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Planar k-cycle resonant graphs with $k = 1, 2$ ☆

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Abstract

A connected graph is said to be k-cycle resonant if, for $1 \leqslant t \leqslant k$, any t disjoint cycles in G are mutually resonant, that is, there is a perfect matching M of G such that each of the t cycles is an M-alternating cycle. The concept of k-cycle resonant graphs was introduced by the present authors in 1994. Some necessary and sufficient conditions for a graph to be k-cycle resonant were also given. In this paper, we improve the proof of the necessary and sufficient conditions for a graph to be k-cycle resonant, and further investigate planar k-cycle resonant graphs with $k = 1, 2$. Some new necessary and sufficient conditions for a planar graph to be 1-cycle resonant and 2-cycle resonant are established.

Keywords: k-cycle resonant graphs; Planar k-cycle resonant graphs; Bridges; Attachment vertices; Cycle-related vertices

1. Introduction

In the topological theory of benzenoid hydrocarbons, a hexagonal system (or benzenoid system) denotes the carbon atom skeleton graph of a benzenoid hydrocarbon, which is a 2-connected plane graph with a plane embedding such that every interior face is bounded by a regular hexagon. A Kekule structure K of a hexagonal system H is also a perfect matching of H. An edge in H is said to be a K-double bond if it belongs to K, otherwise a K-single bond. An edge in H is said to be

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a fixed double (resp, single) bond if it belongs to (resp. does not belong to) every Kekule structure of H. A hexagonal system is said to be normal if it contains no fixed bond. A cycle (or circuit) C in H is said to be conjugated or resonant if there is a Kekule structure K of H such that C is a K-alternating cycle. In the conjugated circuit model [4,5,11,13–15,18,19,28,29,31–34,36–42] conjugated circuits with different sizes have different resonance energies. If the size of a conjugated circuit is equal to $4n+2$, then the smaller the n the larger the resonant energy. So the conjugated hexagon has the largest energy. On the other hand, from a purely empirical standpoint, Clar found that various electronic properties of polycyclic aromatic hydrocarbons can be predicted by appropriately defining an aromatic sextet for their Kekule structures [2,3,6,7,9,10,17,20,30,44,45]. According to Clar's aromatic sextet theory, the Clar formula of a polyhex G (the molecule model of a polycyclic aromatic hydrocarbon) is a set of mutually resonant sextets with the maximum cardinal number, where sextets mean resonant hexagons and a set of mutually resonant sextets means a set of disjoint hexagons for which there is a Kekule structure K so that all of the disjoint hexagons are K-alternating hexagons. In Ref. [43], Zhang Fuji and Chen Rongsi investigated 1-coverable hexagonal systems, each hexagon of which is resonant. A hexagonal system is said to be k-coverable if any k disjoint hexagons of it are mutually resonant. Zheng Maolin [46] first introduced the concept of k-coverable hexagonal systems and investigated their properties and construction. Some necessary and sufficient conditions for a hexagonal system to be k-coverable were given. As a natural generalization of k-coverable hexagonal systems, Guo Xiaofeng and Zhang Fuji introduced the concept of k-cycle resonant graphs [8], and investigated some of their properties. Some simple necessary and sufficient conditions for a graph to be k-cycle resonant were given. In particular, the construction of k-cycle resonant hexagonal systems was completely characterized. It was also shown that in the hexagonal systems with h hexagons obtained from a common parent hexagonal system with h-1 hexagons, k^*-cycle resonant systems have greater resonance energies than 1-cycle resonant systems for $k^*>1$, where k^* is the maximum number of disjoint cycles. Also 1-cycle resonant systems sharing a similar common parent have greater resonance energies than hexagonal systems, which are not 1-cycle resonant. Harary et al. [12] investigated the hexagonal systems all of whose hexagons are simultaneously resonant, that is, there is a perfect matching M in such a hexagonal system H that every hexagon of H is an M-alternating hexagon. It is interesting that the hexagonal systems with all hexagons being simultaneously resonant are also k-cycle resonant, and vice versa, although they have different definitions.

In the present paper, the authors improve the proof of the necessary and sufficient conditions for a graph to be k-cycle resonant, and further investigate general planar k-cycle resonant graphs with $k=1,2$. Some new necessary and sufficient conditions for a graph to be planar 1-cycle resonant graphs or planar 2-cycle resonant graphs are established.

In the investigation of matching theory, Lovasz et al. [1,16,21–27,35] introduced and investigated elementary graphs, 1-extendable graphs, and n-extendable graphs, etc. A graph G is said to be n-extendable if any n independent edges of G are contained in some perfect matching of G. We can similarly call k-cycle resonant graphs as k-cycle extendable graphs. It was proved in Ref. [43] that a hexagonal system H is normal if and only if every hexagon of H is resonant. So 1-coverable hexagonal systems are also 1-extendable and elementary (a graph is said to be elementary if its non-fixed bonds induce a connected spanning subgraph).

