JOURNAL OF NUMBER THEORY 11, 465–471 (1979)

Some Remarks on a Theorem of Montgomery and Vaughan

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Received December 12, 1978

DEDICATED TO PROFESSOR S. CHOWLA ON THE OCCASION OF HIS 70TH BIRTHDAY

1. INTRODUCTION

In [2], H. L. Montgomery and R. C. Vaughan proved the following important

THEOREM 1. *Suppose $R \geqslant 2$; $\lambda_1, \lambda_2, \ldots, \lambda_R$ are distinct real numbers and that $\delta_n = \min_{m \neq n} | \lambda_n - \lambda_m |$. Then if $a_1, a_2, \ldots, a_R$ are complex numbers, we have*

$$\left| \sum_{m \neq n} \sum \frac{a_m \bar{a}_n}{\lambda_m - \lambda_n} \right| \leqslant \frac{3\pi}{2} \sum_n | a_n |^2 \delta_n^{-1}. \tag{1}$$

Remark. We can add any positive constant to each of the λ_n and so we can assume that all the λ_n are positive and distinct. The proof of the theorem is very deep and it is desirable to have a simple proof within the reach of simple calculus.

In almost all applications it suffices to restrict to the special case $\lambda_n = \log(n + \alpha)$ where $0 \leqslant \alpha \leqslant 1$ is fixed and $n = 1, 2, \ldots, R$. Also the constant $3\pi/2$ is not important in many applications. It is the object of this note to supply a very simple proof in this special case with a larger constant in place of $3\pi/2$. Accordingly our main result is

THEOREM 2. *Suppose $R \geqslant 2$, $\lambda_n = \log(n + \alpha)$ where $0 \leqslant \alpha \leqslant 1$ is fixed and $n = 1, 2, \ldots, R$. Let $a_1, \ldots, a_R$ be complex numbers. Then, we have,*

$$\left| \sum_{m \neq n} \sum \frac{a_m \bar{a}_n}{\lambda_m - \lambda_n} \right| \leqslant C \sum n \, | a_n |^2,$$

where C is an absolute numerical constant which is effective.

Remark 1. Instead of the condition $\lambda_n = \log(n + \alpha)$ we can also work with the weaker condition $n(\lambda_{n+1} - \lambda_n)$ is both $\gg 1$ and $\ll 1$. Also no attempt is made to obtain an economical value for the constants such as C.

0022-314X/79/030465–07$02.00/0

Remark 2. Theorem 2 with $\alpha = 0$ and the functional equation of $\zeta(s)$ are together enough to deduce in a simple way the result that for $T \geqslant 2$,

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = \left(\frac{1}{2\pi^2}\right) T(\log T)^4 + O(T(\log T)^3). \tag{3}$$

See [3] for details of proof. The result (3) was first proved by A. E. Ingham by a very complicated method.

Remark 3. In an earlier draft (written long ago) of the present paper I proved that

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \int_0^H |L(\tfrac{1}{2} + it, \chi)|^2 \, dt$$
$$= \frac{\varphi(q)}{q} H \log(qT) + O(T(\log(qT))^\epsilon), \tag{4}$$

where $H > 0$, $\epsilon > 0$, $T = H + 2$ and the 0-constant depends only on ϵ.

Remark 4. The result (3) is generalised (at my suggestion) by V. V. Rane as a hybrid result for L-functions with primitive characters in a nice way. Also the result (4) is sharpened by him to an asymptotic expansion with an error term $O(T^{1/2})$. These results are contained in his doctoral dessertation of Bombay University [4]. In view of the deep researches of R. Balasubramanian and D. R. Heath–Brown I believe that these results can be improved drastically.

THEOREM 3. *If* $\{a_n\}$ *and* $\{b_n\}$ $(n = 1, 2, 3, \ldots, R)$ *are complex numbers where* $R \geqslant 2$ *and* $\lambda_n = \log(n + \alpha)$ *where* $0 \leqslant \alpha \leqslant 1$ *is fixed, then*

$$\left| \sum_{m \neq n} \sum \frac{a_m \bar{b}_n}{\lambda_m - \lambda_n} \right| \leqslant D \left(\sum n \, |a_n|^2\right)^{1/2} \left(\sum n \, |b_n|^2\right)^{1/2}$$

where D is an effective positive numerical constant.

