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Laws of large numbers for periodically and almost periodically correlated processes

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Abstract

This paper gives results related to and including laws of large numbers for (possibly non-harmonizable) periodically and almost periodically correlated processes. These results admit periodically correlated processes that are not continuous in quadratic mean. The idea of a stationarizing random shift is used to show that strong law results for weakly stationary processes may be used to obtain strong law results for such processes.

Key words: Periodically and almost periodically correlated processes; Laws of large numbers; Stationarizing random shift

1. Introduction

A second-order continuous time complex-valued process $X(t)$, $t \in \mathbb{R}$, is called periodically correlated (PC) (see Gladyshev, 1963) with period T if, for every s, t,

$$m(t) = E\{X(t)\} = m(t + T), \tag{1.1}$$

$$R(s,t) = E\{X(s)\overline{X(t)}\} = R(s + T, t + T). \tag{1.2}$$

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A statement equivalent to (1.2) is that, for every t, τ,

$$B(t,\tau) = R(t+\tau,t) = B(t+T,\tau). \tag{1.3}$$

For a review of PC processes, see Yaglom (1987). Similarly, a process is called almost periodically correlated (APC) in the sense of Gladyshev (1963) if the functions $m(t)$ and $R(s,t)$ (or B) are uniformly continuous in (s,t) and for every (s,t), $m(t+\alpha)$ and $R(s+\alpha, t+\alpha)$ (or equivalently $B(\alpha, s-t)$) are uniformly almost periodic (UAP) functions of α. Throughout the paper we assume the mean function vanishes, $m(t) \equiv 0$.

If $X(t)$ is PC and $B(\cdot,\tau) \in L_1[0,T]$ for every τ, then the coefficient functions

$$a_k(\tau) = \frac{1}{T}\int_0^T B(t,\tau)\exp(-\mathrm{i}2\pi kt/T)\,\mathrm{d}t \tag{1.4}$$

exist for every k and τ. Further, they have the representation

$$a_k(\tau) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma\tau)\,\mathrm{d}G_k(\gamma), \tag{1.5}$$

where $G_0(\gamma)$ is non-decreasing and all the $G_k(\gamma)$ are of bounded variation with $\int_{-\infty}^{\infty} |\mathrm{d}G_k(\gamma)| \le \int_{-\infty}^{\infty} \mathrm{d}G_0(\gamma) < \infty$, if and only if $a_0(\tau)$ is continuous at $\tau = 0$ (see Hurd, 1974a). Note that $a_0(\tau)$ may be continuous even when $X(t)$ is not continuous in quadratic mean (q.m.). The sequence of functions $\{G_k(\gamma)\}$ may be interpreted as the non-stationary spectrum of the process $X(t)$ in the sense that $B(t,\tau) = R(t+\tau,t)$ is given for each τ by a Fourier series of the form

$$B(t,\tau) = R(t+\tau,t) \sim \sum_k a_k(\tau)\exp(\mathrm{i}2\pi kt/T), \tag{1.6}$$

where the sense of convergence depends on the smoothness of $R(t+\tau,t)$ in the variable t.

If $X(t)$ is APC, the coefficient functions

$$a(\lambda,\tau) = \lim_{A\to\infty} \tfrac{1}{2A}\int_{-A}^{A} B(t,\tau)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t \tag{1.7}$$

exist for each λ and τ and the set $\Lambda = \{\lambda\colon a(\lambda,\tau) \neq 0 \text{ for some } \tau\} = \{\lambda_k\}$ is countable (Gladyshev, 1963; Hurd, 1991) and contains 0; we set $\lambda_0 = 0$. As in the PC case, the Fourier coefficient functions are themselves Fourier transforms,

$$a(\lambda_k,\tau) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma\tau)\,\mathrm{d}G_k(\gamma), \tag{1.8}$$

with $G_0(\gamma)$ non-decreasing and all the $G_k(\gamma)$ of bounded variation with $\int_{-\infty}^{\infty} |\mathrm{d}G_k(\gamma)| \le \int_{-\infty}^{\infty} \mathrm{d}G_0(\gamma) < \infty$, if and only if $a(0,\tau)$ is continuous at $\tau = 0$ (Hurd, 1991). Similarly, the sequence $\{G_k(\gamma)\}$ may be interpreted as the non-stationary spectrum of the process $X(t)$ in the sense that $B(t,\tau) = R(t+\tau,t)$ has, for each τ, a Fourier series of the form

$$B(t,\tau) = R(t+\tau,t) \sim \sum_{\lambda_k\in\Lambda} a(\lambda_k,\tau)\exp(\mathrm{i}\lambda_k t), \tag{1.9}$$

where the convergence is in the sense of UAP functions (Corduneanu, 1989).

This paper is concerned with the q.m. and a.s. convergence of the averages

$$J_{A,X}(\lambda,\omega)=\frac{1}{A}\int_0^A X(t,\omega)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t. \tag{1.10}$$

In the PC case it will not be assumed that X is q.m. continuous; instead the weaker assumption will be made that $X(t,\omega)$ is jointly measurable in t,ω and that $B(\cdot,0)\in L_1[0,T]$. In the APC case, X is q.m. continuous and thus has a version jointly measurable in t,ω which will be considered. Then the integral in (1.10) is a well-defined Lebesgue integral and has a finite second moment since

$$E\left\{\int_0^A |X(t)|\,\mathrm{d}t\right\}^2 \le E\left\{A\int_0^A |X(t)|^2\,\mathrm{d}t\right\}=A\int_0^A B(t,0)\,\mathrm{d}t<\infty .$$

We review some weak laws of large numbers first and then describe the sense in which strong laws are obtained.

If $X(t)$ is q.m. continuous and weakly stationary, it has the representation

$$X(t)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda t)\,\mathrm{d}\xi(\lambda), \tag{1.11}$$

where the second-order spectral process ξ has orthogonal increments and is right q.m. continuous. The correlation of $X(t)$ may be expressed as

$$R(t+\tau,t)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda\tau)\,\mathrm{d}F_\xi(\lambda), \tag{1.12}$$

where the spectral distribution $F_\xi(\lambda)$ is right continuous, bounded and non-decreasing with $E\{|\xi(\lambda_1)-\xi(\lambda_2)|^2\}=F_\xi(\lambda_1)-F_\xi(\lambda_2)$, $\lambda_2<\lambda_1$. For each λ, the average $J_{A,X}(\lambda)$ converges in q.m. to the jump $\xi(\lambda)-\xi(\lambda-)$ of the spectral process at λ, and $E\{|\xi(\lambda)-\xi(\lambda-)|^2\}=F_\xi(\lambda)-F_\xi(\lambda-)$, the jump of the spectral distribution F_ξ at λ (see Loève, 1963).

If $X(t)$ is (strongly) harmonizable in the sense of Loève (1963), the representation (1.11) is still valid but $\xi(\lambda)$ does not have orthogonal increments (unless $X(t)$ is stationary). The correlation $R(s,t)$ then has the representation

$$R(s,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda_1 s-\mathrm{i}\lambda_2 t)r_\xi(\mathrm{d}\lambda_1,\mathrm{d}\lambda_2), \tag{1.13}$$

with $r_\xi\{(a,b]\times(c,d]\}=E\{[\xi(b)-\xi(a)]\overline{[\xi(d)-\xi(c)]}\}$. Again for each λ, $J_{A,X}(\lambda)$ converges in q.m. to $\xi(\lambda)-\xi(\lambda-)$ and $E|\xi(\lambda)-\xi(\lambda-)|^2=r_\xi\{(\lambda,\lambda)\}$, the r_ξ measure of the point (λ,λ) (Loève, 1963, Sec. 34.4). If $X(t)$ is harmonizable and either PC or APC, the spectral distribution function $G_0(\lambda)$ may be identified with the restriction of r_ξ to the main diagonal, so $E|\xi(\lambda)-\xi(\lambda-)|^2=G_0(\lambda)-G_0(\lambda-)$ (see Honda, 1982; Hurd, 1989, 1991).

