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A simplified analytical approach for pricing discretely-sampled variance swaps with stochastic volatility

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ABSTRACT

Pricing variance swaps under stochastic volatility with discretely-sampled realized variance has been a hot subject pursued recently; quite a few papers have already been published (Zhu and Lian (2009, 2011, [11,4]); Swishchuk and Li (2011) [5]). In this paper, we present a simplified approach to price discretely-sampled variance swaps. Compared with the approach presented by Zhu and Lian (2011) [4], an important feature of our approach is that there is no need for the introduction of a new state variable and the utilization of the generalized Fourier transform. This has significantly simplified the solution procedure and will thus enable researchers to view this type of problems from a different angle.

1. Introduction

Pricing variance swaps under stochastic volatility with discretely-sampled realized variance has been pursued by many researchers recently; various approaches have been proposed. Of course, this is mainly due the substantially increased trading activities of volatility-related derivatives in the past few years[1]. In the literature, there have been two types of valuation approaches, numerical methods and analytical methods. The focus of this paper is to provide a simplified analytical approach to price discretely-sampled variance swaps.

The most influential pioneer works were from [1,2], who have shown how to theoretically replicate a variance swap by a portfolio of standard options. Without requiring to specify the function of volatility process, their models and analytical formulae are indeed very attractive. However, as pointed out by Carr and Corso [3], the replication strategy has a drawback that the sampling time of a variance swap is assumed to be continuous rather than discrete; such an assumption implies that the results obtained from a continuous model can only be viewed as an approximation for the real cases in financial practice, in which all contacts are written with the realized variance being evaluated on a set of discrete sampling points. Another drawback is that this strategy also requires options with a continuum of exercise prices, which is not actually available in a marketplace. The more recent trend is to adopt stochastic volatility models. As pointed out in Zhu and Lian [4] (referred to as ZL1 hereafter), the analytic approaches in this category can be divided into two subcategories, the first of which shares a common assumption that the realized variance is approximated with a continuously-sampled one, which has greatly increased the mathematical tractability, while those in the second subcategory try to directly address the "discretely-sampled" nature of variance swaps.

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1 http://cfe.cboe.com/education/finaleuromoneyvarpaper.pdf

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Due to the limited space here, readers who are interested in the papers in the first subcategory are referred to the review given in ZL1. In addition, it should be mentioned that the most recent paper in this subcategory is presented by Swishchuk and Li [5], who studied valuation of variance swaps under stochastic volatility with delay and jumps and derived an analytic closed-form formula for pricing variance swaps under the assumption of continuous sampling.

On the other hand, in the second subcategory, Broadie and Jain [6] presented a closed-form solution for volatility as well as variance swaps with discrete sampling. They also examined the effects of jumps and stochastic volatility on the price of volatility and variance swaps by comparing calculated prices under various models such as the Black–Scholes model, the Heston stochastic volatility model, the Merton [7] jump diffusion model and the Bates [8] and Scott [9] stochastic volatility and jump model. Zhu and Lian [4] also presented an approach to obtain a closed-form formula for variance swaps based on the discretely-sampled realized variance with the realized variance being defined as the sum of the squared percentage increment of the underlying price. Unlike Broadie and Jain's approach [6], ZL1's approach is based on Little and Pant's approach [10] and they found a simple formula by solving the governing PDE (Partial Differential Equation) system directly. Moreover, their approach is more versatile in terms of dealing with different forms of realized variance. For example, using ZL1's approach, not only can the case with realized variance defined in terms of squared percentage return be dealt with as shown in [4], the case with realized variance defined in terms of squared log return can be dealt with as well, as demonstrated in Zhu and Lian [11]. However, there is actually a simplified approach than that presented by ZL1, which would lead to exactly the same results for the percentage return case, but the procedure with which the generalized Fourier transform needs to be utilized as in ZL1 can be completely avoided. This is the subject of the current paper.

The remainder of the paper is organized as follows. In Section 2, the details of our newly proposed solution approach are given and our results are shown to be identically the same as those obtained by ZL1. In Section 3, we further discuss some major differences between our current approach and that of ZL1, followed by a brief conclusion in Section 4.

