, and assume that v_r is in column i, v_s is in column j, and $j-i \geqslant 2$.

Case 1. e_2 is an edge of C.

Then P_2 is between $z=-1$ and $z=0$ or between $z=1$ and $z=0$; all x-steps of P_2 are in the plane $z=0$; and every y-step of P_2 uses a constant x-coordinate which is a multiple of 3. Note that P_1 is in the half-space $z \geqslant 0$ and all x-steps are taken in the half-space $z \geqslant 1$. Any y-step of P_1 uses a constant x-coordinator which is not a multiple of 3, except it corresponds to some $v_\ell v_\ell^*$. Hence, P_1 and P_2 are disjoint except possibly at their ends.

Case 2. e_2 is a U-edge with $J(e_2)=0$.

By the planarity of G, either (i) $R_{e_2} \subset R_{e_1}$ or (ii) the interiors of R_{e_2} and R_{e_1} are disjoint. Since that P_2 does not use any x-step, P_2 lies entirely in a plane $x = x_0$. First, assume that (i) occurs. Then $3i \leqslant x_0 \leqslant 3j$. If $3i < x_0 < 3j$ then because $t(e_2) \leqslant k+1$, steps (5)–(7) in the construction of P_1 guarantee that P_1 does not intersect P_2 (P_1 jumps over P_2). If $x_0 = 3i$ or $x_0 = 3j$ then P_2 is contained entirely in the one side of the plane $x = x_0$ divided by the line containing the intersection of P_1 with the plane $x = x_0$ (which is a single line segment). So P_1 and P_2 do not intersect except possibly at their ends. Now assume that (ii) occurs. Then either $x_0 \leqslant 3i$ or $x_0 \geqslant 3j$. If $x_0 = 3i$ or $x_0 = 3j$, then the same argument as for (i) applies. If $x_0 < 3i$ or $x_0 > 3j$, then P_1 and P_2 can be separated by a plane $x = 3i - \epsilon$ or $x = 3j + \epsilon$ for some small $\epsilon > 0$, and hence, P_1 and P_2 are disjoint.

Case 3. e_2 a U-edge with $J(e_2) = 1$.

As before we have either (i) $R_{e_2} \subset R_{e_1}$ or (ii) the interiors of R_{e_2} and R_{e_1} are disjoint. Note that by 4.8, P_2 is contained entirely between the two planes $x = 3q$ and $x = 3(q+1)$ for some integer $q \geqslant 0$. Also note that (i) implies $3i \leqslant 3q$ and $3(q+1) \leqslant 3j$, and (ii) implies $3j \leqslant 3q$ or $3(q+1) \leqslant 3i$. Now a similar argument as used in Case 2 can be applied to show that P_1 and P_2 are disjoint except possibly at their ends.

Case 4. e_2 is a U-edge with $J(e_2) \geqslant 2$.

We may assume that e_2 jumps from column i' to column j' with $j' - i' \geqslant 2$. By planarity and by symmetry, we have the following possibilities: $i \neq i'$ and $j \neq j'$; $i = i'$ and $j = j'$; $i = i'$ and $j < j'$; or $i' < i$ and $j = j'$.

Subcase 4(a). $i \neq i'$ and $j \neq j'$.

By symmetry, let $j \leqslant j'$. If $j < i'$, then P_1 and P_2 are separated by the plane $x = 3j + 1$, and so, P_1 and P_2 are disjoint.

Now assume that $j = i'$. Then P_1 lies in the half space $x \leqslant 3i'$ and P_2 lies in the half space $x \geqslant 3i'$. The only part of P_1 that intersects the plane $x = 3i'$ is a line segment consisting of only z-steps on top of some v_ℓ (or the y-step $v_\ell v_\ell^*$ and z-steps on top of v_ℓ^*), and is not used in P_2 (except v_ℓ). So P_1 and P_2 are disjoint except possibly at their ends.

We may therefore assume that $i' < j < j'$. By planarity of G, $i' \leqslant i < j < j'$. If $i' < i$ then P_1 and P_2 are separated by the surface $S = S_1 \cup S_2 \cup S_3$ defined by $S_1 = \{(x, y, z)\colon z = 2k + J(e_2) - 0.5, 3i - 0.5 \leqslant x \leqslant 3j + 0.5\}$, $S_2 = \{(x, y, z)\colon\ z \leqslant 2k + J(e_2) - 0.5, x = 3i - 0.5\}$, $S_3 = \{(x, y, z)\colon\ z \leqslant 2k + J(e_2) - 0.5, x = 3j + 0.5\}$. So P_1 and P_2 are disjoint.