2. Some related results of k-cycle resonant graphs

Definition 1. A graph G is said to be k-cycle resonant or k-cycle extendable if G contains at least k $(\geqslant 1)$ disjoint cycles and, for $1 \leqslant t \leqslant k$, any t disjoint cycles in G are mutually resonant, that is, there is a Kekule structure K of G such that the t disjoint cycles are K-alternating cycles.

The following theorems were given in Ref. [8].

Theorem A (Guo Xiaofeng and Zhang Fuji [8]). *Let G be a k-cycle resonant graph. Then*

(1) *G is bipartite.*
(2) *For $1 \leqslant t \leqslant k$ and any t disjoint cycles $C_1, C_2, \ldots, C_t$ in G, $G - \bigcup_{i=1}^{t} C_i$ contains no odd component.*
(3) *Any two 2-connected blocks in G have no common vertex.*

A **block** of a connected graph G is either a maximal 2-connected subgraph of G or a cut edge of G, and a 2-connected block of G is a maximal 2-connected subgraph of G. Two 2-connected blocks in a connected component of G have at most one common vertex which must be a cut vertex of G.

Theorem B (Guo Xiaofeng and Zhang Fuji [8]). *Let G be a k-cycle resonant graph. Then G is elementary or 1-extendable if and only if G is 2-connected.*

A path P in a graph G is said to be a *chain* if all internal vertices of P are of degree 2 in G and the degree of any end vertex of P is not equal to two in G. A hexagonal system is said to be a catacondensed hexagonal system if every vertex of it lies on the boundary.

Theorem C (Guo Xiaofeng and Zhang Fuji [8]). *A hexagonal system H is k^*-cycle resonant if and only if H is a catacondensed hexagonal system with no chain of even length, where k^* is the maximum number of disjoint cycles in H.*

Theorem 3.1 in Ref. [8] gave some sufficient and necessary conditions for a graph to be k-cycle resonant: "A connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is bipartite and, for $1 \leqslant t \leqslant k$ and any t disjoint cycles $C_1, C_2, \ldots, C_t$ in G, $G - \bigcup_{i=1}^{t} C_i$ contains no odd component."

However, the theorem has a negligence. In fact, the sufficient and necessary conditions are valid if G is 2-connected or G has a perfect matching. A referee (Klein) of

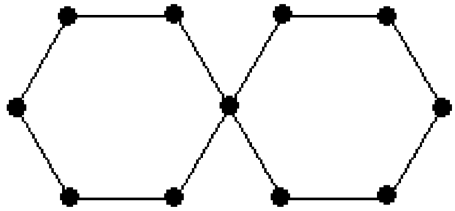

Fig. 1. Graph G consists of two cycles C_1 and C_2 with a common vertex v, where C_1 and C_2 are two 2-connected blocks of G.

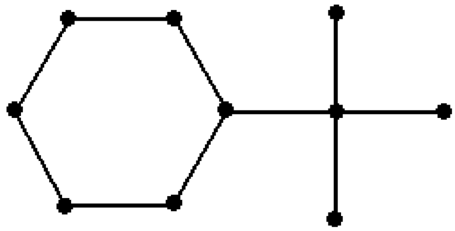

Fig. 2.

the present paper gave an example to show that a graph G with no perfect matching, which is not 2-connected, satisfies the conditions of the theorem but it is not 1-cycle resonant (see Figs. 1 and 2).

Therefore, the theorem given in Ref. [8] should be revised as follows.

Theorem D. *A 2-connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is bipartite and, for* $1 \leqslant t \leqslant k$ *and any t disjoint cycles* $C_1, C_2, \ldots, C_t$ *in G,* $G - \bigcup_{i=1}^{t} C_i$ *contains no odd component.*

The proof of Theorem D is the same as the proof of Theorem 3.1 given in Ref. [8]. In that proof, the condition that "G is 2-connected" had been implicitly used.

For general cases, we have the following.

Theorem E. *A connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is a bipartite graph with perfect matchings and, for* $1 \leqslant t \leqslant k$ *and any t disjoint cycles* $C_1, C_2, \ldots, C_t$ in G, $G - \bigcup_{i=1}^{t} C_i$ *contains no odd component.*

Proof. The necessity is evident. We need only to prove the sufficiency.

Suppose that G is a bipartite graph with perfect matchings and, for $1 \leqslant t \leqslant k$ and any t disjoint cycles $C_1, C_2, \ldots, C_t$ in G, $G - \bigcup_{i=1}^{t} C_i$ contains no odd component. If G is 2-connected, by Theorem D, then G is k-cycle resonant. Hence we assume that G is not 2-connected and $G_1, G_2, \ldots, G_s$ are 2-connected blocks of G.

Claim 1. *Any two* 2*-connected blocks of* G *have no common vertex.*

By Theorem A (3), Claim 1 holds immediately.