2. Our new solution approach

In this section, for the sake of completeness of the paper and easiness for the readers, we shall briefly describe the Heston model [12], we adopt to describe the dynamics of the underlying asset first. Then, we shall show, in detail, our new solution approach based on the dimension-reduction technique initially proposed by Schwartz [13].

2.1. The Heston model

ZL1 has clearly demonstrated the need to use a stochastic volatility model in pricing volatility-based financial derivatives, such as variance swaps discussed in this paper. For this reason, we shall begin this section with a brief description of the Heston stochastic volatility model [12], which we adopt to demonstrate our approach. In this model, the dynamics of the underlying asset S_t is assumed to follow the diffusion process with a stochastic instantaneous variance v_t,

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t^S \\ dv_t = \kappa(\theta - v_t)dt + \sigma_V \sqrt{v_t} dB_t^V \end{cases} \tag{2.1}$$

where μ is the expected return of the underlying asset, θ is the long-term mean of variance, κ is a mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility. The two Wiener processes dB_t^S and dB_t^V describe the random noise in asset and variance respectively. They are assumed to be correlated with a constant correlation coefficient ρ, that is $(dB_t^S, dB_t^V) = \rho dt$. The stochastic volatility process is the familiar squared-root process. To ensure the variance is always positive, it is required that $2\kappa\theta \geq \sigma_V^2$ (see [14,12]).

According to the existence theorem of equivalent martingale measure, we are able to change the real probability measure to a risk-neutral probability measure and describe the processes as:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{B}_t^S \\ dv_t = \kappa^*(\theta^* - v_t)dt + \sigma_V \sqrt{v_t} d\widetilde{B}_t^V \end{cases} \tag{2.2}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa\theta}{\kappa+\lambda}$ are the risk-neutral parameters and the new parameter λ is the premium of volatility risk [12]. For the rest of this paper, our analysis will be based on the risk-neutral probability measure. The conditional expectation at time t is denoted by $E_t^Q = E^Q[\cdot \mid \mathcal{F}_t]$, where $\mathcal{F}_t$ is the filtration up to time t.

2.2. Variance swaps

Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The long position of a variance swap pays a fixed delivery price at expiry and receives the floating amounts of annualized realized variance, whereas the short position is just the opposite. Thus it can be easily used for investors to gain exposure to volatility risk.

Usually, the value of a variance swap at expiry can be written as $V_T = (\sigma_R^2 - K_{var}) \times L$, where the σ_R^2 is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, and L is the notional amount of the swap in dollars per annualized volatility point squared. The T is the life time of the contract.

At the beginning of a contract, it is clearly specified the details of how the realized variance should be calculated. Important factors contributing to the calculation of the realized variance include the underlying asset(or assets), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the variance. A typical formula for the measure of realized variance is

$$\sigma_R^2 = \frac{AF}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2 \tag{2.3}$$

where S_{t_i} is the closing price of the underlying asset at the i-th observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming that there are 252 trading days in one year, if every week then $AF = 52$, if every month then $AF = 12$ and so on. We assume equally-spaced discrete observations in this paper so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = E_t^Q[e^{-r(T-t)}(\sigma_R^2 - K_{var})L]$. This should be zero at the beginning of the contract since there is no cost to enter into a swap. Therefore, the fair variance delivery price can be easily defined as $K_{var} = E_0^Q[\sigma_R^2]$, after setting the value of $V_t = 0$ initially. The variance swap valuation problem is therefore reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

2.3. Our simplified analytical approach

Like ZL1's approach, our simplified solution approach begins with taking the expectation of σ_R^2 in (2.3). Since

$$E_0^Q[\sigma_R^2] = E_0^Q\left[\frac{1}{N\Delta t}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right]\times 100^2 = \frac{1}{N\Delta t}\sum_{i=1}^{N}E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right]\times 100^2, \tag{2.4}$$

the problem of pricing variance swap is reduced to calculating the N conditional expectations in the form of:

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \tag{2.5}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t (i = 1, \ldots, N)$. In the rest of this section, we will focus our main attention on calculating the expectation of this expression. In the process of calculating this expectation, i is regarded as a constant. And hence both t_i and t_{i-1} are regarded as known constants.