Subcase 4(b). $i' = i$ and $j = j'$.

Therefore, $e_1, e_2 \in E_i^+ \cap E_j^-$. So by planarity, we may assume $t_i^+(e_2) > t_i^+(e_1)$ and $t_j^-(e_2) > t_j^-(e_1)$. Furthermore, assume that $e_1 = v_r v_s$ and $e_2 = v_{r'} v_{s'}$.

Let us follow the path P_1 as defined in 4.10 and show that none of the embedding steps leads to an intersection with P_2. We only state the argument for the 11 steps when e_1 is adjacent to both $v_{r-1}v_r$ and $v_{s-1}v_s$ in G and e_2 is adjacent to both $v_{r'-1}v_{r'}$ and $v_{s'-1}v_{s'}$ in G. The other cases are the same with v_r^* or v_s^* replacing v_r or v_s, respectively, and adding steps (0) or (12) which will not cause intersections because $y(v_r)$, $y(v_r^*)$, $y(v_{r\prime})$, and $y(v_{r\prime}^*)$ are all distinct.

(1) P_1 moves from $(3i, y(v_r), 0)$ to $(3i, y(v_r), t_i^+(e_1))$ using only z-steps. The only piece of P_2 in the plane $x = 3i$ is the line segment from $(3i, y(v_{r'}), 0)$ to $(3i, y(v_{r'}), t_i^+(e_2))$ using only z-steps. Since $y(v_r)$ and $y(v_{r'})$ are distinct, there is no intersection.

(2)–(4) P_1 moves from $(3i, y(v_r), t_i^+(e_1))$ to $(3i + 1, y(v_r), t_i^+(e_1))$ using one x^+-step, then to $(3i + 1, Y(e_1), t_i^+(e_1))$ using y-steps, and then to $(3i + 2, Y(e_1), t_i^+(e_1))$ using a single x^+-step. All these steps occur in the space defined by $A = \{(x, y, z)\colon 3i < x < 3i + 2\}$ (with the exception of the endpoints). The only pieces of P_2 in A are also the three line segments generated by steps (2)–(4) moving from $(3i, y(v_{r'}), t_i^+(e_2))$ to $(3i + 2, Y(e_2), t_i^+(e_2))$. Since $t_i^+(e_1) \neq t_i^+(e_2)$ there is no intersection.

(5)–(7) P_1 moves from $(3i + 2, Y(e_1), t_i^+(e_1))$ to $(3i + 2, Y(e_1), 2k + J(e_1))$ using z^+-steps, then to $(3j - 2, Y(e_1), 2k + J(e_1))$ using x^+-steps, and then to $(3j - 2, Y(e_1), t_j^-(e_1))$ using z^--steps. All these steps occur in the space defined by $A = \{(x, y, z)\colon 3i + 2 \leqslant x \leqslant 3j - 2\}$. The only pieces of P_2 in A are also the three line segments generated by steps (5)–(7) moving from $(3i + 2, Y(e_2), t_i^+(e_2))$ to $(3j - 2, Y(e_2), t_j^-(e_2))$. Since $Y(e_1) \neq Y(e_2)$ there is no intersection.

(8)–(10) The argument is similar to the one given for steps (2)–(4).

(11) The argument is similar to the one given for (1).

Hence, P_1 and P_2 are disjoint except possibly at their ends.

We have two cases remaining: $i = i'$ and $j < j'$, and $i' < i$ and $j = j'$. These two cases can be taken care of in the same way as for Subcase 2. We omit the details. □

4.12. *The embedding of B-edges in the half space $z \leqslant 0$.*

The embedding of B-edges can be done in the same way as for U-edges: We repeat 4.6–4.10 with B-edges replacing U-edges, and z^+-step (respectively, z^--step) replacing z^--step (respectively, z^+-step).

Remark. By using similar arguments for U-edges as we used for the B-edges, we can show that the lattice paths representing B-edges and edges on C are disjoint except at their ends. Now assume that e_1 is a U-edge and e_2 is a B-edge. By our embedding scheme, P_1 is in the half space $z \geqslant 0$ and all x-steps are taken in $z \geqslant 1$, and P_2 is in $z \leqslant 0$ and all x-steps are taken in $z \leqslant -1$. Moreover the only y-steps of P_1 and P_2 in the plane $z = 0$ correspond to the single y-steps of type $v_\ell v_\ell^*$. Hence, P_1 and P_2 are disjoint except possibly at their ends.

