ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 281 (2004) 342–365

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Computing Kazhdan–Lusztig cells for unequal parameters

Meinolf Geck

Institut Girard Desargues, bat. Jean Braconnier, Université Lyon 1, 21 av. Claude Bernard, F-69622 Villeurbanne cedex, France

Received 20 February 2004

Communicated by Gunter Malle

Abstract

Following Lusztig, we consider a Coxeter group W together with a weight function L. This gives rise to the pre-order relation $\leqslant_L$ and the corresponding partition of W into left cells. We introduce an equivalence relation on weight functions such that, in particular, $\leqslant_L$ is constant on equivalent classes. We shall work this out explicitly for W of type F_4 and check that several of Lusztig's conjectures concerning left cells with unequal parameters hold in this case, even for those parameters which do not admit a geometric interpretation. The proofs involve some explicit computations using CHEVIE.

1. Introduction

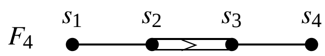
This paper is concerned with the computation of the Kazhdan–Lusztig polynomials, the left cells and the corresponding representations of a finite Coxeter group W with respect to a weight function L. Following Lusztig [15], a weight function on W is a function $L: W \to \mathbb{Z}$ such that $L(ww') = L(w) + L(w')$ whenever $l(ww') = l(w) + l(w')$ where l is the length function on W. As in most parts of [15], we shall only consider weight functions such that $L(w) > 0$ for all $w \neq 1$.

E-mail address: geck@desargues.univ-lyon1.fr.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.07.029

The case where L is constant on the generators of W is known as the equal parameter case. If, moreover, W is a finite Weyl group, then there is a geometric interpretation for the Kazhdan–Lusztig polynomials and this leads to many deep properties for which no elementary proofs are known (see [12,14]). Recently, Lusztig [15] has formulated a number of precise conjectures in the general case of unequal parameters. Furthermore, Lusztig proposes a geometric interpretation at least for those weight functions which arise in the representation theory of finite groups of Lie type. (The complete list of these L is given in [8, Table II, p. 35].)

One of our aims here is to show that some of Lusztig's conjectures hold for W of type F_4 and any weight function, even for those L which do not admit a geometric interpretation. In type F_4, with generators and diagram given by the diagram below, a weight function L is specified by two positive integers $a := L(s_1) = L(s_2) > 0$ and $b := L(s_3) = L(s_4) > 0$:



By explicit computations using the CHEVIE-system [6], we obtain the following results.

Theorem 1.1. *Let W be of type F_4 and L any weight function on W with $L(w) > 0$ for $w \neq 1$. Then the left cell representations of W (with respect to L) are precisely the constructible representations, as defined by Lusztig* [15, Chapter 22].

The above result is conjectured to hold in general by Lusztig [15, §22.29]. As far as the partition of W into left cells is concerned, we shall see that there are only four essentially different cases, according to whether $b = a$, $b = 2a$, $2a > b > a$, or $b > 2a$; see Corollary 4.8 and Remark 4.9.

Theorem 1.2. *Let W be of type F_4 and L any weight function on W with $L(w) > 0$ for $w \neq 1$. For $w \in W$, we define $\Delta(w) \in \mathbb{Z}_{\geqslant 0}$ and $0 \neq n_w \in \mathbb{Z}$ by the condition*

$$P^*_{1,w} = n_w v^{-\Delta(w)} + \text{strictly smaller powers of } v; \quad \text{see Lusztig [15, 14.1].}$$

Let C be a left cell of W (with respect to L). Then the function $w \mapsto \Delta(w)$ reaches its minimum at exactly one element of C, denoted by $d_C \in C$. We have $d_C^2 = 1$ and $n_{d_C} = \pm 1$.

(For the definition of $P^*_{y,w}$, see Section 2.) The elements d_C are the *distinguished involutions* whose existence is predicted by Lusztig [15, Conjectures 14.2 (P1, P6, P13)]. The following result is also part of those conjectures (P4, P9).

Theorem 1.3. *Let W be of type F_4 and L any weight function on W with $L(w) > 0$ for $w \neq 1$. For any $y, w \in W$, we have the following implication*:

$$y \leqslant_L w \quad \text{and} \quad y \sim_{LR} w \quad \Longrightarrow \quad y \sim_L w.$$

(For the definition of the relations $\leqslant_L$, $\sim_L$, $\sim_{LR}$, see Section 2.) The proofs of the above three theorems will be given in Section 4 (see Corollary 4.8).