Using the fact that $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ and $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable, we apply the tower property to the conditional expectation in (2.5) and this gives us a double conditional expectation as follows:

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] = E_0^Q\left[E_{t_{i-1}}^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right]\right] = E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}(E_{t_{i-1}}^Q[S_{t_i}^2] - 2S_{t_{i-1}}E_{t_{i-1}}^Q[S_{t_i}]) + 1\right]. \tag{2.6}$$

The two conditional expectations with respect to $\mathcal{F}_{t_{i-1}}$ on the RHS of (2.6), i.e. $E_{t_{i-1}}^Q[S_{t_i}]$ and $E_{t_{i-1}}^Q[S_{t_i}^2]$, can be computed by using the following proposition with $\gamma = 1$ and $\gamma = 2$, respectively.

Proposition 2.1. *For any given $\gamma \in \mathbb{R}\backslash\{0\}$, if S_t follows the dynamics described in (2.2) and the parameters satisfy the inequality $(\kappa^* - \gamma\rho\sigma_V)^2 > \gamma(\gamma-1)\sigma_V^2$, then the conditional expectation of $Y_t = S_t^\gamma$ is given by*

$$E_{t_{i-1}}^Q[Y_t] = E^Q[Y_t|(Y_{t_{i-1}} = y, v_{t_{i-1}} = v)] = ye^{\hat{C}(\gamma,t_i-t)+\hat{D}(\gamma,t_i-t)v}, \tag{2.7}$$

for all $t \in [t_{i-1}, t_i]$ and $(y, v) \in (0,\infty)\times(0,\infty)$, with

$$\begin{cases} \hat{C}(\gamma,\tau) = \gamma r\tau + \dfrac{\kappa^*\theta^*}{\sigma_V^2}\left((\hat{a}(\gamma)+\hat{b}(\gamma))\tau - 2\ln\left(\dfrac{\hat{b}(\gamma)-\hat{a}(\gamma)+(\hat{a}(\gamma)+\hat{b}(\gamma))e^{\hat{b}(\gamma)\tau}}{2\hat{b}(\gamma)}\right)\right), \\ \hat{D}(\gamma,\tau) = \dfrac{\gamma(\gamma-1)(e^{\hat{b}(\gamma)\tau}-1)}{\hat{b}(\gamma)-\hat{a}(\gamma)+(\hat{a}(\gamma)+\hat{b}(\gamma))e^{\hat{b}(\gamma)\tau}} \quad \textit{for all } \tau \geq 0, \\ \hat{a}(\gamma) = \kappa^* - \gamma\rho\sigma_V, \quad \textit{and} \quad \hat{b}(\gamma) = \sqrt{\hat{a}^2(\gamma)-\gamma(\gamma-1)\sigma_V^2}. \end{cases} \tag{2.8}$$

The proof of this proposition is left in Appendix. It should be remarked that the inequality imposed in Proposition 2.1 is a sufficient condition for a global solution in affine form of (2.7). For pricing variance swaps, only two γ values ($\gamma = 1$ and

$\gamma = 2$) are needed, which have restricted a global solution in affine form with $\kappa^* > (\sqrt{2} + 2\rho)\sigma_V$ in the parameter space. Of course, this condition is independent of an earlier condition, $2\kappa\theta \geq \sigma_V^2$, imposed for the stochastic processes (2.7). For $\kappa^* \leq (\sqrt{2} + 2\rho)\sigma_V$, only local solutions in affine form exist, as shown in Appendix. This suggests that if the Heston model with a slow mean reversion rate (i.e., with $\kappa^* \leq (\sqrt{2} + 2\rho)\sigma_V$) is adopted to price discretely sampled variance swaps with the realized variance defined by (2.3), one has to either seek an analytical solution in a different form or directly resort to numerical solutions of the PDEs as presented in [10].

Utilizing Proposition 2.1 with $\gamma = 1$ and $\gamma = 2$, respectively, we can compute the two conditional expectations

$$E^Q_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}} e^{\hat{C}(1,\Delta t) + \hat{D}(1,\Delta t) v_{t_{i-1}}} = S_{t_{i-1}} e^{r\Delta t}, \tag{2.9}$$

$$E^Q_{t_{i-1}}[S^2_{t_i}] = S^2_{t_{i-1}}\, e^{\hat{C}(2,\Delta t) + \hat{D}(2,\Delta t) v_{t_{i-1}}}. \tag{2.10}$$

Consequently, $E^Q_0[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$ can be easily found as

$$E^Q_0\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] = E^Q_0[e^{\hat{C}(2,\Delta t) + \hat{D}(2,\Delta t) v_{t_{i-1}}} - 2e^{r\Delta t} + 1], \tag{2.11}$$

by substituting (2.9) and (2.10) into (2.6).