Claim 2. *All the edges not belonging to any* 2*-connected block of* G *but having an end vertex in a* 2*-connected block of* G *are fixed single bonds.*

Let e be such an edge with an end vertex v being in a 2-connected block G_i of G. Let C_i be a cycle in G_i containing v. Then the connected component of G-C_i containing the end vertex of e other than v must be an even component. This means that e does not belong to any perfect matching of G, that is, e is a fixed single bond.

Claim 3. *The forest* $G-\bigcup_{i=1}^{s} V(G_i)$ *has a unique perfect matching which is contained in every perfect matching of* G *and so are all fixed double bonds.*

Obviously.

Claim 4. $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$ *has no odd component*, *where* $1 \leqslant t \leqslant k$ *and* C_j, $j = 1,2,\ldots,t$, *are disjoint cycles in* G.

Suppose that $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$ has odd components. Let H^* be the component of $G - \bigcup_{j=1}^{t} V(C_j)$, which is an even component and consists of some odd components in $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$, say $H_1, H_2, \ldots$, and some even components in $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$, and some components in $G - \bigcup_{i=1}^{s} V(G_i)$. Since the forest $G - \bigcup_{i=1}^{s} V(G_i)$ has a unique perfect matching, every component of it is also even. Hence, the even component H^* will contain an even number of odd components of $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$. There is a cut edge e of H^* such that H^*-e has a component, say H_1^*, containing only one odd component of $\bigcup_{i=1}^{s} G_i - \bigcup_{j=1}^{t} V(C_j)$, say H_1, and e has one end vertex v in H_1. Let $H_2^* = H^* - V(H_1^*)$. Then both H_1^* and H_2^* are odd components of H^*-e. Note that e is also a cut edge of G. Let C_0' be a cycle in G containing v, let H' be the component of $G - V(C_0')$ containing H_2, and let $C_1', C_2', \ldots, C_s'$ be the cycles in $\bigcup_{i=1}^{t} C_i$ contained in H'. Then H_2^* is also an odd component of $G - \bigcup_{j=0}^{s} V(C_j')$. This contradicts our assumption.

Claim 5. *For any* 2*-connected block* G_i *in* G *with the maximum number* k_i^* *of disjoint cycles*, *if* $k_i^* < k$, G_i *is* k^**-cycle resonant*, *otherwise* G_i *is* k*-cycle resonant*.

The conclusion of Claim 5 holds by Claim 4 and Theorem D.

From Claim 5, we have that $\bigcup_{i=1}^{s} G_i$ is k-cycle resonant, that is, for $1 \leqslant t \leqslant k$ and any t disjoint cycles in G, there is a perfect matching M^* of $\bigcup_{i=1}^{s} G_i$ such that each of the t cycles is an M^*-alternating cycle. The union of M^* and the unique perfect matching of $G - \bigcup_{i=1}^{s} V(G_i)$ is just a perfect matching of G. This means that G is also k-cycle resonant. $\square$

Based on Theorem B, Theorem E, and Claims 3, and 5, we have the following.

Theorem F. *Let G be a k-cycle resonant graph. Then*,

(i) *for a 2-connected block G' of G with the maximum number k^* of disjoint cycles, if $k^* \leqslant k$, G' is k^*-cycle resonant, otherwise G' is k-cycle resonant*;
(ii) *the forest induced by all the vertices of G not in any 2-connected block of G has a unique perfect matching*.

The above theorems imply that a non-2-connected k-cycle resonant graph can be constructed from some disjoint 2-connected k^* (or k)-cycle resonant graphs and a forest with a perfect matching by adding some edges between the 2-connected graphs and the forest so that the resultant graph is connected and the added edges are cut edges. Hence, we need only to consider 2-connected k-cycle resonant graphs.

Before continuing, we give some terminology and notations.

Let G be a connected graph, and H a subgraph of G. A vertex in H is said to be an *attachment vertex* of H if it is incident with an edge in $G - E(H)$. The set of all attachment vertices of H is denoted by $V_{\mathrm{A}}(H)$. A *bridge* B of H in G is either an edge in $G - E(H)$ with two end vertices being in H, or a subgraph of G induced by all the edges in a connected component B' of $G - V(H)$ together with all the edges with an end vertex in B' and the other in H. The vertices in $V(B) \cap V(H)$ are also attachment vertices of B to H. A bridge with k attachment vertices is called a *k-bridge*.

The attachment vertices of a k-bridge B of a cycle C in G divide C into k edge-disjoint paths, called the segments of B. Two bridges of C *avoid* one another if all the attachment vertices of one bridge lie in a single segment of the other bridge, otherwise they *overlap*. Two bridges B and B^* of C are *skew* if there are four distinct vertices on C, in the cyclic order u, u^*, v, v^*, such that u and v are attachment vertices of B, u^* and v^* are attachment vertices of B^*.

As defined at the fore, a block of a connected graph G is either a maximal 2-connected subgraph of G or a cut edge of G. The *block graph* of G, denoted by $b(G)$, is the graph whose vertices are blocks of G and two vertices of $b(G)$ are adjacent if the corresponding blocks in G have a common vertex. The set of internal vertices of a chain P in G is denoted by $V_{\mathrm{I}}(P)$.