In type F_4, there is a geometric interpretation for the cases where $(a,b) \in \{(1,1),(1,2),(1,4)\}$; see [8, Table II, p. 35]. To deal with arbitrary values for a and b, we have to provide a theoretical argument which shows that it is enough to consider only those L where the values on the generators are bounded by a constant which can be explicitly computed in terms of W. More precisely, in Definition 2.13, we introduce (for general W) an equivalence relation on the set of weight functions, called "generic equivalence." Two generically equivalent weight functions give rise to the same partition of W into left cells, the same left pre-order relation and the same set of left cell representations. In Corollary 3.6, we show that any weight function is generically equivalent to a weight function whose values on the generators are bounded by a constant which can be computed efficiently.

It should be noted that the relation of "generic equivalence" is very strong. As far as applications are concerned, one is interested in a weaker equivalence relation: we say that two weight functions are "cell-equivalent" if they give rise to the same partition of W into left cells. The notion of "generic equivalence" merely provides a convenient technical tool for proving "cell-equivalence."

Lusztig's results [15] on dihedral groups are interpreted in this framework in Example 2.12. Conjecture 2.17 (found independently by Bonnafé) would yield a complete description of the cell-equivalence classes of weight functions in type B_n. In any case, cell-equivalence classes seem to be organised in a rather smooth way.

Both the results in type F_4 and the evidence for the conjecture on type B_n are based on a CHEVIE-program which we have developed, for computing the Kazhdan–Lusztig polynomials, the M-polynomials, and the pre-order relations $\leqslant_L$, $\leqslant_{LR}$ for a finite Coxeter group W and any choice of the parameters (either given by independent indeterminates and a monomial order on them, or given by a weight function). For example, this program systematically computes the polynomials $P^*_{y,w}$ for all pairs $y < w$ in W; it also computes all incidences of the Kazhdan–Lusztig pre-order relation $y \leqslant_L w$. The program automatically checks some of Lusztig's conjectures (in particular, the properties expressed in the above three theorems) and computes the characters carried by the various left cells. These programs have already been used in the computations reported in [7, §11.3] and [5, §7]. To my knowledge, the first such programs (for Kazhdan–Lusztig polynomials in the unequal parameter case) were written by K. Bremke [3] who used them to compute W-graphs for the irreducible representations of certain Iwahori–Hecke algebras of type F_4. We only remark that, in the case of equal parameters, there is already a rather sophisticated theory for the computation of Kazhdan–Lusztig polynomials; see Alvis [1] and Ducloux [4].

2. Total orderings and weight functions

The basic references for this section are [10] and [15]. In the latter reference, Lusztig studies the left cells of a Coxeter group W with respect to a weight function L on W. In the former reference, Lusztig considers a more abstract setting where left cells are defined with respect to an abelian group and a total order on it. We will see in this section that the more abstract setting can be used to show that two given weight functions actually give rise to the same partition of W into left cells. (A similar argument has already been used, for example, in [2].) This will provide the theoretical argument for showing that, in order to determine the left cells for all possible weight functions on W, it is actually enough to consider a certain finite number of weight functions.

We begin by recalling the basic setting for the definition of Kazhdan–Lusztig polynomials and left cells. Let W be a Coxeter group, with generating set S. Let Γ be an abelian group (written multiplicatively) and $\mathbf{A} = \mathbb{Z}[\Gamma]$ be the group algebra of Γ over $\mathbb{Z}$. Let $\{v_s \mid s \in S\} \subset \Gamma$ be a subset such that $v_s = v_t$ whenever $s, t \in S$ are conjugate in W. Then we have a corresponding generic Iwahori–Hecke algebra $\mathbf{H}$, with $\mathbf{A}$-basis $\{\mathbf{T}_w \mid w \in W\}$ and multiplication given by the rule

$$\mathbf{T}_s\,\mathbf{T}_w = \begin{cases} \mathbf{T}_{sw}, & \text{if } l(sw) > l(w), \\ \mathbf{T}_{sw} + \left(v_s - v_s^{-1}\right)\mathbf{T}_w, & \text{if } l(sw) < l(w); \end{cases} \tag{2.1}$$

here $l : W \to \mathbb{N}_0$ denotes the usual length function on W with respect to S. (Note that the above elements $\mathbf{T}_w$ are denoted $\widetilde{T}_w$ in [10].)

Let $a \mapsto \bar{a}$ be the involution of $\mathbb{Z}[\Gamma]$ which takes g to g^{-1} for any $g \in \Gamma$. We extend it to a map $\mathbf{H} \to \mathbf{H}$, $h \mapsto \bar{h}$, by the formula

$$\overline{\sum_{w \in W} a_w \mathbf{T}_w} = \sum_{w \in W} \bar{a}_w \mathbf{T}_{w^{-1}}^{-1} \quad \left(a_w \in \mathbb{Z}[\Gamma]\right). \tag{2.2}$$

Then $h \mapsto \bar{h}$ is in fact a ring involution.