Thus, the output of this algorithm is a lattice graph F which is an embedding of G in the cubic lattice. The following theorem asserts that F is ambient isotopic to G.

4.13. Theorem. *The lattice graph F is ambient isotopic to G.*

Proof. It is easy to see that we can isotope the lattice paths representing the edges of C to the plane $z = 0$ by their projection in the z-direction.

Next we need to deform the lattice paths representing U-edges of G one at a time until it coincides with the corresponding U-edges of G. First, we deform the lattice paths onto which the jump 0 U-edges are embedded: starting with the U-edges of the lowest level, and increasing the level by one at a time. If two U-edges are of the same level, then it does not matter which corresponding lattice path will be deformed first into the plane $z = 0$. Then we deform the lattice paths representing U-edges in E_j^- in the order $j = 2, \dots, k-1$: starting with the U-edges of the lowest t_j^--value. For each lattice path to deform the t_j^--value of the corresponding U-edge increases by one until E_j^- is exhausted. When that happens we move to E_{j+1}^-. The critical observation is that this can be done at any stage, since if we are looking at a lattice path P whose corresponding edge has the lowest level-value or t_j^--value in the remaining lattice paths that have not been isotoped back to G, then there are no lattice paths of F "between" P and the plane $z = 0$. That is, the deformation of the lattice path into the edge of G will encounter no obstruction from another lattice path of F that has not already been isotoped back in the plane $z = 0$. The isotopy of U-edges can be done entirely in the half-space $z \geqslant 0$.

Finally we deform the B-edges in a similar way, entirely in the half-space $z \leqslant 0$. □

By 4.5, 4.7, 4.8, and 4.10, the length of each lattice path so constructed is bounded above by $17k - 3$. Since there are a total of n U-edges and B-edges, the total number of lattice edges to embed the U-edges and B-edges is bounded above by $(17k-3)n$. It follows that the length of F is bounded above by

$$17nk + 2n + 11k = 17n\lceil\sqrt{n}\,\rceil + 2n + 11\lceil\sqrt{n}\,\rceil < 17n^{3/2} + 19n + 11\sqrt{n} + 11.$$

Or, if one prefers a simpler form, we may bound this by $18n^{3/2} + 13n$ for $n \geqslant 50$. Thus, we have the following theorem.

4.14. Theorem. *Let G be a Hamiltonian RP-graph with n vertices, then G can be embedded onto a lattice graph F such that $L(F) \leqslant 17n^{3/2} + 19n + 11\sqrt{n} + 11$.*

5. Rope length of knots and links

We can now use the embedding of Hamiltonian RP-graphs to prove results about the rope length of knot. First we start with lattice knots.

5.1. Theorem. *Let K be a knot or a link, and assume that K has a Hamiltonian RP-graph G with n vertices. Then we can embed K into the cubic lattice with a total length at most $17n^{3/2} + 21n + 11\sqrt{n} + 11$.*

Proof. By applying Theorem 4.13, we can embed G onto a lattice graph F. For a vertex v_G of G let v be the corresponding vertex of F. The opposite edges of G at v_G are represented by lattice paths using opposite lattice edges at v which correspond to y-steps and z-steps. Let vv_1, vv_2, vv_3, vv_4 denote the lattice edges contained in F at v such that vv_1 corresponds to a y^--step from v, vv_2 corresponds to a z^+-step from v, vv_3 corresponds to a y^+-step from v, vv_4 corresponds to a z^--step from v. See Fig. 18(a). Then the following lattice paths are not used by F except for their ends: lattice path L_v from v_1 to v_3 obtained

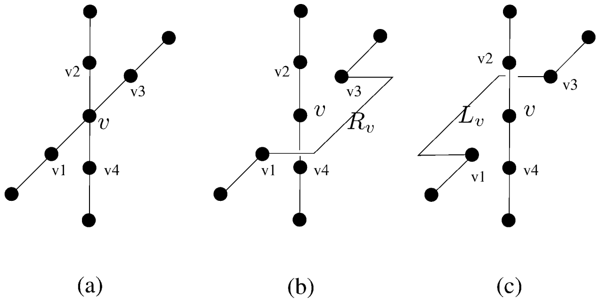

Fig. 18. Changing F into the embedding of a knot or link.

by taking one x^--step, two y^+-steps, and one x^+-step (see Fig. 18(b); and lattice path R_v from v_1 to v_3 obtained by taking one x^+-step, two y^+-steps, and one x^--step, see Fig. 18(c).