Lemma 1. *Let $G=(V,E)$ be a 2-connected graph, P a chain in G, and B a 2-connected subgraph of G with exactly two attachment vertices. Then any block of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) has exactly two attachment vertices in G, and the block graph of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) is a path.*

Proof. The attachment vertices of a block of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) are cut vertices of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) but are not cut vertices of G, since G is 2-connected. So, if a block of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) has at least three attachment vertices in G, then $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) would also have at least three attachment vertices, contradicting that $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) and B have the same attachment vertices in G. Now it follows that the block graph of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) is a path. □

For a 2-connected subgraph B in G with exactly two attachment vertices and a chain P, the blocks of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) containing an attachment vertex of B (resp. P) are called the *end blocks* of $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$). We also call $G[E - E(B)]$ (resp. $G - V_{\mathrm{I}}(P)$) the complement of B (resp. P) in G denoted by $\bar{B}$ (resp. $\bar{P}$).

The following lemma is obvious.w

Lemma 2. *Let G be a 2-connected graph, and P a chain in G. Then any end block of $\bar{P}$ is 2-connected.*

Lemma 3. *Let G be a 2-connected graph, B a bridge of a cycle C in G. Then $\bar{B}$ is 2-connected.*

Proof. $\bar{B}$ consists of the cycle C and some bridges of C. Since G is 2-connected, any bridge B_i of C has at least two attachment vertices which are not cutvertices of $\bar{B}$. Any vertex of B_i not on C is not a cut vertex of $\bar{B}$, otherwise it would be also a cut vertex of G, a contradiction. Hence $\bar{B}$ has no cut vertex, that is, $\bar{B}$ is 2-connected. □

Two paths in a graph G are said to be *internal disjoint* if any common vertex of them is an end vertex of the two paths.

For a bipartite graph G, we always colour vertices of G white and black so that any two adjacent vertices have different colours.

3. Planar 1-cycle resonant graphs

We first give several equivalent propositions.

Theorem 1. *Let G be a 2-connected bipartite planar graph. Then the following statements are equivalent*:

(i) *G is 1-cycle resonant.*
(ii) *For any cycle C in G, $G - V(C)$ has no odd component.*
(iii) *For any cycle C in G, any bridge of C has exactly two attachment vertices which have different colours.*
(iv) *For any cycle C in G, any two bridges of C avoid one another. Moreover, for any 2-connected subgraph B of G with exactly two attachment vertices, the attachment vertices of B have different colours.*

Proof. By Theorem D, (i) and (ii) are equivalent. We need only to prove statements (ii), (iii) and (iv) are also equivalent.

Since G is planar, we may assume G is a plane graph embedded in a plane.

(ii) ⇒ (iii). Let B is a bridge of C. Without loss of generality, we assume B is in the interior of C. Suppose that B has at least three attachment vertices, say v_1, v_2, v_3. Then there is a vertex u in B such that in B there are three internal-disjoint paths starting from u and terminating at v_1, v_2, v_3, respectively, say P_1, P_2, P_3. Let C_i, $i=1,2,3$, denote the cycle consisting of P_{i+1}, P_{i+2}, and the v_{i+1}-v_{i+2} segment on C not containing v_i, where if $i+1$ or $i+2$ is greater than 3 then their values are taken modulo 3. Let $C_i' = C_{i+1} \Delta C_{i+2}$, where $C_{i+1} \Delta C_{i+2}$ denote the symmetric difference of the edge sets of C_{i+1} and C_{i+2}. Let H_i, $i=1,2,3$, be the connected component of $G - C_i'$ containing an internal vertex of P_i. Then all components of $H_i - V(P_i)$ are also components of $G - C_{i+1}$ (resp. $G - C_{i+2}$). By (ii), H_i and all components of $G - C_{i+1}$ (resp. $G - C_{i+2}$) are even components, and so P_i contains an even number of vertices too. Hence $P_i \cup P_{i+1} \cup P_{i+2} - \{v_i, v_{i+1}, v_{i+2}\}$ contains an odd number of vertices and $B - V(C)$ would be an odd component of $G - C$, contradicting (ii). Now it follows that B has exactly two attachment vertices since G is 2-connected.

Moreover, we can assert that the attachment vertices, say v_1, v_2, of B have different colours. Otherwise, any $v_1 - v_2$ path in B has an odd number of vertices. Let C^* be the cycle consisting of a $v_1 - v_2$ path in B and a $v_1 - v_2$ segment on C. Then, since $B - V(C)$ is an even component of $G - C$, $B - V(C^*)$ has an odd number of vertices, and so contains an odd component which is also an odd component of $G - C^*$, again a contradiction.

(iii) $\Rightarrow$ (iv). Suppose that there are two bridges B_1 and B_2 of C which are not mutually avoided and B_2 is in the exterior of C. Let u, v (resp. u^*, v^*) be the attachment vertices of B_1 (resp. B_2). Their cyclic order on C is u, u^*, v, v^*. Let C^* be the cycle consisting of a $u^* - v^*$ path in B_2 and the $u^* - v^*$ segment on C containing u. Then a bridge of C^* containing B_1 has at least three attachment vertices, contradicting (iii).