Now assume that we have chosen a total ordering of Γ. This is specified by a multiplicatively closed subset $\Gamma_+ \subseteq \Gamma \setminus \{1\}$ such that we have $\Gamma = \Gamma_+ \amalg \{1\} \amalg \Gamma_-$, where $\Gamma_- = \{g^{-1} \mid g \in \Gamma_+\}$. Furthermore, we assume that

$$\{v_s \mid s \in S\} \subset \Gamma_+. \tag{2.3}$$

Given a total ordering of Γ as above, we have a corresponding *Kazhdan–Lusztig basis* of $\mathbf{H}$, which we denote by $\{\mathbf{C}_w \mid w \in W\}$. (Note that this basis is denoted by C'_w in [10].) The basis element $\mathbf{C}_w$ is uniquely determined by the conditions that

$$\overline{\mathbf{C}}_w = \mathbf{C}_w \quad \text{and} \quad \mathbf{C}_w = \mathbf{T}_w + \sum_{\substack{y \in W \\ y < w}} \mathbf{P}^*_{y,w}\,\mathbf{T}_y, \tag{2.4}$$

where $\mathbf{P}^*_{y,w} \in \mathbb{Z}[\Gamma_-]$ for $y < w$. Here, $\leqslant$ denotes the Bruhat–Chevalley order on W. We shall also set $\mathbf{P}^*_{w,w} = 1$ for all $w \in W$. For any $w \in W$ we set $v_w := v_{s_1} \cdots v_{s_p}$ where $w = s_1 \cdots s_p$ with $s_i \in S$ is a reduced expression. Then we actually have

$$\mathbf{P}_{y,w} := v_w v_y^{-1}\,\mathbf{P}^*_{y,w} \quad \text{lies in } \mathbb{Z}\left[v_t^2 \;\middle|\; t \in S\right] \text{ and has constant term 1;} \tag{2.5}$$

see Lemma 3.2 below. We have the following multiplication formulas. Let $w \in W$ and $s \in S$. Then

$$\mathbf{T}_s \mathbf{C}_w = \begin{cases} \mathbf{C}_{sw} - v_s \mathbf{C}_w + \displaystyle\sum_{\substack{y < w \\ sy < y}} \mathbf{M}^s_{y,w}\,\mathbf{C}_y, & \text{if } sw > w, \\ v_s \mathbf{C}_w, & \text{if } sw < w, \end{cases} \tag{2.6}$$

where the coefficients $\mathbf{M}^s_{y,w} \in \mathbf{A}$ are such that $\overline{\mathbf{M}}^s_{y,w} = \mathbf{M}^s_{y,w}$. Given $y, w \in W$ and $s \in W$, we write $y \leqslant_{L,s} w$ if the following conditions are satisfied:

$$w = sy > y \quad \text{or} \quad sy < y < w < sw \quad \text{and} \quad \mathbf{M}^s_{y,w} \neq 0. \tag{2.7}$$

The Kazhdan–Lusztig left preorder $\leqslant_L$ is the transitive closure of the above relation, that is, given $y, w \in W$ we have $y \leqslant_L w$ if $y = w$ or if there exists a sequence $y = y_0, y_1, \ldots, y_n = w$ of elements in W and a sequence $s_1, \ldots, s_n$ of generators in S such that $y_{i-1} \leqslant_{L,s_i} y_i$ for $1 \leqslant i \leqslant n$ (See [10, §6].) Thus, we have $\mathbf{H}\,\mathbf{C}_w \subseteq \sum_{y \leqslant_L w} \mathbf{A}\mathbf{C}_y$ for any $w \in W$. The equivalence relation associated with $\leqslant_L$ will be denoted by $\sim_L$ and the corresponding equivalence classes are called the *left cells* of W. Similarly, we write $y \leqslant_{LR} w$ if $y = w$ or if there is a chain of elements $y = y_0, y_1, \ldots, y_n = w$ in W such that, for each i, we have $y_{i-1} \leqslant_L y_i$ or $y_{i-1}^{-1} \leqslant_L y_i^{-1}$. The equivalence relation associated with $\leqslant_{LR}$ will be denoted by $\sim_{LR}$ and the corresponding equivalence classes are called the *two-sided cells* of W. Each two-sided cell is a union of left cells and a union of right cells. Consider the following statement:

$$y \leqslant_L w \quad \text{and} \quad y \sim_{LR} w \quad \Longrightarrow \quad y \sim_L w. \tag{L}$$

This is known to be true in certain cases where there is a geometric interpretation for the parameters (for example, the equal-parameter case where $v_s = v_t$ for all $s \neq t$ in S); see [15, Chapter 14] for more details. The above property plays an important role in certain representation-theoretic constructions; see [11, Chapter 5]. Lusztig [15, 14.2] conjectures that (L) holds in the general unequal parameter case. It would imply that the two-sided cells are the *minimal* subsets of W which are at the same time unions of left cells and union of right cells.

