JOURNAL OF ALGEBRA **192,** 200–208 (1997)
ARTICLE NO. JA966978

Indecomposable Decompositions of Finitely Presented Pure-Injective Modules

José L. Gómez Pardo

Departamento de Alxebra, Universidade de Santiago, 15771 Santiago de Compostela

and

Pedro A. Guil Asensio

Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain

Communicated by Kent R. Fuller

Received February 13, 1996

I. INTRODUCTION

It is well known that pure-injective (= algebraically compact) modules behave well in relation to direct sum decompositions. Any indecomposable pure-injective module has local endomorphism ring [14] and, more generally, any pure-injective module has the exchange property [15]. But, in contrast with Σ-pure-injective modules, which always have indecomposable decompositions [14], pure-injective modules need not have this property. In fact, if all pure-injective right R-modules have such a decomposition, then R is a right pure-semisimple ring by [13]. Obviously, even a right pure-injective ring R need not have an indecomposable decomposition—think of the endomorphism ring of an infinite-dimensional vector space. It is clear from the preceding considerations that, given a right pure-injective ring R, such a decomposition exists for R_R if and only if R is semiperfect. However, while a right Σ-pure-injective ring is semiprimary with maximum condition on annihilator right ideals, a right pure-injective ring is only Von Neumann regular modulo the radical with the idempotent-lifting property

0021-8693/97 $25.00

in general [14]. It is thus natural to ask for conditions for a right pure-injective ring R to be semiperfect.

On the other hand, taking a clue from Osofsky's theorem that shows that if every cyclic right R-module is injective then R is semisimple [10], we study as in [5] completely pure-injective modules, i.e., modules M such that every pure quotient of M is pure-injective. Similarly, a ring R will be called right completely pure-injective when R_R is completely pure-injective. The class of completely pure-injective modules falls between the classes of pure-injective and Σ-pure-injective modules. However, there are commutative completely pure-injective rings which are not perfect, e.g., the ring of p-adic integers. In fact, if R is semiperfect, then the pure quotients of R_R are direct summands and so a semiperfect right pure-injective ring is completely pure-injective. The preceding considerations and Osofsky's theorem thus lead to the following question: Are right completely pure-injective rings semiperfect?

In this paper, we provide an affirmative answer to this question. In fact, the first of our problems is more intimately related to the second than is obvious at first sight: a right pure-injective ring is semiperfect precisely when it is right completely pure-injective. These conditions can be transferred, using functorial techniques, to the endomorphism ring of a finitely presented pure-injective module, given the following necessary and sufficient conditions for the existence of an indecomposable decomposition for these modules. They constitute our main result. Namely, a finitely presented pure-injective right R-module M has an indecomposable decomposition if and only if it is completely pure-injective; these conditions are in turn equivalent to each pure submodule of M being a direct summand of a direct sum of finitely presented modules. Thus we see that, although finitely presented completely pure-injective modules are rather far from being Σ-pure-injective, in general, they share with the latter modules the key property of being direct sums of modules with local endomorphism rings. In particular, the rings R such that every finitely presented right R-module is completely pure-injective are Krull–Schmidt rings in the sense of [8].

Our method of proof was suggested by Osofsky's proof of her theorem in [10]. The situation is different, however. In fact, our proof gives, in the injective case, a different and probably simpler proof of Osofsky's theorem.

Throughout this paper, all rings R will be associative and with identity, and Mod-R will denote the category of right R-modules. By a module we will usually mean a right R-module and, whenever we want to emphasize the fact that M is a right R-module, we will write M_R. We refer to [1], [9], and [11], for all undefined notions used in the text.

2. RESULTS

We begin with an auxiliary lemma. We refer to [9] for the definition and basic properties of pure-injective envelopes.