Let B be a 2-connected subgraph of G with exactly two attachment vertices, say v_1 and v_2. Then there is a cycle C_B in B containing v_1 and v_2. Since G has no cut vertex, there is a path in $G - E(B)$ with end vertices v_1 and v_2, and so in $G - E(B)$ there is a bridge of C_B, say B', with attachment vertices v_1 and v_2. By (iii), B' has exactly two attachment vertices v_1 and v_2 which have different colours.

(iv) $\Rightarrow$ (iii). Suppose that a bridge B of C has at least three attachment vertices, say v_1, v_2, and v_3. In addition, we may assume B lies in the interior of C. Then there is a vertex u in B such that in B there are three internal-disjoint paths starting from u and terminating at v_1, v_2, v_3, respectively, say P_1, P_2, P_3. Let C^* be the cycle consisting of $P_1 \cup P_2$ and the segment on C containing v_3. Then there are two skew bridges of C^* one of which contains P_3 and the other contains the $v_1 - v_2$ segment of C not on C^*. This contradicts (iv).

Let B be a bridge of C with the attachment vertices v_1 and v_2. Then, by Lemma 3, $\bar{B}$ is a 2-connected subgraph of G with exactly two attachment vertices v_1 and v_2. So v_1 and v_2 have different colours.

(iii) $\Rightarrow$ (ii). Suppose that $G - V(C)$ has an odd component B'. In addition, we suppose B' is a minimum in all such odd components for every cycle C and $G - V(C)$. Then there is a bridge B of C with exactly two attachment vertices v_1 and v_2, which contains B' and also has an odd number of vertices. Since v_1 and v_2 have different colours, any $v_1 - v_2$ path in B, say P_B, is of odd length. So $B - V(P_B)$ contains an odd component too, say B''. Let C^* be the cycle consisting of P_B and a $v_1 - v_2$ segment on C. Clearly, B'' is also an odd component of $G-V(C^*)$ which has a smaller vertex number than B'. This contradicts the choice of B'. □

Now we can give the following necessary and sufficient conditions for a graph to be planar 1-cycle resonant.

Theorem 2. (1) *A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any bridge of C has exactly two attachment vertices which have different colours.*

(2) *A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any two bridges of C avoid one another and, for any 2-connected subgraph B of G with exactly two attachment vertices, the attachment vertices of B have different colours.*

By Theorem 1(iii) and (iv), Theorem 2(1) holds if and only if Theorem 2(2) holds. We will only prove Theorem 2(1).

Proof of Theorem 2. (1) By Theorem E and Theorem 1, the necessity is obvious. We only need to prove the sufficiency

By the famous Kuratowski's Theorem (see Theorems 9, 10 in Bondy and Murty's book "Graph Theory with Applications" [1]), a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. Suppose graph G satisfies the conditions of the theorem but is not planar. Then G contains a subdivision of K_5 or $K_{3,3}$. It is not difficult to find a cycle C in a subdivision of K_5 or $K_{3,3}$ such that a bridge B of C has at least three attachment vertices, a contradiction. Hence G is a 2-connected bipartite planar graph. By Theorem 1(i) and (iii), G is planar 1-cycle resonant. □

From Theorems 2(1) and (2) we have the following corollary, which is useful for the investigation of planar 2-cycle resonant graphs.

Corollary 1. *Let G be a 2-connected planar 1-cycle resonant graph, C a cycle of G, and B a bridge of C. Then*

(i) *B is not 2-connected,*
(ii) *every block of B has exactly two attachment vertices, and the block graph of B is a path,*
(iii) *any 2-connected subgraph of G with exactly two attachment vertices has an even number of vertices, and so does any bridge B of C,*
(iv) *for a cycle C^* in a 2-connected block B^* of B, (a) $B^*-V(C^*)$ has no odd component, (b) if C^* does not contain the attachment vertices, say v_1 and v_2, of B^*, then v_1 and v_2 are contained in a common component of $B^*-V(C^*)$, and $B-V(C^*)$ has no odd component,*
(v) *for any 2-connected subgraph B^* of G with exactly two attachment vertices v_1 and v_2, $B^*-v_1-v_2$ has no odd component.*

Proof. (i) Suppose that B is 2-connected. Let v_1 and v_2 be the attachment vertices of B. Then there is a cycle C^* in B which contains v_1 and v_2. C^* is divided as two segments by v_1 and v_2, say P_1 and P_2, each of which is not an edge (otherwise this edge is a bridge of C, a contradiction). We assert that there is no such path in $B - v_1 - v_2$ with an end vertex on $P_1 - v_1 - v_2$ and the other on $P_2 - v_1 - v_2$ which is internal disjoint with P_1 and P_2. Otherwise, let C' be the cycle consisting of P_1 and a $v_1 - v_2$ segment on C. The bridge of C' containing P_2 would have at least three attachment vertices, contradicting that G is 1-cycle resonant by Theorems 1 and 2. However, if there is no such path in $B - v_1 - v_2$ connecting P_1 and P_2, then $B - v_1 - v_2$ is not connected. This also contradicts that B is a bridge of C.