In this paper we derive weak law results for PC and APC processes without the assumption of harmonizability, and then use strong law results of Gaposhkin for weakly stationary processes to obtain strong law results for such processes. We obtain our results under the assumption $m(t) \equiv 0$, and note that they may be applied to the estimation of the Fourier coefficients of $m(t)$ in the following manner. In the PC case, assume $m(t) \not\equiv 0$ and is in $L_1[0,T]$ so the Fourier coefficients $m_k = T^{-1} \int_0^T m(t)\exp(-i2\pi kt/T)\,dt$ exist. If the zero mean process $X - m$ satisfies $J_{A,X-m}(\lambda_k) \to 0$ as $A \to \infty$ with $\lambda_k = 2\pi k/T$, then $J_{A,X}(\lambda_k) \to m_k$ as $A \to \infty$. A similar result holds in the APC case where $m(t)$ is a UAP function. We now summarize the results.

In the case of PC processes, we find (Theorem 1) that $J_{A,X}(\lambda)$ converges in q.m. to a limit η_λ with $E|\eta_\lambda|^2 = G_0(\lambda) - G_0(\lambda -)$, assuming only that $X(t)$ is measurable with $B(\cdot,0) \in L_1[0,T]$ and that $a_0(\tau)$ is continuous at the origin. Since $X(t)$ may be neither stationary nor harmonizable, a question arises concerning the existence of a spectral process with which η_λ may be identified. As a partial answer we relate η_λ to the spectral process $\zeta(\lambda)$ of a weakly stationary process $Y(t) = X(t + \Theta)$ formed by shifting $X(t)$ by the random time Θ, taken to be independent of X and uniformly distributed over $[0,T]$, by $\eta_\lambda \exp(i\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda -)$. Theorem 2 shows that if either $J_{A,X}(\lambda)$ or $J_{A,Y}(\lambda)$ converge a.s., then the other does also, and $J_{A,X}(\lambda)$ and $J_{A,Y}(\lambda)\exp(-i\lambda\Theta(\omega))$ have the same limit (a.s.). Thus, any condition that suffices for the strong law for Y will give the strong law for X.

If $X(t)$ is APC, then under the additional hypothesis $\sum \lambda_k^{-2} < \infty$ (excluding any $\lambda_k = 0$), we show (Theorem 3) that $J_{A,X}(\lambda)$ converges in q.m. to a limit η_λ with $E|\eta_\lambda|^2 = G_0(\lambda) - G_0(\lambda -)$. An example is given that shows the condition $\sum \lambda_k^{-2}$ to be not necessary. When 0 is not a limit point of the set of frequencies Λ, we show (Theorem 4) that a random variable Θ, independent of $X(t)$, may be constructed for which $Y(t) = X(t + \Theta)$ is weakly stationary, and as in the PC case, $\eta_\lambda \exp(i\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda -)$ where $\zeta(\lambda)$ is the spectral process of $Y(t)$ (Proposition 2). Theorem 5 extends Theorem 2 to APC processes and to Θ having finite moment of order larger than 1.

2. Periodically correlated processes

We begin with a simple example of a PC process that is discontinuous in q.m.: an amplitude modulation of a stationary process, $X(t,\omega) = f(t)Z(t,\omega)$, where $Z(t)$ is a q.m. continuous weakly stationary process and $f(t)$ is a T-periodic function in $L_2[0,T]$. Then $X(t)$ is a PC process with $B(\cdot,\tau) \in L_1[0,T]$ for every τ and is q.m. discontinuous if $f(t)$ is discontinuous.

The following lemma shows that when $B(\cdot,0) \in L_1[0,T]$, the integrals $\int_u^v B(t,\tau)\,dt$ may be approximated, with uniformly bounded error, by $(v-u)a_0(\tau)$. This will play a crucial role in proving the weak law for PC processes.

Lemma 1. *If $X(t)$ is PC with period T and $B(\cdot,0) \in L_1[0,T]$, then for any u, v and τ,*

$$\left| \int_u^v B(t,\tau)\,dt - (v-u)a_0(\tau) \right| \le Ta_0(0). \tag{2.1}$$

Proof. Assume $u < v$ and let n be the largest integer $\leq (v-u)/T$ so we may write $v - u = nT + \delta T$ where $0 \leq \delta < 1$. Then from the periodicity of $B(\cdot, \tau)$ and (1.4), we find

$$\int_u^v B(t,\tau)\,\mathrm{d}t - (v-u)a_0(\tau) = \left\{ \int_0^{(n+\delta)T} - (n+\delta)\int_0^T \right\} B(t,\tau)\,\mathrm{d}t$$

$$= \left\{ \int_0^{\delta T} - \delta \int_0^T \right\} B(t,\tau)\,\mathrm{d}t$$

$$= \left\{ (1-\delta)\int_0^{\delta T} - \delta \int_{\delta T}^T \right\} B(t,\tau)\,\mathrm{d}t.$$

Taking absolute value, the LHS of (2.1) is

$$\leq \int_0^T |B(t,\tau)|\,\mathrm{d}t \leq \int_0^T B^{1/2}(t+\tau,0)B^{1/2}(t,0)\,\mathrm{d}t \leq Ta_0(0), \tag{2.2}$$

where we applied the Schwarz inequality first in the probability space to $B(t,\tau) = E\{X(t+\tau)\overline{X(t)}\}$, and then in $L_2[0,T]$. □

We now give the weak law for a class of PC processes that includes processes of the form $X(t,\omega) = f(t)Z(t,\omega)$ discussed above.

Theorem 1. *If $X(t)$ is a measurable PC process with period T, $B(\cdot, 0) \in L_1[0,T]$ and $a_0(\tau)$ continuous at $\tau = 0$, then for each λ, as $A \to \infty$, $J_{A,X}(\lambda)$ converges in q.m. to a random variable η_λ and $E\{|\eta_\lambda|^2\} = G_0(\lambda) - G_0(\lambda -)$.*

Proof. In order to show that $J_{A,X}(\lambda)$ converges in q.m., we show that

$$E\{J_{A,X}(\lambda)\overline{J_{B,X}(\lambda)}\} = \frac{1}{AB}\int_0^A\int_0^B R(s,t)\exp(\mathrm{i}\lambda t - \mathrm{i}\lambda s)\,\mathrm{d}s\,\mathrm{d}t \tag{2.3}$$

converges to a limit as A and B tend independently to ∞. Assuming $A < B$ and using the transformation $\tau = s - t$, we obtain

$$E\{J_{A,X}(\lambda)\overline{J_{B,X}(\lambda)}\} = \frac{1}{AB}\int_0^A\int_0^B B(t,s-t)\exp[\mathrm{i}\lambda(t-s)]\,\mathrm{d}s\,\mathrm{d}t$$

$$= \frac{1}{AB}\left\{ \int_{\tau=-A}^0 \int_{t=-\tau}^A + \int_{\tau=0}^{B-A}\int_{t=0}^A \right.$$

$$\left. + \int_{\tau=B-A}^B \int_{t=0}^{B-\tau} \right\} B(t,\tau)\exp(-\mathrm{i}\lambda\tau)\,\mathrm{d}t\,\mathrm{d}\tau$$

$$\equiv I_1 + I_2 + I_3. \tag{2.4}$$

The use of Lemma 1 in each of these three integrals produces

$$I_1 = \frac{1}{AB}\int_{-A}^{0} (A+\tau)a_0(\tau)\exp(-\mathrm{i}\lambda\tau)\,\mathrm{d}\tau + \frac{1}{AB}\int_{-A}^{0} \varepsilon_1(\tau)\,\mathrm{d}\tau, \tag{2.5}$$