We fix an orientation in the projection to keep track of the under and over crossings. At each vertex v of F that corresponds to a vertex of G, we replace the lattice path v_1vv_3 by L_v, or by R_v depending on whether v_1v_3 is an under-strand or over-strand in the projection of the knot K as shown in Fig. 18.

Therefore, at each crossing two additional x-steps are needed and the total length of the lattice embedding increases by $2n$ when the lattice graph F is changed into a lattice embedding of the knot or link K. $\square$

By Theorem 5.1, we get the following theorem by simply substituting $4 \cdot Cr(K)$ for n.

5.2. Theorem. *Let K be a knot or link. If K is minimally Hamiltonian then we can embed K into the cubic lattice with a length at most* $17(Cr(K))^{3/2} + 21Cr(K) + 11\sqrt{Cr(K)} + 11$.

By Theorems 5.1 and 3.6, we have the following theorem.

5.3. Theorem. *Let K be a knot or link. Then K can be embedded into the cubic lattice with length at most* $136(Cr(K))^{3/2} + 84Cr(K) + 22\sqrt{Cr(K)} + 11$.

Since a lattice knot or link can be changed into a $C^{1,1}$ knot or link of thickness $1/2$ (by replacing the corners where the knot makes turns with suitable quarter circles of radius $1/2$), we have the following:

5.4. Theorem. *Let K be a knot or a link. Then the rope length of K is bounded above by* $34(Cr(K))^{3/2} + 42Cr(K) + 22\sqrt{Cr(K)} + 22$ *if K is minimally Hamiltonian. Otherwise the rope length of K is bounded above by* $272(Cr(K))^{3/2} + 168Cr(K) + 44\sqrt{Cr(K)} + 22$.

6. Further discussions and questions

The main result in this paper is that the rope-length of a knot K is bounded above by $c \cdot (Cr(K))^{3/2}$ for some constant $c > 0$. There is apparently ample room left for the improvement of the constant we obtained here. However, a more important issue is whether one can improve the power $3/2$. We know that there exist a constant $a > 0$ and an infinite family of knots such that the rope-length of each knot K in that family is bounded below by $aCr(K)$ for some constant $a > 0$. What happens between the power 1 and $3/2$? It is apparent from our embedding algorithm that the length of the embedded knot depends on the levels of the edges in $G \backslash C$ where G is a Hamiltonian projection of K (with at most $4 \cdot Cr(K)$ vertices) and C is a Hamilton cycle in G. In fact, we have the following

6.1. Theorem. *Let* $\{K_n\}$ *be a family of knots* (*or links*). *If there exists a constant* $m > 0$ *such that each* K_n *in this family admits a Hamiltonian projection* G_n *with at most* $m \cdot Cr(K_n)$ *vertices and a Hamilton cycle* C_n *such that every edge in* $G_n \backslash C_n$ *has a level number at most* m *with at most* m *exceptions, then there exists a constant* $c > 0$ *such that* $L(K_n) \leqslant c \cdot Cr(K_n)$ *for every* K_n *in this family.*

If the answer to Problem 1.2 is negative, then there must exist an infinite family of knots $\{K_n\}$ such that the condition in the above theorem fails to hold. Since the nature of Problem 1.2 calls for explicit construction of thick knots, we do not have many options other than designing efficient embedding algorithms on the cubic lattice. To this extent, the study of Hamilton cycles in G lends us a rather powerful tool. Many questions can be raised in this regard. For instance, one may ask what kind of knots have projections that would satisfy the condition in the above theorem. One may also explore the possibility of changing the known projections of a family of knots so the new projections would then satisfy the condition of the theorem.

We conclude this paper with the following open questions.

6.2. Question. *Is it true that* $\sup\{L(K)/Cr(K)\} = \infty$ (*where the supremum is taken over all knots and links*)?

6.3. Question. *For any* $1 < p \leqslant 3/2$, *are there a constant* $a > 0$ *and an infinite family of knots and links such that for any member* K *in the family,* $L(K) \geqslant a \cdot (Cr(K))^p$?

6.4. Question. *Is it possible to improve the embedding algorithm in Section* 4 *to give an upper bound* $\mathrm{O}((Cr(K))^p)$ *for some constant* $1 \leqslant p < 3/2$?

None of the questions seems easy to solve, but the authors feel that improvements over the embedding algorithm are most promising and Question 3 above may have an affirmative answer.

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