Formula (2.11) may appear to be different from its counterpart of equation (2.29) in ZL1. However, a careful scrutiny shows that the calculated conditional expectation $E^Q_0[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$ is indeed the same as the conditional expectation $E^Q_0[e^{r\Delta t} f(v_{t_{i-1}})]$ calculated in ZL1. This is because the $\hat{C}(2, \tau)$ and $\hat{D}(2, \tau)$ in our formula (2.8) are related to the $\tilde{C}(\tau)$ and $\tilde{D}(\tau)$ in ZL1, respectively, as

$$\hat{C}(2, \tau) = \tilde{C}(\tau) + r\tau \quad \text{and} \quad \hat{D}(2, \tau) = \tilde{D}(\tau) \tag{2.12}$$

for all $\tau \geq 0$, where $\tilde{C}(\tau)$ and $\tilde{D}(\tau)$ are defined as

$$\begin{cases} \tilde{C}(\tau) = r\tau + \dfrac{\kappa^*\theta^*}{\sigma_V^2}\left((\tilde{a} + \tilde{b})\tau - 2\ln\left(\dfrac{1 - \tilde{g}e^{\tilde{b}\tau}}{1 - \tilde{g}}\right)\right), \\ \tilde{D}(\tau) = \dfrac{\tilde{a} + \tilde{b}}{\sigma_V^2}\left(\dfrac{1 - e^{\tilde{b}\tau}}{1 - \tilde{g}e^{\tilde{b}\tau}}\right), \\ \tilde{a} = \hat{a}(2), \tilde{b} = \hat{b}(2), \quad \text{and} \quad \tilde{g} = \dfrac{\tilde{a} + \tilde{b}}{\tilde{a} - \tilde{b}}. \end{cases} \tag{2.13}$$

A simple substitution of (2.12) into the conditional expectation on the RHS of (2.11) yields equation (2.29) in ZL1, or

$$E^Q_0\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] = E^Q_0[e^{r\Delta t} f(v_{t_{i-1}})] = \int_0^\infty e^{r\Delta t} f(v_{t_{i-1}}) p(v_{t_{i-1}}|v_0) dv_{t_{i-1}}, \tag{2.14}$$

where f is given by

$$f(v) = e^{\tilde{C}(\Delta t) + \tilde{D}(\Delta t)v} + e^{-r\Delta t} - 2. \tag{2.15}$$

Now, with the conditional expectation $E^Q_0[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$ expressed in (2.11), we can directly adopt Proposition 2.2 in ZL1 to obtain the same pricing formula for discretely-sampled variance swaps as

$$K_{var} = E^Q_0[\sigma_R^2] = \frac{e^{r\Delta t}}{T}\left[f(v_0) + \sum_{i=2}^{N} f_i(v_0)\right] \times 100^2, \tag{2.16}$$

where

$$f_i(v_0) = e^{\tilde{C}(\Delta t) + \frac{c_i e^{-\kappa^* t_{i-1}}}{c_i - \tilde{D}(\Delta t)}\tilde{D}(\Delta t) v_0}\left(\frac{c_i}{c_i - \tilde{D}(\Delta t)}\right)^{\frac{2\kappa^*\theta^*}{\sigma_V^2}} + e^{-r\Delta t} - 2, \tag{2.17}$$

with $c_i = \frac{2\kappa^*}{\sigma_V^2(1 - e^{-\kappa^* t_{i-1}})}$.

Since a discussion on the continuous limit of K_{var} when $\Delta t \to 0$, $N \to \infty$, and some numerical examples have already been provided in ZL1 in terms of the utilization of (2.16) in pricing discretely-sampled variance swaps, we shall focus our discussion on some major differences between the two approaches in the next section.

3. Discussions

Having obtained the same results, we would like to point out some major differences between ZL1's approach and our approach presented here.