(ii) From Lemmas 3 and 1, it follows that every block of B has exactly two attachment vertices in G and the block graph of B is a path.

(iii) Let B^* be a 2-connected subgraph of G with exactly two attachment vertices. There is a cycle C in B^* containing two attachment vertices. Then each component of $B - V(C)$ is also a component of $G - V(C)$, and so is an even component. Since G is bipartite by Theorems 1 and 2, C is also an even cycle. Now it follows that B^* has an even number of vertices. Let B be a bridge of C. By Lemma 3, $\bar{B}$ is 2-connected and has exactly two attachment vertices. So $\bar{B}$ has an even number of vertices, and so does B.

(iv) Let C^* be a cycle in a 2-connected block B^* of B.

(a) Suppose that $B^* - V(C^*)$ has an odd component B_0. Then B_0 must contain an attachment vertex of B^*. Otherwise, B_0 is also an odd component of $G - V(C^*)$, contradicting that G is 1-cycle resonant. Since B^* is bipartite and has an even number of vertices by (iii), $B^* - V(C^*)$ has exactly two odd components each of which contains an attachment vertex of B^*. Let C be a cycle in B^* containing the attachment vertices v_1 and v_2, which is divided by v_1 and v_2 into two segments, say P_1 and P_2. Since $V(C^*)$ is a cut set of B^* separating v_1 and v_2, it contains vertices of both P_1 and P_2. This means that there is a path in B^* starting from an internal vertex of P_1 and terminating at an internal vertex of P_2, which is internal disjoint with P_1 and P_2. Hence, we can find a cycle C' in G containing P_1 but not containing edges of P_2, so that the bridge of C' containing P_2 would have at least three attachment vertices, contradicting that G is 1-cycle resonant.

(b) By a similar argument as in the proof of (iv)(a), we can assert that, if C^* does not contain the attachment vertices of B^*, then the attachment vertices of B^* are contained in a same component of $B^* - V(C^*)$. Hence, they are also contained in a same component of $B - V(C^*)$, say B'. Clearly, all the components of $B^* - V(C^*)$ other than B' are also components of $G - V(C^*)$, and so are even components. All the blocks of B other than B^* are contained in B'. Since B has an even number of vertices and C^* is an even cycle, B' is also an even component.

(v) By (iii), B^* has an even number of vertices. So if $B^* - v_1 - v_2$ has an odd component, it has at least two odd components, say B_1 and B_2. In $B^* - V(B_1)$ there is a $v_1 - v_2$ path P. Since G is 2-connected, $\overline{B^*}$ is connected and there is a $v_1 - v_2$ path P^* in $\overline{B^*}$. Let C^* be the cycle in G consisting of P and P^*. Then B_1 is also a component of $G - C^*$, and so is an even component, a contradiction. □

4. 2-Cycle resonant graphs

A vertex u of a graph G is said to be *cycle-related* to another vertex v of G if u is contained in a 2-connected block of G and any cycle containing u must also contain v. If v is also cycle-related to u, then u and v are *mutually cycle-related*.

Property 1. If a vertex u of a connected graph G is cycle-related to another vertex v of G, then u and v belong to a same 2-connected block B in G and all the edges in $G-v$ incident with u are cut edges of $G-v$.

Now we give the following necessary and sufficient conditions for a planar graph to be 2-cycle resonant.

Theorem 3. *A 2-connected graph G is planar 2-cycle resonant if and only if,*

(i) *G is planar 1-cycle resonant,*

(ii) *for a chain P with even length and end vertices v_1 and v_2, $G-V_{\mathrm{I}}(P)$ has exactly two blocks each of which is 2-connected and v_1 and v_2 are cycle-related to the common vertex of the two blocks,*

(iii) *for a chain P with odd length and end vertices v_1 and v_2 such that $G-V_{\mathrm{I}}(P)$ is not 2-connected, either (a) $G-V_{\mathrm{I}}(P)$ has exactly three blocks, each of which is a 2-connected, and each of v_1 and v_2 is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, or (b) any two 2-connected blocks of $G-V_{\mathrm{I}}(P)$ are disjoint,*

(iv) *for a 2-connected subgraph B_1 of G with exactly two attachment vertices, if $\bar{B}_1$ is not 2-connected and every block of $\bar{B}_1$ is 2-connected, then $\bar{B}_1$ has exactly three blocks, say B_2,B_3,B_4, and the attachment vertices of each of B_1,B_2,B_3,B_4 are mutually cycle-related in the block.*

Proof. First we address the necessity.

(i) By the definition of k-cycle resonance graphs, a 2-cycle resonant graph G is also 1-cycle resonant.