COROLLARY 1. *Let* $q \geqslant 1$ *be an integer. Then, for* $1 \leqslant j \leqslant q$, *we have,*

$$\left| \sum_{\substack{m \neq n \\ m \equiv n \equiv j (\bmod q)}} \sum \frac{a_m \bar{b}_n}{\lambda_m - \lambda_n} \right|$$
$$\leqslant D_1 \left(\frac{|a_j|^2}{\log((q + j)/j)} + \frac{1}{q} \sum_{\substack{n \equiv j (\bmod q) \\ n > q}} n \, |a_n|^2 \right)^{1/2}$$
$$\times \left(\frac{|b_j|^2}{\log((q + j)/j)} + \frac{1}{q} \sum_{\substack{n \equiv j (\bmod q) \\ n > q}} n \, |b_n|^2 \right)^{1/2}$$

where D_1 *is an effective numerical constant.*

COROLLARY 2. *Let $q \geqslant 1$ be an integer and χ run through all Dirichlet characters* mod q. *Then we have*

$$\Big| \sum_{\chi \bmod q} \Big(\sum_{m \neq n} \frac{a_m \bar{b}_n \chi(m) \overline{\chi(n)}}{\lambda_m - \lambda_n} \Big) \Big|$$
$$\leqslant D_2 \varphi(q) \Big(\sum_{1 \leqslant j \leqslant q} \frac{|a_j|^2}{\log(2q/j)} + \frac{1}{q} \sum_{\substack{n > q \\ (n,q)=1}} n \,|a_n|^2 \Big)^{1/2}$$
$$\times \Big(\sum_{\substack{1 \leqslant j \leqslant q \\ (j,q)=1}} \frac{|b_j|^2}{\log(2q/j)} + \frac{1}{q} \sum_{\substack{n > q \\ (n,q)=1}} n \,|b_n|^2 \Big)^{1/2}$$

where D_2 is an effective numerical constant.

Remark. The corollaries can be deduced from Theorem 3 as follows. The contribution from the terms for which either $m = j$ or $n = j$ can be estimated thus: we have

$$\int_0^T \Big(a_j e^{-i\lambda_j t} \sum_{\substack{n \equiv j (\bmod q) \\ n > q}} \bar{b}_n e^{i\lambda_n t} \Big)\, dt = E(T) - E(0)$$

where

$$E(T) = \sum \frac{a_j \bar{b}_n e^{iT(\lambda_n - \lambda_j)}}{i(\lambda_n - \lambda_j)}$$

under obvious restrictions on the sum. Integrating from 0 to 1 with respect to T we have

$$|E(0)| \leqslant \sum \frac{|a_j b_n|}{(\log(n/j))^2} + \int_0^1 dT \int_{-T}^{T} \Big| a_j \sum \bar{b}_n e^{i\lambda_n t} \Big|\, dt$$
$$\leqslant \sum \frac{|a_j b_n|}{(\log(n/j))^2} + |a_j| \int_0^1 (2T)^{1/2} \Big(\int_{-T}^{T} \Big| \sum \bar{b}_n e^{i\lambda_n t} \Big|^2 dt \Big)^{1/2} dT$$
$$\leqslant \sum \frac{|a_j b_n|}{(\log(n/j))^2} + 2\,|a_j| \Big(\int_0^1 dT \int_{-T}^{T} \Big| \sum \bar{b}_n e^{i\lambda_n t} \Big|^2 dt \Big)^{1/2}$$
$$\leqslant \sum \frac{|a_j b_n|}{(\log(n/j))^2} + 2\,|a_j| \Big(\frac{1}{q} \sum n\,|b_n|^2 + \frac{2D}{q} \sum n\,|b_n|^2 \Big)^{1/2}$$

(by using the result to follow; also note $n > q$).