$$I_2 = \frac{1}{AB}\int_{0}^{B-A} A a_0(\tau)\exp(-\mathrm{i}\lambda\tau)\,\mathrm{d}\tau + \frac{1}{AB}\int_{0}^{B-A} \varepsilon_2(\tau)\,\mathrm{d}\tau, \tag{2.6}$$

$$I_3 = \frac{1}{AB}\int_{B-A}^{B} (B-\tau)a_0(\tau)\exp(-\mathrm{i}\lambda\tau)\,\mathrm{d}\tau + \frac{1}{AB}\int_{B-A}^{B} \varepsilon_3(\tau)\,\mathrm{d}\tau, \tag{2.7}$$

where in each case $\varepsilon_j(\tau)$ is a measurable function of τ and $|\varepsilon_j(\tau)| \le Ta_0(0)$. Combining the leftmost integrals in these three expressions (use (2.4) with $B(t,\tau)$ replaced with $a_0(\tau)$) yields

$$f_1(A,B) = \frac{1}{AB}\int_0^A\int_0^B a_0(s-t)\exp[\mathrm{i}\lambda(t-s)]\,\mathrm{d}s\,\mathrm{d}t.$$

Now from (1.5) we have

$$\begin{aligned} f_1(A,B) &= \int_{-\infty}^{\infty}\left\{\frac{1}{A}\int_0^A \mathrm{e}^{\mathrm{i}(\lambda-u)t}\,\mathrm{d}t\right\}\left\{\frac{1}{B}\int_0^B \mathrm{e}^{-\mathrm{i}(\lambda-u)s}\,\mathrm{d}s\right\}\mathrm{d}G_0(u) \\ &= G_0(\lambda) - G_0(-\lambda) + \int_{u\neq\lambda} \frac{\mathrm{e}^{\mathrm{i}(\lambda-u)A}-1}{\mathrm{i}(\lambda-u)A}\,\frac{\mathrm{e}^{-\mathrm{i}(\lambda-u)B}-1}{-\mathrm{i}(\lambda-u)B}\,\mathrm{d}G_0(u) \\ &\to G_0(\lambda) - G_0(-\lambda) \quad \text{as } A,B\to\infty, \end{aligned}$$

since the integrand converges boundedly to zero. Similarly, combining the rightmost integrals in (2.5)–(2.7) gives

$$f_2(A,B) = \frac{1}{AB}\left\{\int_{-A}^{0}\varepsilon_1(\tau)\,\mathrm{d}\tau + \int_0^{B-A}\varepsilon_2(\tau)\,\mathrm{d}\tau + \int_{B-A}^{B}\varepsilon_3(\tau)\,\mathrm{d}\tau\right\}.$$

In view of the bounds $|\varepsilon_j(\tau)| \le Ta_0(0)$ it follows that

$$|f_2(A,B)| \le \frac{A+B}{AB}Ta_0(0)$$

and so $f_2(A,B) \to 0$ as $A\to\infty$ and $B\to\infty$ independently. It follows that $E\{|J_{A,X}(\lambda) - J_{B,X}(\lambda)|^2\} \to 0$ as $A,B\to\infty$ and thus $J_{A,X}(\lambda)$ converges in q.m. to a random variable η_λ with $E\{|\eta_\lambda|^2\} = G_0(\lambda) - G_0(-\lambda)$. □

The weak law obtained from Theorem 1 is: *For measurable PC processes with* $B(\cdot,0)\in L_1[0,T]$, *and* $a_0(\tau)$ *continuous at* $\tau = 0$, *we have* $A^{-1}\int_0^A X(t)\,\mathrm{d}t \to 0$ *in q.m. as* $A\to\infty$ *if and only if* $G_0(\lambda)$ *is continuous at* $\lambda = 0$.

In regard to the example $X(t) = f(t)Z(t)$, note that $B(t,\tau) = f(t+\tau)\overline{f(t)}R_Z(\tau)$ and so $B(\cdot,\tau)\in L_1[0,T]$ as $f(\cdot)\in L_2[0,T]$. Also, since $a_k(\tau) = c_k(\tau)R_Z(\tau)$ where

$c_k(\tau) = T^{-1}\int_0^T f(t+\tau)\overline{f(t)}\exp(-\mathrm{i}2\pi kt/T)\,\mathrm{d}t$, and the $c_k(\tau)$ are continuous, then $a_0(\tau)$ is continuous at $\tau = 0$. Finally, to get the condition on G_0 in terms of the spectral distribution F_Z, we denote by $\{f_n, n \in \mathbb{Z}\}$ the Fourier coefficients of f. It follows that $G_k(\lambda) = \sum_n f_n \bar{f}_{n-k} F_Z(\lambda - 2\pi n/T)$ and so $G_0(\lambda)$ is continuous at $\lambda = 0$ if and only if F_Z is continuous at all the points $\{2\pi n/T,\ n \in \mathbb{Z}\}$.

When the process $X(t)$ is stationary or harmonizable, the limiting random variable η_λ described in Theorem 1 is the jump $\eta_\lambda = \xi(\lambda) - \xi(\lambda -)$ of the spectral process appearing in (1.11). Although processes satisfying the hypotheses of Theorem 1 are not necessarily weakly stationary or harmonizable, there is an associated weakly stationary process that may be formed by a random time shift $X(t+\Theta)$, where Θ is independent of X and uniformly distributed over the interval $[0, T]$ (Hurd, 1974b). For the construction of such a Θ independent of X the probability space may have to be enlarged, and if so, the original probability space will be replaced by the enlarged one without further notice. The correlation of $X(t+\Theta)$ is $a_0(\tau)$, and since it has a spectral representation

$$X(t+\Theta) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda t)\,\mathrm{d}\zeta(\lambda),$$

it is natural to enquire about the connection between η_λ and $\zeta(\lambda)$.

Proposition 1. *Suppose $X(t)$ is PC with period T and satisfies the hypotheses of Theorem 1. If Θ is independent of X and uniformly distributed over the interval $[0, T]$ then $\eta_\lambda \exp(\mathrm{i}\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda -)$ a.s. where $\zeta(\lambda)$ is the spectral process of $X(t+\Theta)$.*

Proof. First fix any real number $\theta \in [0, T]$. Then

$$E\left\{\left|\frac{1}{A}\int_0^A X(t+\theta)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t - \eta_\lambda \exp(\mathrm{i}\lambda\theta)\right|^2\right\}$$

$$= E\left\{\left|\frac{1}{A}\int_\theta^{A+\theta} X(s)\exp(-\mathrm{i}\lambda s)\,\mathrm{d}s - \eta_\lambda\right|^2\right\},$$

where η_λ is the limit whose existence is guaranteed by Theorem 1, and adding and subtracting $A^{-1}\int_0^A X(s)\exp(-\mathrm{i}\lambda s)\,\mathrm{d}s$, and using $|z+w|^2 \le 2(|z|^2 + |w|^2)$, we find

$$\le 2E\left\{\left|\frac{1}{A}\left(\int_\theta^{A+\theta} - \int_0^A\right) X(s)\exp(-\mathrm{i}\lambda s)\,\mathrm{d}s\right|^2\right\}$$

$$+ 2E\left\{\left|\frac{1}{A}\int_0^A X(s)\exp(-\mathrm{i}\lambda s)\,\mathrm{d}s - \eta_\lambda\right|^2\right\}$$

$$\triangleq I(A,\theta) + J(A). \tag{2.8}$$

By Theorem 1, $J(A) \to 0$ as $A \to \infty$. Putting $D = (\theta, A + \theta) \triangle (0, A)$ we have

$$I(A,\theta) \le \frac{2}{A^2}\int_D\int_D E\{|X(s)X(t)|\}\,\mathrm{d}s\,\mathrm{d}t \le \frac{2}{A^2}\left(\int_D B^{1/2}(s,0)\,\mathrm{d}s\right)^2$$

$$\le \frac{2}{A^2}|D|\int_D B(s,0)\,\mathrm{d}s = \frac{8T}{A^2}\int_0^T B(s,0)\,\mathrm{d}s. \tag{2.9}$$