LEMMA 2.1. *Let R be a right pure-injective ring and Q, T right ideals of R such that $Q \cap T = 0$ and $Q \oplus T$ is pure in R_R. If P_Q, P_T are pure-injective envelopes of Q, T, respectively, contained in R_R, then $P_Q \cap P_T = 0$ and $P_Q \oplus P_T$ is a direct summand of R_R.*

Proof. Using functor ring techniques [6] (see also [11, 52.3] or [3, Lemma 2.1]), R can be represented as the endomorphism ring of an injective object E of a Grothendieck category $\mathcal{C}$ such that the following properties hold. The functor $\mathrm{Hom}_{\mathcal{C}}(E, -)$: $\mathcal{C} \to \text{Mod-}R$ has a left adjoint $- \otimes_R E$: Mod-$R \to \mathcal{C}$ which takes pure exact sequences of Mod-R into (pure) exact sequences of $\mathcal{C}$. Moreover, a right R-module X is pure-injective if and only if $X \otimes_R E$ is injective in $\mathcal{C}$, and if $PE(X)$ is a pure-injective envelope of X in Mod-R, then $PE(X) \otimes_R E$ is an injective envelope of $X \otimes_R E$ in $\mathcal{C}$. Thus we see that if we identify R with $\mathrm{End}_{\mathcal{C}}(E)$ and P_Q is a pure-injective envelope of Q contained in R, we have that $P_Q = \mathrm{Hom}_{\mathcal{C}}(E, P_Q \otimes E)$, where $P_Q \otimes E$ is an injective envelope of $Q \otimes_R E$ contained in E. Similarly, $P_T = \mathrm{Hom}_{\mathcal{C}}(E, P_T \otimes E)$ with $P_T \otimes E$ an injective envelope of $T \otimes_R E$ contained in E. On the other hand, since $Q \oplus T$ is pure in R_R, we have that $Q \otimes_R E$ and $T \otimes_R E$ are subobjects of E such that $(Q \otimes_R E) \cap (T \otimes_R E) = 0$. Therefore, the injective envelopes of these subobjects have zero intersection, that is, $(P_Q \otimes_R E) \cap (P_T \otimes_R E) = 0$. From this it follows that $P_Q \cap P_T = 0$. Obviously, $P_Q \oplus P_T$ is then a pure-injective envelope of the pure right ideal $Q \oplus T$ and hence a direct summand of R_R. ∎

THEOREM 2.2. *Let R be a right pure-injective ring. If R is not semiperfect, then there exist pure right ideals N, L of R and a homomorphism φ: $N \to R/L$ that cannot be extended to R.*

Proof. Let $J = J(R)$ be the radical of R. Then R/J is Von Neumann regular and idempotents lift modulo J [14, Theorem 9]. Furthermore, any countable set of orthogonal idempotents of R/J can be lifted to an orthogonal set of idempotents of R (see, e.g., [1, Exercise 27.1]). Thus our hypothesis implies that there exists an infinite set $\{e_i\}_I$ of orthogonal idempotents in R.

Now let $L = \oplus_{i \in I}(e_i R)$. Then, for each finite subset $F \subseteq I$, $\oplus_F(e_i R) = (\Sigma_F e_i)R$ is a direct summand of R_R and so L is a pure right ideal.

Next consider a nonempty subset A of I and let $A' = I - A$. Since the idempotents $\{e_i\}_{i \in A}$ are orthogonal, it is clear that $\oplus_A(e_i R)$ and $\oplus_{A'}(e_i R)$ are right ideals of R that satisfy the hypotheses of Lemma 2.1. Thus there exist pure-injective envelopes P_A of $\oplus_A(e_i R)$ and $P_{A'}$ of $\oplus_{A'}(e_i R)$ and a right ideal T of R such that

$$R_R = P_A \oplus P_{A'} \oplus T.$$

Then there exists an idempotent element $e_A \in R$ such that $P_A = e_A R$ and $P_{A'} \oplus T = (1 - e_A)R$. Furthermore, we have that for each $i \in A$, $e_i \in e_A R$ and so $e_A e_i = e_i$, while for $i \in A'$, $e_i \in (1 - e_A)R$ and hence $e_A e_i = 0$.

Let us now write $I = \bigcup_{A \in \mathcal{A}} A$ as an infinite union of infinite pairwise disjoint subsets. As in [10] we have that, by Zorn's lemma, there exists a maximal subset $\mathcal{K} \subseteq 2^I$ with respect to the properties:

1. $\mathcal{A} \subseteq \mathcal{K}$.
2. $|A| \geq \aleph_0$ for each $A \in \mathcal{K}$.
3. $|A \cap B| < \aleph_0$ for each $A, B \in \mathcal{K}$, $A \neq B$.