Firstly, both approaches involve solving PDEs to obtain Formula (2.19) with two steps. While the first step is to evaluate the conditional expectation

$$E^{Q}_{t_{i-1}}\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right], \tag{3.1}$$

the second step is to evaluate the conditional expectation of the new random variable $X = E^{Q}_{t_{i-1}}[(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2]$, i.e. $E^{Q}_{0}[X]$. The main difference between the two approaches lies in the evaluation of (3.1); the evaluations of the conditional expectation involved in the second step are exactly the same (cf. Proposition 2.2 in ZL1). In other words, the value of $f(v)$ defined in (2.15) is exactly the same after the jump condition (2.11) in ZL1 has been imposed.

In order to evaluate the conditional expectation (3.1), ZL1 introduced a new state variable defined by

$$I_t = \int_0^t \delta(t_{i-1}-\tau)S_\tau d\tau \tag{3.2}$$

where $\delta(\cdot)$ is the Dirac delta function. As they have clearly stated, there is a jump in the process I_t over $t = t_{i-1}$. Therefore, they need to solve an associated PDE, which is in a different form from the associated PDE (A.2) and then impose a jump condition at $t = t_{i-1}$. As a result, they had to use the generalized Fourier transform method to obtain a closed-form formula for this step. Clearly, the success of ZL1's approach largely depends on the successful Fourier inverse transform that is simple and straightforward for this particular case. For some other cases, where the analytical inversion of Fourier transform could not be found, our approach may still work!

Our approach for evaluating the conditional expectation (3.1) is simpler and far more direct than ZL1's approach. There is no need to introduce an additional state variable and consequently, no need to impose a jump condition either. Moreover, in our approach, we can directly apply the general asset valuation theory by Garman [15] to the processes S_t^2 and v_t to obtain the associated PDE (A.2) and the terminal condition (A.3), which can then be analytically solved by using the dimension-reduction technique proposed by Schwartz [13] without resorting to the generalized Fourier transform method. On the other hand, the sample paths of the process I_t defined in ZL1 are discontinuous, which implies, by definition, that the process is not a diffusion process. Therefore, it is not clear if the general asset valuation theory by Garman [15] can be directly used. Of course, this does not mean that the associated PDE (2.7) in ZL1 is wrong, because the general asset valuation theory, through which one obtains an associated PDE from diffusion processes, is only a sufficient condition (see [15]). Quite on the contrary, both ZL1 and Little and Pant [10] demonstrated that the introduction of a new state variable I_t is correct; the resulting PDE (2.7) in ZL1 also contains a delta function in front of the diffusive term in the direction of I_t.

An important point should be remarked before leaving this section. Using Formula (2.7) in Proposition 2.1, one can derive an explicit formula for every conditional moment of the underlying asset price, $E^{Q}_{t_{i-1}}[S^{\gamma}_{t_i}]$, under the Heston model (2.2). For example, with $\gamma = 1, 2, 3, 4$, the conditional variance, conditional skewness and conditional kurtosis of S_{t_i} with respect to $\mathcal{F}_{t_{i-1}}$ can be easily worked out in explicit forms as shown in the second half of Appendix. These explicit formulae can be used, in a similar fashion of what has been presented in this paper, to price derivatives based on higher moments such as skewness swaps and kurtosis swaps discussed in Schoutens [16]. They can of course also be very useful in analysing asymptotic properties of the underlying asset prices under the Heston model (2.2).

4. Conclusions

In this paper, we have presented a simplified analytical approach for pricing variance swaps with the realized variance defined on discrete sampling points. We have verified this approach by demonstrating that exactly the same formula as that obtained in Zhu and Lian [4] can be obtained when the underlying asset is assumed to follow Heston's two-factor stochastic volatility model. Unlike the approach presented by ZL1, an important feature of our approach is that there is no need for the introduction of a new state variable and the utilization of the generalized Fourier transform. Instead, we apply the general asset valuation theory to obtain the associated PDE, which is solved by employing the dimension-reduction technique. Comparison between our approach and ZL1's approach is made and the advantages of latter are pointed out.