Now since Θ is uniformly distributed on $[0, T]$ independently of X, it follows that for $A > T$ we have

$$E\left\{\left|\frac{1}{A}\int_0^T X(t+\Theta)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t - \eta_\lambda \exp(\mathrm{i}\lambda\Theta)\right|^2\right\}$$

$$= E[E\{|\cdots|^2|\Theta\}]$$

$$= \frac{1}{T}\int_0^T E\left\{\left|\frac{1}{A}\int_0^A X(t+\theta)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t - \eta_\lambda \exp(\mathrm{i}\lambda\theta)\right|^2\right\}\mathrm{d}\theta$$

$$\le \frac{1}{T}\int_0^T I(A,\theta)\,\mathrm{d}\theta + J(A) \le \frac{8T}{A^2}\int_0^T B(s,0)\,\mathrm{d}s + J(A) \to 0 \quad \text{as } A \to \infty. \tag{2.10}$$

But we also know that $A^{-1}\int_0^A X(t+\Theta)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t \to \zeta(\lambda) - \zeta(\lambda-)$ in q.m. for the weakly stationary process $X(t+\Theta)$ (Doob, 1953, p. 489) and so evidently $\eta_\lambda \exp(\mathrm{i}\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda-)$ a.s. □

The preceding result permits an identification of the limit of $J_{A,X}(\lambda)$ with the spectral process of $X(t+\Theta)$. But the random shift notion leads to even stronger results because the shifted process $Y(t,\omega) = X(t+\Theta(\omega),\omega)$ has the same sample paths as $X(t,\omega)$ shifted by $\Theta(\omega)$, which for the case at hand is assumed to be in the interval $[0, T]$. The following theorem shows that if either $J_{A,X}(\lambda)$ or $J_{A,Y}(\lambda)$ converge a.s., then the other does also, and $J_{A,X}(\lambda)$ and $J_{A,Y}(\lambda)\exp(\text{-i}\lambda\Theta(\omega))$ have the same limit (a.s.).

Theorem 2. *Suppose $X(t)$ is PC with period T and satisfies the hypotheses of Theorem* 1. *If $Y(t) = X(t+\Theta)$ where Θ is independent of X and uniformly distributed over the interval $[0,T]$, then if either $J_{A,X}(\lambda)$ or $J_{A,Y}(\lambda)$ converge a.s., the other does also and* $\lim_{A\to\infty} J_{A,X}(\lambda) = \exp(-\mathrm{i}\lambda\Theta)\lim_{A\to\infty} J_{A,Y}(\lambda)$ *a.s.*

Proof. The techniques used in this proof are similar to those found in Loève (1963, Section 34.7). In view of

$$J_{A,Y}(\lambda,\omega)\exp(-\mathrm{i}\lambda\Theta(\omega)) - J_{A,X}(\lambda,\omega)$$

$$= \frac{1}{A}\int_A^{A+\Theta(\omega)} X(t,\omega)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t - \frac{1}{A}\int_0^{\Theta(\omega)} X(t,\omega)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t, \tag{2.11}$$

the result follows if we can show these last two terms converge to zero a.s., for every λ. We will actually show that the convergence is uniform in λ a.s.; that is, the exceptional set does not depend on λ. The rightmost term in (2.11) converges to zero a.s., and uniformly in λ, as $A \to \infty$ because for almost every ω, it is upperbounded in absolute value by $\int_0^T |X(t,\omega)|\,dt < \infty$. To prove that the leftmost term converges to zero a.s., and uniformly in λ, we first show that it does so along the particular sequence $A_n = nT$. For if

$$g_n(\lambda,\omega) = \frac{1}{A_n}\int_{A_n}^{A_n+\Theta(\omega)} X(t,\omega)\exp(-i\lambda t)\,dt, \tag{2.12}$$

it may be easily verified that for all λ, $E\{|g_n(\lambda)|^2\} \le a_0(0)n^{-2}$ and so $\sum_n E\{|g_n(\lambda)|^2\} < \infty$. It follows from the Borel–Cantelli lemma that $\lim_{n\to\infty} g_n(\lambda,\omega) = 0$ a.s. uniformly in λ. To show the general result we set

$$h_n(A,\lambda,\omega) = \frac{1}{A}\int_A^{A+\Theta(\omega)} X(t,\omega)\exp(-i\lambda t)\,dt - g_n(\lambda,\omega),$$

$$h_n(\lambda,\omega) = \sup_{(n-1)T\le A<nT} |h_n(A,\lambda,\omega)|.$$

If we can show that for all λ,

$$\sum_n E\{h_n(\lambda)^2\} < \infty, \tag{2.13}$$

then by the Borel–Cantelli lemma, $\lim_{n\to\infty} h_n(\lambda,\omega) = 0$ a.s. uniformly in λ, and hence the leftmost term in (2.11) converges to zero a.s., uniformly in λ. To establish (2.13) we may write (for $(n-1)T \le A < nT$ and $A+\Theta > nT$), with $I(t,\lambda,\omega) = X(t,\omega)\exp(-i\lambda t)$,

$$\begin{aligned} h_n(A,\lambda,\omega) &= \frac{1}{A}\int_A^{nT} I(t,\lambda,\omega)\,dt + \left(\frac{1}{A} - \frac{1}{nT}\right)\int_{nT}^{A+\Theta} I(t,\lambda,\omega)\,dt - \frac{1}{nT}\int_{A+\Theta}^{nT+\Theta} I(t,\lambda,\omega)\,dt \\ &\triangleq J_1(n,A,\lambda,\omega) + J_2(n,A,\lambda,\omega) + J_3(n,A,\lambda,\omega), \end{aligned}$$

where if $A+\Theta \le nT$ the middle integral is replaced by its negative. It follows that

$$E\{|h_n(\lambda)|^2\} \le \sum_{j=1}^{3} 3E\left\{\sup_{(n-1)T\le A<nT} |J_j(n,A,\lambda)|^2\right\} \tag{2.14}$$

and, for $J_1(n,A,\lambda,\omega)$,

$$\begin{aligned} E\left\{\sup_{(n-1)T\le A<nT} |J_1(n,A,\lambda)|^2\right\} &\le E\left\{\left(\frac{1}{(n-1)T}\int_{(n-1)T}^{nT} |X(t)|\,dt\right)^2\right\} \\ &\le \frac{1}{(n-1)^2} E\left\{\frac{1}{T}\int_{(n-1)T}^{nT} |X(t)|^2\,dt\right\} \\ &= \frac{1}{(n-1)^2}\, a_0(0). \end{aligned}$$

Similar inequalities for $J_2(n, A, \lambda, \omega)$ and $J_3(n, A, \lambda, \omega)$ show that $E\{|h_n(\lambda)|^2\}$ is $O(n^{-2})$ for all λ and hence the desired summability of (2.13) is established. □

Theorem 2 shows that any condition that suffices for the strong law for Y gives the strong law for X. Since $Y(t)$ is wide sense stationary with correlation function given by (1.5) with $k = 0$, the conditions of Gaposhkin (1977) on $a_0(\tau)$ or $G_0(\lambda)$ will suffice for the a.s. convergence to 0 of $J_{A,Y}(\lambda)$. For example, if $G_0(\lambda)$ is continuous at $\lambda = 0$ and there is a $\lambda' > 0$ for which

$$\int_{0<|\lambda|<\lambda'} \left(\log\log\frac{1}{|\lambda|}\right)^2 \mathrm{d}G_0(\lambda) < \infty, \tag{2.15}$$

then the strong law holds for $Y(t)$ and therefore for $X(t)$. The preceding complements a recent result by Honda (1990) on the strong law of large numbers for *Gaussian* PC processes. Gaposhkin (1977) also gives sufficient conditions for the strong law for processes that are quasi-stationary in the sense $m(t) = 0$, $R(t,t)$ is bounded and $|R(t+\tau, t)| \le \phi(\tau)$ where $\phi(\tau) \downarrow 0$ as $\tau \uparrow \infty$. If $\int_1^\infty \tau^{-1}\phi(\tau)\mathrm{d}\tau < \infty$, then the strong law holds for $X(t)$.