Let N be the right ideal of R defined by $N = \sum_{A \in \mathcal{K}} e_A R$. We claim that N is a pure right ideal. To see this consider $A, B \in \mathcal{K}$, $A \neq B$. Then $A \cap B$ is finite, say $A \cap B = \{i_1, \ldots, i_r\}$. Observe now that if $C = B - \{i_1, \ldots, i_r\}$, then $C \cap A = \varnothing$ and, as $e_B R$ is a pure-injective envelope of $\oplus_B(e_i R)$, we may write $e_B R = e_{i_1} R \oplus \cdots \oplus e_{i_r} R \oplus X_B$, where X_B is a pure-injective envelope of $\oplus_C(e_i R)$. Since $e_A R$ is a pure-injective envelope of $\oplus_A(e_i R)$, it follows from Lemma 2.1 that $e_A R \cap X_B = 0$ and that $e_A R + e_B R = e_A R + X_B$ is a direct summand of R_R. By induction we obtain that if $A_1, \ldots, A_n \in \mathcal{K}$, then $e_{A_1} R + \cdots + e_{A_n} R$ is a direct summand of R_R. Thus N is a direct limit of these direct summands and hence a pure right ideal of R.

Next, consider the quotient module $N/L = \sum_{A \in \mathcal{K}}((e_A R + L)/L)$. We claim that this sum is direct. Suppose, then, that $A, B_1, \ldots, B_s \in \mathcal{K}$ are different. As we have just seen $\sum_{j=1}^{s} e_{B_j} R$ is a direct summand of R_R, so that we may write $\sum_{j=1}^{s} e_{B_j} R = gR$ with $g \in R$ an idempotent. Furthermore, gR is a pure-injective envelope of $\oplus_B(e_i R)$ with $B = \bigcup_{j=1}^{s} B_j$. To show that $e_A R \cap gR \subseteq L$, observe that $A \cap B = A \cap (\bigcup_{j=1}^{s} B_j) = \bigcup_{j=1}^{s}(A \cap B_j)$ is finite set, say $A \cap B = \{k_1, \ldots, k_r\}$. Then we have as before that $e_B R = e_{k_1} R \oplus \cdots \oplus e_{k_r} R \oplus X_B$, where $e_A R \cap X_B = 0$. Since $e_{k_1} R \oplus \cdots \oplus e_{k_r} R \subseteq e_A R$ we obtain by modularity that

$$\begin{aligned} e_A R \cap gR &= e_A R \cap \left(e_{k_1} R \oplus \cdots \oplus e_{k_r} R \oplus X_B\right) \\ &= e_{k_1} R \oplus \cdots \oplus e_{k_r} R \subseteq L. \end{aligned}$$

Thus the sum is indeed direct.

Now, we finish our argument as in the proof of [10, Theorem]. We define a homomorphism $\psi\colon N/L \to R/L$ by $\psi(e_A + L) = e_A + L$ if $A \in$

$\mathcal{A}$ and $\psi(e_A + L) = L$ if $A \notin \mathcal{A}$. Let $\pi\colon N \to N/L$ be the canonical projection and let $\varphi = \psi \circ \pi\colon N \to R/L$. Assume, by contradiction, that φ has an extension $\overline{\varphi}\colon R \to R/L$, so that $\overline{\varphi} \circ u = \varphi$, where $u\colon N \to R$ denotes the inclusion. Let then $x \in R$ such that $\overline{\varphi}(1) = x + L$. We have, for each $A \in \mathcal{K}$,

$$xe_A + L = \overline{\varphi}(e_A) = \varphi(e_A) + \begin{cases} e_A + L & \text{if } A \in \mathcal{A}, \\ L & \text{if } A \notin \mathcal{A}. \end{cases}$$