(ii) For a chain P with even length and end vertices v_1 and v_2, $G-V_{\mathrm{I}}(P)$ is not 2-connected. Otherwise, in $G-V_{\mathrm{I}}(P)$ there is a cycle C containing v_1 and v_2 such that $P-v_1-v_2$ is an odd component of $G-V(C)$, contradicting that G is 1-cycle resonant. Hence $G-V_{\mathrm{I}}(P)$ has at least two end blocks, say B_1 and B_2, which are 2-connected by Lemma 2. Assume $G-V_{\mathrm{I}}(P)$ has at least three blocks or has exactly two blocks such that v_i, $i=1$ or 2, is not cycle-related to the other attachment vertex of B_i. Then there are two disjoint cycles C_1 and C_2 in $B_1\cup B_2$ containing v_1 and v_2, respectively, such that $P-v_1-v_2$ is an odd component of $G-V(C_1\cup C_2)$, contradicting that G is 2-cycle resonant.

(iii) For a chain P with odd length such that $G-V_{\mathrm{I}}(P)$ is not 2-connected, $G-V_{\mathrm{I}}(P)$ has an odd number of blocks since two attachment vertices of every block of $G-V_{\mathrm{I}}(P)$ have different colours by Theorems 1, 2.1, and 2.2. In addition, any chain P_i in $G-V_{\mathrm{I}}(P)$ induced by non-2-connected blocks of $G-V_{\mathrm{I}}(P)$ must also be of odd length. Otherwise, $G - V_{\rm I}(P_i)$ contains at least two 2-connected blocks (the end blocks of $G - V_{\rm I}(P)$) and other blocks lying in the maximal chain P, contradicting (ii). If $G - V_{\rm I}(P)$ has exactly three blocks, say B_1, B_2, B_3 (where B_1 and B_2 contain v_1 and v_2, respectively), each of which is 2-connected, but (iii) (a) is not true, then in $B_1 \cup B_3$ or $B_2 \cup B_3$, say $B_1 \cup B_3$, there are two disjoint cycles C_1 and C_2 such that C_1 contains v_1 and C_2 contains the common vertex of B_2 and B_3. Then the component of $G - V(C_1 \cup C_2)$ containing $V_{\rm I}(P)$ and $V(B_2) \setminus V(B_3)$ is an odd component, contradicting that G is 2-cycle resonant. If $G - V_{\rm I}(P)$ has exactly three blocks, say B_1, B_2, B_3 (where B_1 and B_2 contain v_1 and v_2, respectively), and B_3 is not 2-connected, the only two 2-connected blocks B_1 and B_2 are disjoint. If $G - V_{\rm I}(P)$ has at least five blocks and there are two 2-connected blocks $B_{\rm r}$ and $B_{\rm s}$ which have a common vertex, without loss of generality, assume that $B_{\rm r}$ is an end block of $G - V_{\rm I}(P)$ containing v_1 and $B_{\rm t}$ is the other end block of $G - V_{\rm I}(P)$. Then there is a cycle $C_{\rm s}$ in $B_{\rm s}$ containing the common vertex of $B_{\rm r}$ and $B_{\rm s}$ and a cycle $C_{\rm t}$ in $B_{\rm t}$ containing v_2 which are disjoint and the component of $G - V(C_{\rm s} \cup C_{\rm t})$ containing v_1 is an odd component, a contradiction.

(iv) For a 2-connected subgraph B_1 of G with exactly two attachment vertices, by Lemma 1, let $B_2, B_3, \ldots, B_{\rm t}$ be blocks of $\bar{B}_1$ such that B_i and B_{i+1}, $i = 2, 3, \ldots, t-1$, have a common vertex. By Theorem 2.2, $\bar{B}_1$ has an odd number of blocks. If $\bar{B}_1$ is not 2-connected, and every block of $\bar{B}_1$ is 2-connected, we assert $\bar{B}_1$ has exactly three blocks. Otherwise, $t \geqslant 6$. Let C_1 (resp. C_4) be a cycle in B_1 (resp. B_4) containing two attachment vertices of B_1 (resp. B_4). Then C_1 and C_4 are disjoint and the component of $G - V(C_1 \cup C_4)$ containing the common vertex of B_2 and B_3 is an odd component, contradicting that G is 2-cycle resonant. Moreover, the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block. Otherwise, assume that there is a cycle C_1 in B_1 containing only an attachment vertex of B_1, say the common vertex of B_1 and B_2. Let C_4 be a cycle in B_4 containing two attachment vertices of B_4. Then C_1 and C_4 are disjoint and the component of $G - V(C_1 \cup C_4)$ containing the common vertex of B_2 and B_3 is an odd component, again a contradiction.

To address the sufficiency, suppose that G is not 2-cycle resonant. Then there are two disjoint cycles C_1 and C_2 in G such that $G - V(C_1 \cup C_2)$ contains an odd component B'. Since G is 1-cycle resonant, $G - C_1$ has no odd component. Let B be a bridge of C_1 containing C_2. Then B' must be contained in B, and is an odd component of $B - V_{\rm A}(B) - V(C_2)$. We can choose such disjoint cycles C_1 and C_2 so that the bridge B of C_1 containing C_2 is minimal. By Corollary 1, B is not 2-connected. Hence, since the number of blocks of B must be odd, B has at least three blocks, in which the block containing C_2, say B^*, is 2-connected.