For the example $X(t) = f(t)Z(t)$ we may contrast the two sufficient conditions for the strong law. The spectral condition is that $G_0(\lambda) = \sum |f_n|^2 F_Z(\lambda - 2\pi n/T)$ must be continuous at $\lambda = 0$ and satisfy (2.15). On the other hand, if $f(t)$ is bounded and continuous, and if $|R_Z(\tau)| \le \phi(\tau)$ with $\phi(\tau) \downarrow 0$ as $\tau \uparrow \infty$ and $\int_1^\infty \tau^{-1}\phi(\tau)\mathrm{d}\tau < \infty$, then $X(t)$ is quasi-stationary and the strong law holds for $X(t)$. It may be noted that the spectral condition constrains $F_Z(\lambda)$ only at the points $\{\lambda = 2\pi n/T\colon f_n \neq 0\}$, thus permitting $F_Z(\lambda)$ to have jumps elsewhere and in this event, $R_Z(\tau)$ will oscillate as $\tau \to \infty$, thus violating the condition of quasi-stationarity.

3. Almost periodically correlated processes

As noted in the introduction, if $X(t)$ is APC and harmonizable, then we can conclude a weak law of large numbers based on the harmonizability assumption. But the APC processes are not all harmonizable because they contain the continuous PC processes which are not all harmonizable, as pointed out by Gladyshev (1963). Our approach is to obtain conditions yielding an approximation similar to (2.1), and the rest will follow. But the proof of Lemma 1 relies heavily on the periodicity and so an alternative proof is needed for APC processes. We have found a method of proof, subject to a condition on the frequencies Λ, which is developed in Lemmas 2 and 3.

First, Lemma 2 gives a condition under which the average $A^{-1}\int_a^{a+A} f(x)\mathrm{d}x$ of a complex valued UAP function $f(x)$ converges as $A \to \infty$ to its limit $M\{f\}$ at a rate $O(A^{-1})$ uniformly in the variable a; for an arbitrary UAP function, the convergence is uniform in a, but not necessarily $O(A^{-1})$. For the UAP function f, denote $\Lambda_f = \{\lambda_j\colon M\{f(x)\exp(-\mathrm{i}\lambda_j x)\} \neq 0\}$ and recall that Λ_f is countable.

Lemma 2. *If $f(x)$ is UAP and $\sum_{0 \neq \lambda_j \in \Lambda_f} \lambda_j^{-2} < \infty$, then there exists a finite constant K such that for arbitrary a and for $A > 0$,*

$$\left| \int_a^{a+A} f(x)\,\mathrm{d}x - AM\{f\} \right| < K. \tag{3.1}$$

Proof. To prove this claim we recall from the theory of UAP functions (Corduneanu, 1968, 1989, p. 41) that there exists a sequence of trigonometric polynomials

$$f_n(x) = \sum_{\lambda_j \in \Lambda_n} a_{j,n} \exp(\mathrm{i}\lambda_j x), \tag{3.2}$$

converging uniformly to $f(x)$ where the finite set $\Lambda_n \uparrow \Lambda_f$ and for every j,

$$\lim_{n \to \infty} a_{j,n} = a_j \equiv M\{f(x)\exp(-\mathrm{i}\lambda_j x)\}.$$

Therefore, it suffices to show (3.1) for f_n. Then

$$\frac{1}{A}\int_a^{a+A} f_n(x)\,\mathrm{d}x = a_{0,n} + \sum_{0 \neq \lambda_j \in \Lambda_n} a_{j,n} \mathrm{e}^{\mathrm{i}\lambda_j a} \frac{\mathrm{e}^{\mathrm{i}\lambda_j A} - 1}{\mathrm{i}\lambda_j A} \xrightarrow[A \to \infty]{} a_{0,n} = M\{f_n\},$$

where $a_{0,n} = 0$ if $0 \notin \Lambda_n$, and

$$\left| \frac{1}{A}\int_a^{a+A} f_n - M\{f_n\} \right| \leq \sum_{0 \neq \lambda_j \in \Lambda_n} |a_{j,n}| \frac{2}{|\lambda_j| A}$$
$$\leq \frac{2}{A}\left(\sum_{\lambda_j \in \Lambda_n} |a_{j,n}|^2 \right)^{1/2} \left(\sum_{0 \neq \lambda_j \in \Lambda_n} \lambda_j^{-2} \right)^{1/2}. \tag{3.3}$$

Finally, $f_n \to f$ implies that $M\{|f_n|^2\} = \sum_{\lambda_j \in \Lambda_n} |a_{j,n}|^2 \to M\{|f|^2\} = \sum_{j \in \mathbb{Z}} |a_j|^2 < \infty$, because the Fourier coefficients of UAP functions are square summable. This leads to

$$\left| \frac{1}{A}\int_a^{a+A} f_n - M\{f_n\} \right| \leq \frac{3}{A}(M\{|f|^2\})^{1/2} \left(\sum_{0 \neq \lambda_j \in \Lambda_f} \lambda_j^{-2} \right)^{1/2}$$

for sufficiently large n, which implies (3.1). □

For APC processes, the development in Lemma 2 may be applied to each UAP function $B(\cdot, \tau)$, $\tau \in \mathbb{R}$, to obtain the required approximation. We recall that if Λ_τ is the set of frequencies for $B(\cdot, \tau)$ then $\Lambda = \bigcup_{\tau \in \mathbb{R}} \Lambda_\tau$ is countable.

Lemma 3. *If $X(t)$ is APC and $\sum_{0 \neq \lambda_j \in \Lambda} \lambda_j^{-2} < \infty$, then there exists a finite $K > 0$, such that for every u, v and τ,*

$$\left| \int_u^v B(t,\tau)\,\mathrm{d}t - (v-u)a(0,\tau) \right| < K. \tag{3.4}$$

Proof. Using (3.2) for the UAP function $B(\cdot, \tau)$, with $\Lambda_{n,\tau} \uparrow \Lambda_\tau \subset \Lambda$, we have

$$B_n(t,\tau) = \sum_{\lambda_j \in \Lambda_{n,\tau}} a_n(\lambda_j, \tau) e^{i\lambda_j t} = a_n(0,\tau) + \sum_{0 \neq \lambda_j \in \Lambda_{n,\tau}} a_n(\lambda_j, \tau) e^{i\lambda_j t},$$

and by (3.3),

$$\left| \int_u^v B_n(t,\tau)\,dt - a_n(0,\tau)(v-u) \right| \leq 2\left(\sum_{\lambda_j \in \Lambda_{n,\tau}} |a_n(\lambda_j,\tau)|^2 \right)^{1/2} \left(\sum_{0 \neq \lambda_j \in \Lambda_{n,\tau}} \lambda_j^{-2} \right)^{1/2}.$$

Letting $n \to \infty$ gives

$$\left| \int_u^v B(t,\tau)\,dt - a(0,\tau)(v-u) \right| \leq 2\left(\sum_{\lambda_j \in \Lambda_\tau} |a(\lambda_j,\tau)|^2 \right)^{1/2} \left(\sum_{0 \neq \lambda_j \in \Lambda_\tau} \lambda_j^{-2} \right)^{1/2}.$$