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Appendix

We now present a brief proof of Proposition 2.1. First, we show that the transformation $Y_t = S_t^{\gamma}$ is well-defined for any $\gamma \in \mathbb{R} \setminus \{0\}$, namely, Y_t is a real-valued stochastic process. From (2.2), for any $t \in [t_{i-1}, t_i]$, one can show by using Itô's lemma that $S_t = S_{t_{i-1}} \exp(\int_{t_{i-1}}^{t} (r - \frac{1}{2} v_s) ds + \int_{t_{i-1}}^{t} \sqrt{v_s} d\tilde{B}_s^S)$. This implies if $S_{t_{i-1}} > 0$ then $Y_t > 0$ with probability one for any $\gamma \in \mathbb{R} \setminus \{0\}$. Next, we show the derivation of Formula (2.7). Applying Itô's lemma to the above transformation gives us

$$dY_t = \left(\gamma r + \frac{1}{2}\gamma(\gamma - 1)v_t\right) Y_t dt + \gamma \sqrt{v_t} Y_t d\tilde{B}_t^S. \tag{A.1}$$

We now consider a two-dimensional Itô process (Y_t, v_t) and a contingent claim $U_i^{(\gamma)}(y, v, t) = E^Q[Y_t | (Y_{t_{i-1}} = y, v_{t_{i-1}} = v)]$ whose payoff at expiry t_i is Y_{t_i} and no cost to enter this claim. Specifically, this contingent claim is known as a forward contract. Following the general asset valuation theory by Garman [15], $U_i^{(\gamma)}$ satisfies the PDE

$$\frac{\partial U_i^{(\gamma)}}{\partial t} + \frac{1}{2}\gamma^2 v y^2 \frac{\partial^2 U_i^{(\gamma)}}{\partial y^2} + \frac{1}{2}\sigma_V^2 v \frac{\partial^2 U_i^{(\gamma)}}{\partial v^2} + \gamma \rho \sigma_V v y \frac{\partial^2 U_i^{(\gamma)}}{\partial y \partial v} + \left(\gamma r + \frac{1}{2}\gamma(\gamma - 1)v\right) y \frac{\partial U_i^{(\gamma)}}{\partial y} + \kappa^*(\theta^* - v)\frac{\partial U_i^{(\gamma)}}{\partial v} = 0, \tag{A.2}$$

subject to the terminal condition

$$U_i^{(\gamma)}(y, v, t_i) = y, \tag{A.3}$$

for all $t \in [t_{i-1}, t_i)$ and $(y, v) \in (0, \infty) \times (0, \infty)$. Let $\tau = t_i - t$. Following Schwartz's solution procedure [13], the solution of the above PDE can be assumed of the form

$$U_i^{(\gamma)}(y, v, t) = y e^{\hat{C}(\gamma, t_i - t) + \hat{D}(\gamma, t_i - t)v}. \tag{A.4}$$

Substituting (A.4) into the PDE (A.2) yields a set of ordinary differential equations (ODE),

$$\frac{d\hat{D}}{d\tau} = \frac{1}{2}\sigma_V^2 \hat{D}^2 + (\gamma \rho \sigma_V - \kappa^*)\hat{D} + \frac{\gamma(\gamma - 1)}{2}, \tag{A.5}$$