Next

$$|a_j| \sum \frac{|b_n|}{(\log(n/j))^2} \leqslant (|a_j|^2)^{1/2} \Big(\sum n\,|b_n|^2 \Big)^{1/2} \Big(\sum \frac{1}{n(\log(n/j))^4} \Big)^{1/2}$$
$$\leqslant \Big(\frac{|a_j|^2}{\log(2q/j)} \Big)^{1/2} \Big(\sum n\,|b_n|^2 \Big)^{1/2} \Big(\frac{1}{q} \sum_{n=2}^{\infty} \frac{1}{n(\log n)^3} \Big)^{1/2}$$
$$\ll \Big(\frac{|a_j|^2}{\log(2q)/j} \Big)^{1/2} \Big(\frac{1}{q} \sum n\,|b_n|^2 \Big)^{1/2}.$$

Similar argument applies to $\sum (\bar{b}_j a_n / i(\lambda_n - \lambda_j))$. Also by Theorem 3

$$\left| \sum_{\substack{m \equiv n \equiv j (\mathrm{mod}\ q) \\ m > q, n > q \\ m \neq n}} \sum \frac{a_m \bar{b}_n}{\lambda_m - \lambda_n} \right| \leqslant \frac{D}{q} \Bigg(\sum_{\substack{n > q \\ n \equiv j (\mathrm{mod}\ q)}} n \mid a_n \mid^2 \Bigg)^{1/2} \Bigg(\sum_{\substack{n > q \\ n \equiv j (\mathrm{mod}\ q)}} n \mid b_n \mid^2 \Bigg)^{1/2}$$

as can be seen by writing $\lambda_n - \log q$ in place by λ_n. This completes the deduction of Corollary 1 from Theorem 3. Corollary 2 follows easily from Corollary 1.

We now proceed to state a hybrid large sieve analytic (conjectural) asymptotic formula. The reason for the conjectures can be traced to Theorems 8 and 11 on pages 24 and 31 of Bombieri's paper [1].

CONJECTURE. *Let $\{a_n\}$ and $\{b_n\}$ $(n = 1, 2, \ldots, R;\ R \geqslant 2)$ be $2R$ complex numbers. Let $0 < \lambda_1 < \lambda_2 < \cdots$ and put $\delta_n = \min_{m \neq n} \mid \lambda_n - \lambda_m \mid$. Next let*

$$A_\epsilon = \sum ((Q^{2+\epsilon} + \delta_n^{-1-\epsilon}) \mid a_n \mid^2)$$

and

$$B_\epsilon = \sum ((Q^{2+\epsilon} + \delta_n^{-1-\epsilon}) \mid b_n \mid^2) \qquad \textit{where} \quad \epsilon \geqslant 0.$$

Then there exist effective constants D_3 (defending only on ϵ) and D_4 (a numerical constant) such that

$$\left| \sum_{q \leqslant Q} \sum_{\chi \bmod q} \left(\sum_{m \neq n} \sum \frac{a_m \bar{b}_n \chi(m) \overline{\chi(n)}}{\lambda_m - \lambda_n} \right) \right| \leqslant D_3 (A_\epsilon B_\epsilon)^{1/2}, \quad (\epsilon > 0),$$

and

$$\left| \sum_{q \leqslant Q} \log \left(\frac{2Q}{q} \right) \sum_{\chi \bmod q}{}^{*} \left(\sum_{m \neq n} \sum \frac{a_m \bar{b}_n \chi(m) \overline{\chi(n)}}{\lambda_m - \lambda_n} \right) \right| \leqslant D_4 (A_0 B_0)^{1/2}$$

where χ runs over all Dirichlet characters mod q *and the asterisk indicates the restriction to primitive characters* mod q.

Remark 1. We believe that at least the conjecture involving * is true with some or no modifications when $\lambda_n = \log n$.

Remark 2. Following the method of my paper [3], and assuming the truth of the conjecture, I can obtain asymptotic formulae (with error terms uniform in Q and T) for

$$\sum_{q \leqslant Q} \log \left(\frac{2Q}{q} \right) \sum_{\chi \bmod q} \int_T^{2T} \mid L(\tfrac{1}{2} + it, \chi) \mid^{2k} dt$$

and

$$\sum_{q \leqslant Q} \log \left(\frac{2Q}{q} \right) \sum_{\chi \bmod q}{}^{*} \int_T^{2T} \mid L(\tfrac{1}{2} + it, \chi) \mid^{2k} dt$$

for $k = 1, 2, 3$ and 4 under suitable constraints on Q and T. The details will appear elsewhere.

Remark 3. It is easy to prove asymptotic formulae for $\sum_{x \bmod q} | L(\frac{1}{2}, \chi)|^{2k}$ for $k = 1$. When q is a prime and $k = 1$ the sum is $\sim q \log q$. It will be of some interest to prove (or disprove!) that when $k = 2$ and q is prime then the sum is $\sim (2\pi^2)^{-1} q(\log q)^4$. We may ask other questions.