Then (3.4) follows since for all τ we have $\sum_{0 \neq \lambda_j \in \Lambda_\tau} \lambda_j^{-2} \leq \sum_{0 \neq \lambda_j \in \Lambda} \lambda_j^{-2}$, and

$$\sum_{\lambda_j \in \Lambda_\tau} |a(\lambda_j,\tau)|^2 = \lim_{A \to \infty} \frac{1}{A} \int_0^A |B(t,\tau)|^2\,dt \leq \lim_{A \to \infty} \frac{1}{A} \int_0^A B(t+\tau,0) B(t,0)\,dt$$

$$= \lim_{A \to \infty} \frac{1}{A} \int_0^A B^2(t,0)\,dt = \sum_{\lambda_j \in \Lambda_0} |a(\lambda_j,0)|^2,$$

where we used $B^2(t,\tau) \leq B(t+\tau,0)B(t,0)$, and the Schwarz inequality in $L_2[0,T]$. □

In view of Lemma 3, the results of Theorem 1 remains true for APC processes whenever $\sum_{0 \neq \lambda_j \in \Lambda} \lambda_j^{-2} < \infty$, a condition that is always satisfied for PC processes.

Theorem 3. *If $X(t)$ is APC and $\sum_{0 \neq \lambda_j \in \Lambda} \lambda_j^{-2} < \infty$, then for each λ, as $A \to \infty$, $J_{A,X}(\lambda)$ converges in q.m. to a random variable η_λ where* $E\{|\eta_\lambda|^2\} = G_0(\lambda) - G_0(\lambda-)$.

Proof. We are reminded that APC processes are continuous in q.m., have bounded second moments and $a(0,\tau)$ is continuous at $\tau = 0$. Thus, $J_{A,X}(\lambda)$ may be interpreted either as a Lebesgue integral a.s. or as a q.m. integral in the Riemann sense and $a(0,\tau)$ has the representation (1.8). The remainder of the proof follows exactly the proof of Theorem 1 with the bound $Ta_0(0)$ of Lemma 1 replaced with K from Lemma 3. □

Theorem 3 gives the following weak law for APC processes that satisfy $\sum \lambda_k^{-2} < \infty$: *$J_{A,X}(0) \to 0$ in q.m. if and only if $G_0(\lambda)$ is continuous at $\lambda = 0$.*

That the condition of square summability in Theorem 3 is not necessary is illustrated by the example of almost periodic amplitude modulation of a stationary process: $X(t) = f(t)Z(t)$. Let $f(t)$ be continuous, bounded and AP so that $f(t) \sim \sum_k f_k e^{i\lambda_k t}$, $\sum_k |f_k|^2 < \infty$, and $Z(t)$ be q.m. continuous weakly stationary so that

$Z(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\zeta(\lambda)$ where ζ has orthogonal increments, $E|d\zeta(\lambda)|^2 = dF_Z(\lambda)$ and F_Z is non-decreasing, right-continuous and bounded. Then

$$J_{A,X}(\lambda) = \int_{-\infty}^{\infty} \left\{ \frac{1}{A} \int_0^A e^{it(u-\lambda)} f(t)\, dt \right\} d\zeta(u).$$

The integrand tends as $A \to \infty$ to f_k if $\lambda - u = \lambda_k \in \Lambda_f$ and to 0 otherwise and is bounded in absolute value by $\sup_{t\in\mathbb{R}} |f(t)| < \infty$, thus

$$\frac{1}{A} \int_0^A e^{it(u-\lambda)} f(t)\, dt \xrightarrow[A\to\infty]{} \sum_k f_k 1_{\{\lambda-\lambda_k\}}(u) \quad \text{in } L^2(dF).$$

It follows that the limit of $J_{A,X}(\lambda)$ always exists in q.m.

$$J_{A,X}(\lambda) \xrightarrow[A\to\infty]{} \sum_k f_k \zeta(\{\lambda - \lambda_k\}) \triangleq \eta_\lambda$$

and $E\{|\eta_\lambda|^2\} = \sum_k |f_k|^2 \{F_Z(\lambda - \lambda_k) - F_Z(\lambda - \lambda_k^-)\}$. In particular the weak law $J_{A,X}(0) \to 0$ in q.m. is satisfied if and only if the spectral distribution $F_Z(\lambda)$ of the stationary process $Z(t)$ is continuous at the frequencies $\lambda_k \in \Lambda_f$ of the AP amplitude modulation $f(t)$ (since $dF_Z(-\lambda) = dF_Z(\lambda)$). In this case the frequencies Λ are obtained from Λ_f as follows. From (1.7) we have

$$a(\lambda, \tau) = R_Z(\tau) \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t+\tau)\overline{f(t)} e^{-i\lambda t} dt = R_Z(\tau) \sum_{\substack{k \ j \\ \lambda_k - \lambda_j = \lambda}} \sum f_k \bar{f}_j e^{i\lambda_k \tau}$$

so that $\Lambda = \{\lambda = \lambda_k - \lambda_j : \lambda_k, \lambda_j \in \Lambda_f\}$. In particular $a(0,\tau) = R_Z(\tau) \sum_k |f_k|^2 e^{i\lambda_k \tau}$ and if we take $\lambda_k = ck^{1/2}$ then $\lambda_k - \lambda_0 = ck^{1/2}$ and $\sum_{k \neq 0} (\lambda_k - \lambda_0)^{-2} = \infty$ so that $\sum_{0 \neq \lambda_n \in \Lambda} \lambda_n^{-2} = \infty$. Thus, in this example the conclusion of Theorem 3 holds true without the square summability assumption.

The random shift interpretation may also be applied to APC processes since they can also be stationarized by a random time shift.

Theorem 4. *If $X(t)$ is APC, and 0 is not a limit point of Λ, the distribution of a real random variable Θ, independent of $X(t)$, may be chosen so that $X(t + \Theta)$ is weakly stationary with correlation function $a(0, \tau)$.*

Proof. Put $Y(t) = X(t + \Theta)$. Since Θ is independent of $X(t)$ we have

$$R_Y(s,t) = E\{X(s+\Theta)\bar{X}(t+\Theta)\} = E\{E[X(s+\Theta)\bar{X}(t+\Theta)|\Theta]\}$$

$$= \int_{-\infty}^{\infty} E[X(s+\theta)\bar{X}(t+\theta)]\mu(d\theta) = \int_{-\infty}^{\infty} R(s+\theta, t+\theta)\mu(d\theta)$$

or

$$B_Y(t,\tau) = \int_{-\infty}^{\infty} B(t+\theta, \tau)\mu(d\theta), \tag{3.5}$$

where μ is the distribution of Θ. To show that $Y(t)$ is weakly stationary we will show that $R_Y(u,v) = R_Y(u-v)$, or equivalently that $B_Y(t,\tau) = B_Y(\tau)$. First, we may conclude that $Y(t)$ is APC. To see this, the inequality

$$|B_Y(t+\alpha,\tau) - B_Y(t,\tau)| \le \int_{-\infty}^{\infty} |B(t+\alpha+\theta,\tau) - B(t+\theta,\tau)|\mu(\mathrm{d}\theta)$$

$$\le \sup_{\theta\in\mathbb{R}} |B(\theta+\alpha,\tau) - B(\theta,\tau)|$$

shows that

$$\left\{\alpha\colon \sup_{\theta\in\mathbb{R}}|B(\theta+\alpha,\tau) - B(\theta,\tau)| < \varepsilon\right\} \subseteq \left\{\alpha\colon \sup_{t\in\mathbb{R}}|B_Y(t+\alpha,\tau) - B_Y(t,\tau)| < \varepsilon\right\}.$$

Therefore, the larger set is relatively dense, since the smaller set is. Since $B_Y(t,\tau)$ is also seen to be uniformly continuous in (t,τ), we conclude (Corduneanu, 1968, 1989, p. 14) that $B(\cdot,\tau)$ is UAP for every τ and so $Y(t)$ is APC.