$$\frac{d\hat{C}}{d\tau} = \gamma r + \kappa^* \theta^* \hat{D}, \tag{A.6}$$

subject to the initial conditions

$$\hat{D}(\gamma, 0) = 0, \tag{A.7}$$

$$\hat{C}(\gamma, 0) = 0. \tag{A.8}$$

There are three cases, in the parameter space, that need to be considered separately, depending on the values of $\hat{c}(\gamma) = (\kappa^* - \gamma\rho\sigma_V)^2 - \gamma(\gamma - 1)\sigma_V^2$ and $\hat{a}(\gamma) = \kappa^* - \gamma\rho\sigma_V$. Case 1: $\hat{c}(\gamma) > 0$. In this case a global solution can be easily found as $\hat{D}(\gamma, \tau)$ expressed in (2.8). Case 2: $\hat{c}(\gamma) = 0$, which needs to be further divided into two sub-cases: Case 2.1: $\hat{c}(\gamma) = 0$ and $\hat{a}(\gamma) \geq 0$ and Case 2.2: $\hat{c}(\gamma) = 0$ and $\hat{a}(\gamma) < 0$. In Case 2.1, a global solution can be obtained as $\hat{D}_1(\gamma, \tau) = \frac{\gamma(\gamma-1)\tau}{(2+\hat{a}(\gamma)\tau)}$ for all $\tau \in [0, \infty)$. Applying the L'Hôpital's rule, one can easily show that $\lim_{\hat{b}(\gamma)\to 0^+} \hat{D}(\gamma, \tau) = \lim_{\hat{c}(\gamma)\to 0^+} \hat{D}(\gamma, \tau) = \hat{D}_1(\gamma, \tau)$ for all $\tau \in [0, \infty)$. This implies that Case 2.1 can be regarded as a special case of Case 1 in which $\hat{b}(\gamma)$ approaches zero from above. In Case 2.2, on the other hand, we obtain the same $\hat{D}_1(\gamma, \tau)$ as a local solution only for all $\tau \in [0, -2/\hat{a}(\gamma))$. Case 3: $\hat{c}(\gamma) < 0$. This case also yields a local solution as $\hat{D}_2(\gamma, \tau) = \frac{1}{\sigma_V^2}[\hat{a}(\gamma) - \sqrt{-\hat{c}(\gamma)} \cot(\sqrt{-\hat{c}(\gamma)}\tau - \psi(\gamma))]$ for all $\tau \in [0, \zeta(\gamma))$ where $\zeta(\gamma) = (\psi(\gamma) + \pi)/\sqrt{-\hat{c}(\gamma)}$ and $\psi(\gamma) = \cot^{-1}(-\hat{a}(\gamma)/\sqrt{-\hat{c}(\gamma)})$.

Once $\hat{D}(\gamma, \tau)$ is found, $\hat{C}(\gamma, \tau)$ can be easily found through integrating (A.6) subject to (A.8), i.e. $\hat{C}(\gamma, \tau) = \int_0^{\tau} (\gamma r + \kappa^* \theta^* \hat{D}(\gamma, s)) ds$. In Case 1, the explicit form of $\hat{C}(\gamma, \tau)$, as a global solution, is given in (2.8). The two $\hat{C}_1(\gamma, \tau)$ solutions corresponding to $\hat{D}_1(\gamma, \tau)$ for both Cases 2.1 and 2.2 are of the same form $\hat{C}_1(\gamma, \tau) = \gamma r \tau + \frac{\kappa^* \theta^*}{\sigma_V^2}[\hat{a}(\gamma)\tau + 2\ln(\frac{2}{2+\hat{a}(\gamma)\tau})]$. However, for Case, 2.1, this is a global solution with the domain $\tau \in [0, \infty)$, whereas it is only a local solution for Case, 2.2 with the domain $\tau \in [0, -2/\hat{a}(\gamma))$. In Case 3, a local solution corresponding to $\hat{D}_2(\gamma, \tau)$ can be obtained as $\hat{C}_2(\gamma, \tau) = \gamma r \tau + \frac{\kappa^* \theta^*}{\sigma_V^2}[\hat{a}(\gamma)\tau + \ln(\frac{\sin(\psi(\gamma))}{\sin(\psi(\gamma) - \sqrt{-\hat{c}(\gamma)}\tau)})]$ for all $\tau \in [0, \zeta(\gamma))$.

Once the proof of Proposition 2.1 is completed, we would also want to point out that, using Formula (2.7) in Proposition 2.1, the conditional variance, conditional skewness and conditional kurtosis of S_{t_i} with respect to $\mathcal{F}_{t_{i-1}}$ can be easily found, with $\gamma = 1, 2, 3, 4$, as

$$Var[S_{t_i}|\mathcal{F}_{t_{i-1}}] = E^Q_{t_{i-1}}[S^2_{t_i}] - (E^Q_{t_{i-1}}[S_{t_i}])^2 = S^2_{t_{i-1}} e^{2r\Delta t}(e^{\bar{C}(2,\Delta t)+\hat{D}(2,\Delta t)v_{t_{i-1}}} - 1), \tag{A.9}$$

$$Skew[S_{t_i}|\mathcal{F}_{t_{i-1}}] = \frac{E^Q_{t_{i-1}}[(S_{t_i} - E^Q_{t_{i-1}}[S_{t_i}])^3]}{(Var[S_{t_i}|\mathcal{F}_{t_{i-1}}])^{\frac{3}{2}}} = \frac{\sum_{k=0}^{3}\binom{3}{3-k}(-1)^k e^{\bar{C}(3-k,\Delta t)+\hat{D}(3-k,\Delta t)v_{t_{i-1}}}}{(e^{\bar{C}(2,\Delta t)+\hat{D}(2,\Delta t)v_{t_{i-1}}} - 1)^{\frac{3}{2}}}, \tag{A.10}$$