2. A Simple Proof of Theorem 3

We begin with

Lemma 1. *We have*

$$\left| \sum_{m \neq n}\sum \frac{a_m \bar{a}_n}{m-n} \right| \leqslant \pi \sum | a_n |^2.$$

Proof. $2\pi \int_0^1 (\int_0^y | \sum a_m e^{2\pi i m x} |^2 \, dx) \, dy = \pi \sum | a_n |^2 - E/i$, where E/i is the real number for which $| E | \leqslant \pi \sum | a_n |^2$ is to be proved. Note that since the integrand is nonnegative, $2\pi \int_0^1 (\int_{-y}^y | \sum a_n e^{2\pi i n x} |^2 \, dx) \, dy = 2\pi \sum | a_n |^2$ is an upper bound. Thus

$$0 \leqslant \pi \sum | a_n |^2 - \frac{E}{i} \leqslant 2\pi \sum | a_n |^2$$

and this proves the lemma.

Lemma 2. *We have*

$$\left| \sum_{m \neq n}\sum \frac{a_m \bar{b}_n}{m-n} \right| \leqslant 3\pi \left(\sum | a_n |^2\right)^{1/2} \left(\sum | b_n |^2\right)^{1/2}.$$

Proof. $2\pi \int_0^1 (\int_0^y (\sum a_m e^{2\pi i m x})(\sum \bar{b}_n e^{-2\pi i n x}) \, dx) \, dy = \pi \sum (a_m \bar{b}_m) - E/i$ gives the result since by Holder's inequality

$$\left| \pi \sum (a_m \bar{b}_m) - \frac{E}{i} \right| \leqslant 2\pi \left(\int_0^1 \left(\int_0^y \left| \sum a_m e^{2\pi i m x} \right|^2 dx\right) dy\right)^{1/2}$$
$$\times \left(\int_0^1 \left(\int_0^y \left| \sum \bar{b}_n e^{-2\pi i n x} \right|^2 dx\right) dy\right)^{1/2}$$
$$\leqslant 2\pi \left(\sum | a_n |^2\right)^{1/2} \left(\sum | b_n |^2\right)^{1/2}$$

on using Lemma 1. This completes the proof of Lemma 2.

We next deduce Theorem 3 from Lemma 2 as follows. We divide the range $1 \leqslant n \leqslant R$ by introducing intervals $I_i = (2^{i-1}, 2^i)$ and the pairs (m, n) with $m \neq n$ into those lying in $I_i \times I_j$. We now start with

$$\int_0^1 \left(\int_0^y \left(\sum a_m e^{2\pi i \lambda_m x} \right) \left(\sum \bar{b}_n e^{-2\pi i \lambda_n x} \right) dx \right) dy$$

$$= \frac{1}{2} \sum a_m \bar{b}_m - \frac{E}{2\pi i} + \sum_{\substack{k,l \\ k \geqslant 1, l \geqslant 1}} \frac{1}{2\pi i} \sum_{(m,n) \in I_k \times I_l} \int_0^1 \frac{a_m \bar{b}_n e^{2\pi i (\lambda_m - \lambda_n) y}}{\lambda_m - \lambda_n} dy$$

where E is the quantity for which we seek an upper bound, and hence we have the fundamental inequality

$$\left| \frac{E}{2\pi i} \right| \leqslant \frac{1}{2} \sum | a_m \bar{b}_m | + \frac{1}{2\pi} \sum_{k,l} \left| \sum_{(m,n) \in I_k \times I_l} \int_0^1 \frac{a_m \bar{b}_n e^{2\pi i (\lambda_m - \lambda_n) y}}{\lambda_m - \lambda_n} dy \right|$$

$$+ \left(\int_0^1 \int_{-y}^y \left| \sum a_m e^{2\pi i \lambda_m x} \right|^2 dx\, dy \right)^{1/2}$$

$$\times \left(\int_0^1 \int_{-y}^y \left| \sum \bar{b}_n e^{-2\pi i \lambda_n x} \right|^2 dx\, dy \right)^{1/2}$$