Claim 1. *In the above assumption, C_2 contains at least one attachment vertex of B^*.*

Otherwise, by Corollary 1(iv) (b), the attachment vertices of B^* are contained in a same component of $B^* - V(C_2)$, and $B - V(C_2)$ has no odd component. But B' is an odd component of $B - V_{\rm A}(B) - V(C_2)$. So B^* is an end block of B, and an odd component of $B - V_{\rm A}(B) - V(C_2)$ not containing any attachment vertex of B^*, say B'' (possible, $B'' = B'$), is contained in B^*. Note that, by Corollary 1(iv), two attachment vertices of B^* are contained in a same component of $B^* - V(C_2)$, so there is a path P

in B with end vertices being the attachment vertices of B, which is disjoint with B''. Let C_1^* be the cycle consisting of P and a segment on C_1. Then B'' is also an odd component of $G - C_1^* - C_2$, and the bridge of C_1^* containing B'' is a proper subgraph of B. This contradicts the choice of C_1 and C_2.

Now we consider the following cases.

Case 1: There is a chain P of even length in B induced by cut edges of B. By condition (ii) and Lemma 2, $G - V_{\mathrm{I}}(P)$ consists of exactly two 2-connected blocks, one of which is $\bar{B}$ and the other is denoted by B^*. In addition, the end vertices of P are cycle-related to the common vertex of $\bar{B}$ and B^*, and C_1 and C_2 are disjoint. Thus C_2 cannot contain the attachment vertices of B^*. This contradicts Claim 1.

Case 2: There is a chain P of odd length in B induced by cut edges of B such that $G - V_{\mathrm{I}}(P)$ is not 2-connected. $G - V_{\mathrm{I}}(P)$ contains at least two 2-connected blocks, one of which is $\bar{B}$ and the other, say B^*, contains the cycle C_2. Hence, any maximal chain in B induced by cut edges of B must be of odd length by condition (ii).

If $G - V_{\mathrm{I}}(P)$ has exactly three blocks each of which is 2-connected, by condition (iii) (a) and Claim 1, B^* and $\bar{B}$ must be the end blocks of $G - V_{\mathrm{I}}(P)$. Otherwise, C_1 and C_2 would have a common vertex, a contradiction. Then either C_2 contains two attachment vertices of B^* or it contains only one attachment vertex of B^* which is not an end vertex of P. In any case, any component of $B^* - C_2$ not containing an end vertex of P is also a component of $B - V_{\mathrm{A}}(B) - C_2$ and $G - C_2$, and so is an even component. Let B'' be the other 2-connected block in B. The components of $B'' - V_{\mathrm{A}}(B'')$ are also components of $B - V_{\mathrm{A}}(B) - C_2$, and, by Corollary 1 (v), $B'' - V_{\mathrm{A}}(B'')$ has no odd component. The only other component of $B - V_{\mathrm{A}}(B) - C_2$, which contains edges on P, must be also even, since both $B - V_{\mathrm{A}}(B)$ and C_2 have an even number of vertices. This also contradicts our assumption.

Similarly, if any two 2-connected blocks of $G - V_{\mathrm{I}}(P)$ are disjoint, $B - V_{\mathrm{A}}(B) - C_2$ has no odd component, and again this is a contradiction.

Case 3: Each block of B is 2-connected. Let $B_1 = \bar{B}$ which is also 2-connected by Lemma 3. Then, by condition (iv), B consists of exactly three 2-connected blocks, say B_2, B_2, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related. Combining Claim 1, we have that C_2 contains two attachment vertices of B^* (one of B_2, B_2, B_4, say B_3, which is disjoint with B_1). Then any component of $B^* - C_2$ is also a component of $B - V_{\mathrm{A}}(B) - C_2$ and $G - C_2$, and so is an even component. The other components of $B - V_{\mathrm{A}}(B) - C_2$ are also components of $B_2 - V_{\mathrm{A}}(B_2)$ and $B_4 - V_{\mathrm{A}}(B_4)$, and so are even by Corollary 1(v). This again contradicts our assumption.

The proof is thus completed. □

5. Conclusion

Previously, k-cycle resonant hexagonal systems were completely characterized in Ref. [8]. General planar k-cycle resonant graphs have a more complex characterization. The present work establishes some necessary and sufficient conditions for a 2-connected graph to be a planar 1-cycle resonant graph or a planar 2-cycle resonant graph. Based on the conditions, an efficient algorithm for determining whether or not a

2-connected graph is planar 1-cycle resonant or 2-cycle resonant can be developed. We shall establish such an algorithm and investigate the ear decomposition of planar 1-cycle resonant graphs and 2-cycle resonant graphs in further works.

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