$$Kurt[S_{t_i}|\mathcal{F}_{t_{i-1}}] = \frac{E^Q_{t_{i-1}}[(S_{t_i} - E^Q_{t_{i-1}}[S_{t_i}])^4]}{(Var[S_{t_i}|\mathcal{F}_{t_{i-1}}])^2} = \frac{\sum_{k=0}^{4}\binom{4}{4-k}(-1)^k e^{\bar{C}(4-k,\Delta t)+\hat{D}(4-k,\Delta t)v_{t_{i-1}}}}{(e^{\bar{C}(2,\Delta t)+\hat{D}(2,\Delta t)v_{t_{i-1}}} - 1)^2}, \tag{A.11}$$

where $\bar{C}(\gamma, \Delta t) = \frac{\kappa^*\theta^*}{\sigma_V^2}[(\hat{a}(\gamma) + \hat{b}(\gamma))\Delta t - 2\ln(\frac{\hat{b}(\gamma)-\hat{a}(\gamma)+(\hat{a}(\gamma)+\hat{b}(\gamma))e^{\hat{b}(\gamma)\Delta t}}{2\hat{b}(\gamma)})]$ for all $\Delta t \geq 0$ and $\binom{n}{k}$ denotes the binomial coefficient. Clearly, (A.9) has reinstated what one naturally expects: under the Heston model (2.2), the variance of the underlying asset price at a given future time depends on both the underlying asset price and instantaneous variance at the current time, whereas the expectation of the underlying asset price depends only on its own value at the current time, as shown in (2.9). On the other hand, (A.10) and (A.11) indicate that both the conditional skewness and conditional kurtosis depend only on the instantaneous variance.

References

[1] P. Carr, D. Madan, in: R. Jarrow (Ed.), Towards a Theory of Volatility Trading, Volatility, Risk Publications, 1998, pp. 417–427.
[2] K. Demeterfi, E. Derman, M. Kamal, J. Zou, More than you ever wanted to know about volatility swaps, Goldman Sachs Quantitative Strategies Research Notes (1999).
[3] P. Carr, A. Corso, Commodity covariance contracting, Energy and Power Risk Management (2001).
[4] S.-P. Zhu, G. Lian, A closed-form exact solution for pricing variance swaps with stochastic volatility, Mathematical Finance 21 (2) (2011) 233–256.
[5] A. Swishchuk, X. Li, Pricing variance swaps for stochastic volatilities with delay and jumps, International Journal of Stochastic Analysis (2011), accepted with Article ID 435145, doi:10.1155/2011/435145.
[6] M. Broadie, A. Jain, The effect of jumps and discrete sampling on volatility and variance swaps, Internaltional Journal of Theoretical and Applied Finance 11 (8) (2008) 761–797.
[7] R.C. Merton, Theory of rational option pricing, Bell Journal of Economic Management Science 4 (1973) 141–183.
[8] D. Bates, Jumps and stochastic volatility: exchange rate processes implicit in Deutsche Mark options, Review of Financial Studies (1996) 69–107.
[9] L. Scott, Pricing stock options in a jump diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods, Mathematical Finance 7 (4) (1997) 413–426.
[10] T. Little, V. Pant, A finite-difference method for the valuation of variance swaps, Journal of Computational Finance 5 (1) (2001) 81–101.
[11] S.-P. Zhu, G. Lian, Pricing variance swaps with stochastic volatility, in: Proceedings of the World Congress on Engineering, vol. 2, 2009, pp. 1359–1364.
[12] S.L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, Review of Financial Studies 6 (2) (1993) 327–343.
[13] E.S. Schwartz, The stochastic behavior of commodity prices: implications for valuation and hedging, The Journal of Finance 52 (3) (1997) 923–973.
[14] J. Cox, J. Ingersoll Jr, S. Ross, A theory of the term structure of interest rates, Econometrica 53 (2) (1985) 385–407.
[15] M. Garman, A general theory of asset valuation under diffusion state processes, in: Working paper NO. 50, University of California, Berkeley, 1977, pp. 1–54.
[16] W. Schoutens, Moment swaps, Quantitative Finance 5 (6) (2005) 525–530.