$$= \Sigma_1 + \Sigma_2 + (\Sigma_3)^{1/2} (\Sigma_4)^{1/2}$$

we remark that if $| k - l | \geqslant 3$ then

$$\left| \sum_{(m,n) \in I_k \times I_l} \int_0^1 \frac{a_m \bar{b}_n e^{2\pi i (\lambda_m - \lambda_n) y}}{\lambda_m - \lambda_n} dy \right|$$

$$\ll \left(\sum_{(m,n) \in I_k \times I_l} | a_m \bar{b}_n | \right) \underset{(m,n) \in I_k \times I_l}{\text{maximum}} (\lambda_m - \lambda_n)^{-2}$$

$$\ll (k - l)^{-2} \sum_{(m,n) \in I_k \times I_l} | a_m \bar{b}_n | \ll (k - l)^{-2} S_k^{1/2} T_l^{1/2}$$

where $S_k = \sum_{n \in I_k} n \, | a_n |^2$ and $T_l = \sum_{n \in I_l} n \, | b_n |^2$. Hence the contribution to Σ_2 from k, l with $| k - l | \geqslant 3$ is $\sum_{|k-l| \geqslant 3} (S_k^{1/2} T_l^{1/2} / (k - l)^2) \ll (\sum_k S_k)^{1/2} (\sum_k T_k)^{1/2}$. Now we consider those terms of Σ_2 with $| k - l | < 3$. A typical term is

$$\int_0^1 \sum_{(m,n) \in I_k \times I_l} \frac{a_m \bar{b}_n e^{2\pi i (\lambda_m - \lambda_n) y}}{\lambda_m - \lambda_n} dy$$

Here the inner sum is

$$N \left(\sum_{(m,n) \in I_k \times I_l} \frac{a_m' \bar{b}_n'}{(N\lambda_m) - (N\lambda_n)} \right)$$

where $a'_m = a_m e^{2\pi i \lambda_m y}$ and $b'_n = b_n e^{2\pi i \lambda_n y}$, and N is any positive number. Observe that if $N = 2^{k+800}$, then the integral parts of $N\lambda_m (m \in (I_k \cup I_l))$ differ each other by at least 3. Also in the denominator we replace $N\lambda_m - N\lambda_n$ by $[N\lambda_m] - [N\lambda_n]$ the consequent error being

$$O\left(N \sum \frac{|a_m b_n|}{([N\lambda_m] - [N\lambda_n])^2}\right)$$

which is easily seen to be $O(S_k^{1/2} T_l^{1/2})$. Next by Lemma 2 we see that

$$N \sum_{(m,n) \in I_k \times I_l} \frac{a'_m \bar{b}'_n}{[N\lambda_m] - [N\lambda_n]} = O(S_k^{1/2} T_l^{1/2}).$$

Thus we see that if $|k - l| < 3$, the contribution of $\sum_{(m,n) \in I_k \times I_l} \cdots$ to Σ_2 is $O(S_k^{1/2} T_l^{1/2})$. Combining all this one sees easily that $\Sigma_2 = O((\sum n | a_n |^2)^{1/2} \times (\sum n | b_n |^2)^{1/2})$. The method of estimation of Σ_2 shows that $\Sigma_3 = \sum | a_n |^2 + O(\sum n | a_n |^2) = O(\sum n | a_n |^2)$ and $\Sigma_4 = O(\sum n | b_n |^2)$. Trivially $\sum | a_n b_n | \leqslant (\sum | a_n |^2)^{1/2} (\sum | b_n |^2)^{1/2}$ and so

$$E = O\left(\left(\sum n | a_n |^2\right)^{1/2} \left(\sum n | b_n |^2\right)^{1/2}\right).$$

This completes the proof of Theorem 3.

Remark. In a paper entitled "on the mean fourth power of the Rieman zeta function and other allied problems," (to appear), R. Balasubramanian deals with asymptotic formulae for $\int_{\sigma=1/2, |t| \leqslant T} | F(s)|^2 \, ds$ where $F(s) = (d^m/ds^m)((\zeta(s))^2)$ ($m \geqslant 0$ being an integer) and also hybrid analogues for L-functions and so on. These are improvements of the results of V. V. Rane.

Note added in proof. It must be mentioned that M. J. Narlikar has (in a paper to appear) improved the hybrid monsquare results of V. V. Rane. These improvements are along the lines of the deep researches of R. Balasubramanian and D. R. Heath-Brown (referred to in remark 4 preceding theorem 3).

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