Therefore we have that $xe_A = e_A + l_A$ with $l_A \in L$ for $A \in \mathcal{A}$, and that $xe_A \in L$ for $A \notin \mathcal{A}$. Let now $A \in \mathcal{A}$. For each $i \in A$, $e_A e_i = e_i$ and so $e_i x e_i = e_i x e_A e_i = e_i(e_A + l_A)e_i = e_i e_A e_i + e_i l_A e_i = e_i + e_i l_A e_i$. Since $l_A \in L$, we obtain that, for almost all $i \in I$, $e_i l_A e_i = 0$. Thus, for almost all $i \in A$, $e_i = e_i x e_i$. Let $A_0 = \{i \in A | e_i = e_i x e_i\}$, which is a cofinite subset of A and hence infinite. For each $A \in \mathcal{A}$, choose a element $c_A \in A_0$ and set $C = \{c_A | A \in \mathcal{A}\}$. Since $\mathcal{A}$ is infinite so is C and by the maximality of $\mathcal{K}$ there exists a set $D \in \mathcal{K}$ such that $D \cap C$ is infinite. It is clear that $D \notin \mathcal{A}$ and so $xe_D \in L$ and hence $e_i x e_D e_i = 0$ for almost all $i \in I$. In particular, $e_i x e_D e_i = 0$ for almost all $i \in D \cap C$. But if $i \in D \cap C$ we have that because $i \in C$, $e_i = e_i x e_i$, and because $i \in D$, $e_i x e_i = e_i x e_D e_i$. Thus we obtain that $e_i = 0$ for almost all $i \in D \cap C$, which is a contradiction and proves and theorem. ▮

COROLLARY 2.3. *Let R be a right pure-injective ring. Then the following conditions are equivalent*:

(i) *R is semiperfect.*

(ii) *R is right completely pure-injective.*

(iii) *Every pure right ideal of R is pure-projective.*

Proof. If R is semiperfect, then by [11, 36.4 (1)], every pure right ideal of R is a direct summand and so conditions (ii) and (iii) hold. Conversely, it follows from Theorem 2.2 that (ii) implies (i). Finally, assume that (iii) holds. Let N and L be pure right ideals of R, $f\colon N \to R/L$ a homomorphism, and $p\colon R \to R/L$ the canonical projection. Then our hypothesis implies that there exists $g\colon N \to R$ such that $f = p \circ g$. Let $u\colon N \to R$ be the inclusion. Since R_R is pure-injective, there exists $h\colon R \to R$ such that $g = h \circ u$. Then $f = p \circ g = p \circ h \circ u$ and, since f has an extension to R we see that R is semiperfect by Theorem 2.2. Thus we have that (iii) implies (i) and this completes the proof. ▮

Remarks. Observe that, since a ring whose cyclic right modules are injective is Von Neumann regular, Osofsky's theorem follows at once from Corollary 2.3.

It is clear that a right pure-injective ring R need not be completely pure-injective: any nonsemisimple regular right self-injective ring provides an example. On the other hand, recall that a right R-module M is Σ-pure-injective when every direct sum of copies of M is pure-injective. Since the pure submodules of Σ-pure-injective modules are direct summands [9, Corollary 8.2], it is clear that every Σ-pure-injective module is completely pure-injective. The converse, however, is far from being true. In fact, while a right pure-injective ring which is either right noetherian or semiperfect is completely pure-injective (since the pure right ideals are direct summands), a completely pure-injective ring need not be Σ-pure-injective even if it is commutative and either noetherian or semiprimary. An example of the first situation is provided by the power series ring $R = K[[X_1, \ldots, X_n]]$ over a field K. It is well known that R is linearly compact and hence pure-injective by [14, Proposition 1]. Since R is also noetherian, it is completely pure-injective as we have already remarked. However, R is not Σ-pure-injective, as it is not semiprimary (see [14]). An example of the second situation was given in [12, Example 11], where a commutative semiprimary pure-injective (and hence completely pure-injective) ring R is constructed which is not Σ-pure-injective.

LEMMA 2.4. *Let M be a finitely presented right R-module and $S = \mathrm{End}(M_R)$. Then the following assertions hold*:

(i) *If X is a set and p: $M^{(X)} \to Q$ a pure epimorphism, then the canonical homomorphism $\mathrm{Hom}_R(M, Q) \otimes_S M \to Q$ is an isomorphism.*

(ii) *If U is a flat right S-module, then the canonical homomorphism $U \to \mathrm{Hom}_R(M, U \otimes_S M)$ is an isomorphism.*

Proof. (i) By [11, 34.2 (2)], Q is a direct limit of finite direct sums of copies of M. If F is a finite set, then the canonical morphism $\mathrm{Hom}_R(M, M^{(F)}) \otimes_S M \to M^{(F)}$ is an isomorphism. Since M is finitely presented, the functor $\mathrm{Hom}_R(M, -)$: Mod-$R \to$ Mod-S preserves direct limits and so $\mathrm{Hom}_R(M, Q) \otimes_S M \to Q$ is also an isomorphism.