It follows that $B_Y(t,\tau)$ has a Fourier series as in (1.9) possessing at most a countable set of frequencies and coefficient functions

$$a_Y(\lambda,\tau) = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} B_Y(t,\tau)\exp(-\mathrm{i}\lambda t)\,\mathrm{d}t$$

$$= \lim_{A\to\infty} \int_{-\infty}^{\infty} \frac{1}{2A}\int_{-A+\theta}^{A+\theta} B(s,\tau)\exp(-\mathrm{i}\lambda s + \mathrm{i}\lambda\theta)\,\mathrm{d}s\mu(\mathrm{d}\theta) \tag{3.6}$$

in view of (3.5). But $(2A)^{-1}\int_{-A+\theta}^{A+\theta} B(s,\tau)\exp(-\mathrm{i}\lambda s)\,\mathrm{d}s$ is a continuous function of θ and, since $B(s,\tau)$ is bounded, converges boundedly in θ to $a(\lambda,\tau)$. Thus, by the bounded convergence theorem, (3.6) becomes

$$a_Y(\lambda,\tau) = a(\lambda,\tau)\int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda\theta)\mu(\mathrm{d}\theta) = a(\lambda,\tau)\Phi_\Theta(\lambda), \tag{3.7}$$

where $\Phi_\Theta(\lambda)$ is the characteristic function of Θ. This expression shows that the set Λ_Y of frequencies of $B_Y(\cdot,\tau)$ is a subset of the set of frequencies Λ of X, and that Y will be weakly stationary if $\Phi_\Theta(\lambda) = 0$ for $0 \neq \lambda \in \Lambda$.

Since 0 is not a limit point of Λ, there exists an interval $[-\varepsilon,\varepsilon]$ that contains only one element of Λ, namely $\lambda = 0$. It is not difficult to choose the distribution of Θ so that $\Phi_\Theta(\lambda) = 0$ for $\lambda \notin [-\varepsilon,\varepsilon]$; e.g., $\Phi_\Theta(\lambda)$ triangular. For such Θ we have $a_Y(\lambda,\tau) = 0$ except for $\lambda = 0$ and so for each τ, the UAP function $B_Y(t,\tau)$ is equal to its 0th Fourier coefficient $a(0,\tau)$ from the uniqueness of the Fourier coefficients of UAP functions; thus $Y(t)$ is weakly stationary and its correlation is $a(0,\tau)$. □

Theorem 4 is motivated by Gardner (1978), who considers a narrower class of APC processes, termed almost cyclostationary. When Θ in Theorem 4 is chosen to have a triangular characteristic function $\Phi_\Theta(\lambda) = \max(0, 1-|\lambda|/\varepsilon)$, its density is

$\phi_\Theta(\theta) = (\varepsilon/2\pi)\{\sin(\varepsilon\theta/2)/(\varepsilon\theta/2)\}^2$ and $E(|\Theta|^p) < \infty$ only when $0 < p < 1$. To come up with a Θ with finite variance take its characteristic function to be the convolution of the triangular characteristic function with itself so that its density is $\phi_\Theta(\theta) = (3\varepsilon/4\pi)\{\sin(\varepsilon\theta/2)/(\varepsilon\theta/2)\}^4$ ($\sim c|\theta|^{-4}$ as $|\theta| \to \infty$) and then $E(|\Theta|^p) < \infty$ for $0 < p < 3$. The following construction, also due to Gardner (1978) and applied to harmonizable APC processes in Hurd (1991), may be used to give a different Θ of finite variance that makes $X(t + \Theta)$ weakly stationary. We first note that it may be shown from (1.7) that $a(-\lambda, \tau) = \bar{a}(\lambda, -\tau)\exp(-i\lambda\tau)$, so that $\lambda \in \Lambda$ implies $-\lambda \in \Lambda$. Let $\Lambda^+ = \{\lambda \in \Lambda\colon \lambda > 0\}$. The random variable Θ is the almost sure limit of the sequence

$$\Theta_n = \sum_{j=1}^{n} \eta_j \tag{3.8}$$

where η_j is uniformly distributed over the interval $[-\pi/\lambda_j, \pi/\lambda_j]$ for $\lambda_j \in \Lambda^+$ and for all n the random variables $\eta_1, \eta_2, \ldots, \eta_n$ are independent and are independent of the process $\{X(t), t \in \mathbb{R}\}$. Under the hypothesis $\sum_{0 \neq \lambda_j \in \Lambda} \lambda_j^{-2} < \infty$ it may be shown that Θ_n converges with probability 1 to a random variable Θ having finite variance (see Hurd, 1991). Also

$$\Phi_\Theta(\lambda) = \prod_{j=1}^{\infty} \frac{\sin(\pi\lambda/\lambda_j)}{\pi\lambda/\lambda_j} \tag{3.9}$$

satisfies $\Phi_\Theta(\lambda_j) = 0$ for $\lambda_j \in \Lambda$ except $\Phi_\Theta(0) = 1$.

Now, as in Proposition 1, we are able to relate the limiting random variable η_λ given by Theorem 3 to the spectral process $\zeta(\lambda)$ associated with $X(t + \Theta)$.

Proposition 2. *If $X(t)$ is APC and for each λ, as $A \to \infty$, $J_{A,X}(\lambda) \to \eta_\lambda$ in q.m., and if Θ is independent of X and $X(t + \Theta)$ is weakly stationary with spectral process $\zeta(\lambda)$, then $\eta_\lambda \exp(i\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda -)$ a.s.*

Proof. From (2.8) and (2.10) we have

$$E\left\{\left|\frac{1}{A}\int_0^A X(t + \Theta)\exp(-i\lambda t)\,dt - \eta_\lambda \exp(i\lambda\Theta)\right|^2\right\} \le E\{I(A, \Theta)\} + J(A). \tag{3.10}$$

By assumption $J(A) \to 0$ as $A \to \infty$. We now show that $E\{I(A, \Theta)\} \to 0$ as $A \to \infty$. Since $B(\cdot, 0)$ is UAP, it is bounded by say $M < \infty$, and by (2.9) we have

$$I(A, \theta) \le \frac{2}{A^2}|D|^2 M = \frac{8M}{A^2}\min(\theta^2, A^2),$$

where we used $|D| \le 2\min(|\theta|, A)$. Thus, for each fixed θ, $I(A, \theta) \to 0$ as $A \to \infty$ and for all θ and A, $I(A, \theta) \le 8M$. It follows that $E\{I(A, \Theta)\} \to 0$ as $A \to \infty$. Hence, (3.10) tends to 0 as $A \to \infty$. But as in Theorem 3, $A^{-1}\int_0^A X(t + \Theta)\exp(-i\lambda t)\,dt \to \zeta(\lambda) - \zeta(\lambda -)$ in q.m. and so $\eta_\lambda \exp(i\lambda\Theta) = \zeta(\lambda) - \zeta(\lambda -)$. □

To use the shift notion to obtain strong laws in the APC case we have to be a little more careful because $\Theta(\omega)$ is not confined to a bounded interval. The following theorem shows that if $\Theta(\omega)$ has finite absolute moment of order larger than one, then a version of Theorem 2 is obtained for APC processes. As the proof uses only the fact that APC processes have bounded variances, the result is stated in this more general setup.