The proof of (ii) is similar, using the fact that a flat module is a direct limit of finitely generated free modules. ∎

It follows from [4, Theorem 1] that a finitely presented completely pure-injective module with Von Neumann regular endomorphism ring has an indecomposable decomposition. In the following corollary we obtain necessary and sufficient conditions for the existence of such a decomposition without any hypothesis on the endomorphism ring. In particular we see that, although the class of completely pure-injective rings is much larger than that of Σ-pure-injective rings, finitely presented completely pure-injective modules share with Σ-pure-injective modules the good behavior vis-à-vis indecomposable decompositions.

COROLLARY 2.5. *Let M be a finitely presented pure-injective right R-module. Then the following conditions are equivalent*:

(i) *M has an indecomposable decomposition.*

(ii) *M is completely pure-injective.*

(iii) *Every pure submodule of M is pure-projective.*

Proof. Let $S = \mathrm{End}(M_R)$. Since the endomorphism ring of an indecomposable pure-injective module is local [14, Theorem 9], condition (i) is equivalent to S being semiperfect by [1, 27.6]. In particular, if (i) holds, every pure right ideal of S is a direct summand. Let now Q be a pure quotient of M_R. By Lemma 2.4, the canonical morphism $\mathrm{Hom}_R(M, Q) \otimes_S M \to Q$ is an isomorphism. Further, it is clear that $\mathrm{Hom}_R(M, Q)$ is a pure quotient, and hence a direct summand, of S_S. Thus we have that $Q \cong \mathrm{Hom}_R(M, Q) \otimes_S M$ is a direct summand of M_R and so it is clear that (i) implies (ii) and (iii).

Assume now that (ii) holds. If $Z = S/X$ is a pure quotient of S_S, then Z is flat and so, by Lemma 2.4, the canonical morphism $\alpha_Z\colon Z \to \mathrm{Hom}_R(M, Z \otimes_S M)$ is an isomorphism. Then let $q\colon U \to V$ be a pure monomorphism in Mod-S and let $f\colon U \to Z$ be a homomorphism. It is clear that $Z \otimes_S M$ is a pure quotient of M and hence, by hypothesis, pure-injective. Since $q \otimes_S M\colon U \otimes_S M \to V \otimes_S M$ is pure in Mod-R, the homomorphism $f \otimes_S M\colon U \otimes_S M \to Z \otimes_S M$ has an extension $h\colon V \otimes_S M \to Z \otimes_S M$, that is, $f \otimes_S M = h \circ (q \otimes_S M)$. By naturality, we have $f = \alpha_Z^{-1} \circ \mathrm{Hom}_R(M, f \otimes_S M) \circ \alpha_U = \alpha_Z^{-1} \circ \mathrm{Hom}_R(M, h) \circ \mathrm{Hom}_R(M, q \otimes_s M) \circ \alpha_u = \alpha_Z^{-1} \circ \mathrm{Hom}_R(M, h) \circ \alpha_V \circ q$, which shows that f extends to V and hence that Z is pure-injective. Thus S is right completely pure-injective and hence semiperfect by Corollary 2.3, so that (ii) implies (i).

Finally, we show that (iii) implies (i). Using Theorem 2.2 it suffices to show that if N and L are pure right ideals of S and $f\colon N \to S/L$ is a homomorphism, then f can be lifted to S. Let $p\colon S \to S/L$ be the canonical projection. Then $N \otimes_S M$ is a pure submodule of M and hence pure-projective. Since $p \otimes_S M$ is a pure epimorphism, we see that $f \otimes_S M$ factors in the form $f \otimes_S M = (p \otimes_S M) \circ h$ with $h\colon N \otimes_S M \to M$. Since M is finitely presented we have, as before, that α_S and $\alpha_{S/L}$ are isomorphisms and hence that $f = \alpha_{S/L}^{-1} \circ \mathrm{Hom}_R(M, f \otimes_S M) \circ \alpha_N = \alpha_{S/L}^{-1} \circ \mathrm{Hom}_R(M, p \otimes_S M) \circ \mathrm{Hom}_R(M, h) \circ \alpha_N = p \circ \alpha_S^{-1} \circ \mathrm{Hom}_R(M, h) \circ \alpha_N$. Thus f has a lifting to S and hence S is semiperfect. ∎

In [8], a ring R is called a Krull−Schmidt ring when every finitely presented right R-module is a direct sum of modules with local endomorphism rings (this condition if left-right symmetric). It follows from Corollary 2.5 that any finitely presented completely pure-injective module has such a decomposition and so we obtain:

COROLLARY 2.6. *Let R be a ring such that every finitely presented right R-module is completely pure-injective. Then R is a Krull–Schmidt ring.*

Remarks. Among the rings that satisfy the hypothesis of the preceding corollary are both left pure-semisimple and right pure-semisimple rings (see [7, 8] for the left case), commutative Σ-pure-injective rings [14], and commutative linearly compact rings. The latter have a Morita duality by [2] and using a standard argument (as in [6, Exercise 7.10]) one can show that every reflexive (and hence every finitely generated) module is pure-injective. These rings need not be perfect and so, in contrast with the other two classes of rings just mentioned, they need not be Σ-pure-injective. As a consequence of the Crawley–Jønsson–Warfield theorem [1, 26.5], Krull–Schmidt rings have the additional property that every pure-projective module is (uniquely by Azumaya's theorem [1, 12.6]) a direct sum of finitely presented modules with local endomorphism rings. This was observed in [14] for commutative Σ-pure-injective rings.

ACKNOWLEDGMENTS

The authors would like to thank Juan Martínez Hernández for his helpful comments.

This work was partially supported by the DGICYT (PB93-0515, Spain). The first author was also partially supported by the European Community (Contract CHRX-CT93-0091) and the Xunta de Galicia (XUGA 10502B94) and the second author by the C. A. de Murcia (PIB 94-25).

REFERENCES

[1] F. W. Anderson and K. R. Fuller, "Rings and categories of modules," 2nd ed., Springer-Verlag, New York, 1992.

[2] P. N. Ánh, Morita duality for commutative rings, *Comm. Algebra* **18** (1990), 1781–1788.

[3] J. L. García and N. V. Dung, Some decomposition properties of injective and pure-injective modules, *Osaka J. Math.* **31** (1994), 95–108.

[4] J. L. Gómez Pardo, N. V. Dung, and R. Wisbauer, Complete pure injectivity and endomorphism rings, *Proc. Amer. Math. Soc.* **118** (1993), 1029–1034.

[5] J. L. Gómez Pardo and P. A. Guil Asensio, Endomorphism rings of completely pure-injective modules, *Proc. Amer. Math. Soc.* **124** (1996), 2301–2309.

[6] L. Gruson and C. U. Jensen, Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, *in* Lecture Notes in Mathematics, Vol. 867, pp. 234–294, Springer-Verlag, Berlin, 1981.

[7] I. Herzog, Finitely presented right modules over a left pure-semisimple ring, *Bull. London Math. Soc.* **26** (1994), 333–338.

[8] I. Herzog, A test for finite representation type, *J. Pure Appl. Algebra* **95** (1994), 151–182.
[9] C. U. Jensen and H. Lenzing, "Model Theoretic Algebra," Gordon and Breach, New York, 1989.
[10] B. L. Osofsky, Noninjective cyclic modules, *Proc. Amer. Math. Soc.* **19** (1968), 1383–1384.
[11] R. Wisbauer, "Grundlagen der Modul- und Ringtheorie," Verlag Reinhard Fischer, München, 1988.
[12] W. Zimmermann, (Σ-) algebraic compactness of rings, *J. Pure Appl. Algebra* **23**, (1982), 319–328.
[13] B. Zimmermann-Huisgen, Rings whose right modules are direct sums of indecomposable modules, *Proc. Amer. Math. Soc.* **77** (1979), 191–197.
[14] B. Zimmermann-Huisgen and W. Zimmermann, Algebraically compact rings and modules, *Math. Z.* **161** (1978), 81–93.
[15] B. Zimmermann-Huisgen and W. Zimmermann, Classes of modules with the exchange property, *J. Algebra* **88** (1984), 416–434.