Theorem 5. *Suppose $X(t)$ is a measurable process and satisfies $E\{|X(t)|^2\} \le M$ for all t, Θ is independent of X with $E(|\Theta|^p) < \infty$ for some $1 < p \le 2$, and $Y(t) = X(t+\Theta)$. Then if either $J_{A,X}(\lambda)$ or $J_{A,Y}(\lambda)$ converge a.s., then the other does also and $\lim_{A\to\infty} J_{A,X}(\lambda) = \exp(-i\lambda\Theta)\lim_{A\to\infty} J_{A,Y}(\lambda)$ a.s.*

Proof. As in Theorem 2 we will show that the two terms on the right-hand side of (2.11) converge to zero a.s. The second term converges to zero a.s. because $\Theta(\omega)$ is finite a.s., and thus for almost every ω, it is upper-bounded in absolute value by $\int_0^{|\Theta(\omega)|} |X(t,\omega)|\,dt < \infty$. To prove that the first term in the right-hand side of (2.11) converges to zero a.s., we first show that it does so for the particular sequence $A_n = nT$. For $g_n(\lambda,\omega)$ given by (2.12) we obtain

$$|g_n(\lambda)| \le \frac{1}{nT}\int_{nT}^{nT+|\Theta|} |X(t)|\,dt$$

and

$$|g_n(\lambda)|^p \le \frac{1}{n^pT^p}|\Theta|^{p/2}\left(\int_{nT}^{nT+|\Theta|} |X(t)|^2\,dt\right)^{p/2}$$

and by the independence of Θ and X, and since $2/p \ge 1$, we have

$$\begin{aligned} E\{|g_n(\lambda)|^p\} &\le \frac{1}{n^pT^p}\int_{-\infty}^{\infty} |\theta|^{p/2} E\left\{\left(\int_{nT}^{nT+\theta} |X(t)|^2\,dt\right)^{p/2}\right\}\mu(d\theta) \\ &\le \frac{1}{n^pT^p}\int_{-\infty}^{\infty} |\theta|^{p/2}\left[E\left\{\int_{nT}^{nT+\theta} |X(t)|^2\,dt\right\}\right]^{p/2}\mu(d\theta) \\ &\le \frac{1}{n^pT^p}\int_{-\infty}^{\infty} |\theta|^{p/2}[|\theta|M]^{p/2}\mu(d\theta) = \frac{M^{p/2}E(|\Theta|^p)}{n^pT^p}, \end{aligned}$$

where $E(|\Theta|^p) < \infty$ by assumption and M is the bound on $E\{|X(t)|^2\} = R(t,t) = B(t,0)$ which is UAP and thus bounded. We conclude that $\sum_n E\{|g_n(\lambda)|^p\} < \infty$, and as before the result follows from the Borel–Cantelli lemma. To show the general result we also proceed, as in the proof of Theorem 2, to show the analog of (2.13), $\sum_n E\{|h_n(\lambda)|^p\} < \infty$, by establishing that $E\{|h_n(\lambda)|^p\}$ is $O(n^{-p})$. Again using the analog of (2.14) based on $|x+y+z|^p \le 3^p(|x|^p + |y|^p + |z|^p)$, the term involving J_1 is seen to be $O(n^{-p})$ because Θ does not come into the argument. The term involving J_3

is also $O(n^{-p})$ because for each ω the integral is over an interval of length less than T. We shall now show $E\{\sup_{(n-1)T\le A<nT}|J_2(n,A,\lambda)|^p\}$ is $O(n^{-2p})$ and thus also $O(n^{-p})$. For $(n-1)T\le A<nT$ we have

$$
\begin{aligned}
|J_2(n,A,\lambda)|^p &\le \left(\frac{1}{A}-\frac{1}{nT}\right)^p\left|\int_{nT}^{A+\Theta}|X(t)|\,\mathrm{d}t\right|^p\\
&\le \left(\frac{1}{(n-1)T}-\frac{1}{nT}\right)^p\left(\int_{(n-1)T-|\Theta|}^{nT+|\Theta|}|X(t)|\,\mathrm{d}t\right)^p\\
&\le \frac{1}{n^p(n-1)^pT^p}(T+2|\Theta|)^{p/2}\left(\int_{(n-1)T-|\Theta|}^{nT+|\Theta|}|X(t)|^2\,\mathrm{d}t\right)^{p/2}
\end{aligned}
$$

and since X and Θ are independent and we obtain

$$
\begin{aligned}
&E\left\{\sup_{(n-1)T\le A<nT}|J_2(n,A,\lambda)|^p\right\}\\
&\le \frac{1}{n^p(n-1)^pT^p}\int_{-\infty}^{\infty}(T+2|\theta|)^{p/2}E\left(\int_{(n-1)T-|\theta|}^{nT+|\theta|}|X(t)|^2\,\mathrm{d}t\right)^{p/2}\mu(\mathrm{d}\theta)\\
&\le \frac{1}{n^p(n-1)^pT^p}\int_{-\infty}^{\infty}(T+2|\theta|)^{p/2}\left\{E\int_{(n-1)T-|\theta|}^{nT+|\theta|}|X(t)|^2\,\mathrm{d}t\right\}^{p/2}\mu(\mathrm{d}\theta)\\
&\le \frac{M^{p/2}E\{(T+2|\Theta|)^p\}}{n^p(n-1)^pT^p}.
\end{aligned}
$$

This establishes the desired summability of $\sum_n E\{|h_n(\lambda)|^p\}<\infty$. □

As is Theorem 2, this result may be applied to argue that any condition that suffices for the strong law for Y will give the strong law for X. For example, if $\sum_{0\neq\lambda_j\in\Lambda}\lambda_j^{-2}<\infty$ and Θ has characteristic function (3.9), then Θ has finite variance and $Y(t)$ is weakly stationary with correlation function $a(0,\tau)$ given by (1.8) and so the conditions of Gaposhkin (1977) on $a(0,\tau)$ or $G_0(\gamma)$ will suffice for the a.s. convergence of $J_{A,X}(\lambda)$.

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References

A.S. Besicovitch, Almost Periodic Functions (Cambridge University Press, London, 1932).
C. Corduneanu, Almost Periodic Functions (Chelsea, New York, 1989); also (Wiley, New York, 1968).

J.L. Doob, Stochastic Processes (Wiley, New York, 1953).
V.F. Gaposhkin, Criteria for the strong law of large numbers for some classes of second order stationary processes and homogeneous random fields, Theory Probab. Appl. 22 (1977) 287–310.
W.A. Gardner, Stationarizable random processes, IEEE Trans. Inform. Theory IT-24 (1978) 8–22.
E.G. Gladyshev, Periodically correlated random sequences, Soviet Math. 2 (1961) 383–388.
E.G. Gladyshev, Periodically and almost periodically correlated random processes with continuous time parameter, Theory Probab. Appl. 8 (1963) 173–177.
I. Honda, On the spectral representation and related properties of periodically correlated stochastic processes, Trans. IEICE Japan E65 (1982) 723–729.
I. Honda, On the ergodicity of Gaussian periodically correlated stochastic processes, Trans. IEICE Japan E73 (1990) 1729–1737.
H.L. Hurd, Periodically correlated processes with discontinuous correlation functions, Theory Probab. Appl. 19 (1974a) 834–838.
H.L. Hurd, Stationarizing properties of random shifts, SIAM J. Appl. Math. 26 (1974b) 203–211.
H.L. Hurd, Representation of strongly harmonizable periodically correlated processes and their covariances, J. Multivariate Anal. 29 (1989) 53–67.
H.L. Hurd, Correlation theory for almost periodically correlated processes, J. Multivariate Anal. 37 (1991) 24–45.
M. Loève, Probability Theory (Van Nostrand, New York, 3rd ed., 1963).
A.M. Yaglom, Correlation Theory of Stationary and Related Random Functions, Vols. 1 and 2 (Springer, New York, 1987).