Each left cell $\mathfrak{C}$ gives rise to a representation of $\mathbf{H}$. This is constructed as follows (see [10, §7]). Let $V_{\mathfrak{C}}$ be an $\mathbf{A}$-module with a free $\mathbf{A}$-basis $\{e_w \mid w \in \mathfrak{C}\}$. Then the action of $\mathbf{T}_s$ ($s \in S$) is given by the formula

$$\mathbf{T}_s.e_w = \begin{cases} e_{sw} + v_s e_w - \displaystyle\sum_{\substack{y<w\\ sy<y}} (-1)^{l(w)-l(y)} \mathbf{M}^s_{y,w} e_y, & \text{if } sw > w, \\ -v_s^{-1} e_w, & \text{if } sw < w, \end{cases} \tag{2.8}$$

where we tacitly assume that $e_y = 0$ if $y \notin \mathfrak{C}$. (The formula (2.8) can be related to the formula (2.6) using a suitable automorphism of $\mathbf{H}$; see [10, §6].) Assume now that W is finite. Upon specialization $v_s \mapsto 1$ ($s \in S$), we obtain a representation of W which is called the representation carried by $\mathfrak{C}$. We denote by $\chi_{\mathfrak{C}}$ the character of that representation, that is, the map $w \mapsto \operatorname{trace}(w|V_{\mathfrak{C}})$. On the other hand, let $\operatorname{Con}(W, \Gamma_+)$ be the set of so-called constructible characters of W, as defined by Lusztig; see [15, Chapter 22] (and also [5, §3], for the general setting with respect to $\Gamma_+ \subset \Gamma$). Consider the following statement:

$$\operatorname{Con}(W, \Gamma_+) = \{\chi_{\mathfrak{C}} \mid \mathfrak{C} \text{ left cell in } W \text{ with respect to } \Gamma_+ \subset \Gamma\}. \tag{C}$$

It is conjectured by Lusztig [15, 22.29] that (C) always holds.[1] This is known to be true in the equal parameter case (see [13]) and some cases with unequal parameters (see, for example, the explicit results on type $I_2(m)$ in [15], on type B_n in [2], and on type F_4 in [5]). The important point about (C) is that the constructible characters can be easily determined by a recursive procedure, using the induction of characters from parabolic subgroups of W.

Summary. Given an abelian group Γ with a total order specified by $\Gamma_+ \subset \Gamma$ and a choice of parameters $\{v_s \mid s \in S\} \subset \Gamma_+$, we obtain

- a collection of polynomials $\mathbf{P}^*_{y,w} \in \mathbb{Z}[\Gamma_-]$ for all $y < w$ in W;
- a collection of polynomials $\mathbf{M}^s_{y,w} \in \mathbb{Z}[\Gamma]$ whenever $sy < y < w < sw$.

These data determine, in a purely combinatorial way, a pre-order relation $\leqslant_L$ on W and the corresponding partition of W into left cells and two-sided cells. Finally, we obtain a set of characters of W (the characters carried by the left cells).

Now let us specialise the above setting to the case where the parameters of the Iwahori–Hecke algebra are given by a weight function. Following [15], a weight function on W is a function $L : W \to \mathbb{Z}$ such that $L(ww') = L(w) + L(w')$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$. Such a function is determined by its values $L(s)$ on S which are subject only to the condition that $L(s) = L(s')$ for any $s \neq s'$ in S such that the order of ss' is finite and odd. (See Matsumoto's lemma [7, §1.2].) We shall only consider weight functions L such that $L(s) > 0$ for all $s \in S$. Let $A = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. We have a corresponding Iwahori–Hecke algebra H with parameters $\{v^{L(s)} \mid s \in S\}$. Thus, H has an A-basis $\{T_w \mid w \in W\}$ and the multiplication is determined by the formula

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ T_{sw} + \big(v^{L(s)} - v^{-L(s)}\big) T_w, & \text{if } l(sw) < l(w). \end{cases} \tag{2.10}$$

Now consider the abelian group $\{v^n \mid n \in \mathbb{Z}\}$ with the total order specified by $\{v^n \mid n > 0\}$. Thus, as above, we have a corresponding Kazhdan–Lusztig basis $\{C_w \mid w \in W\}$ of H. Consequently, we obtain

- a collection of polynomials $P^*_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $y < w$ in W;
- a collection of polynomials $M^s_{y,w} \in \mathbb{Z}[v, v^{-1}]$ whenever $sy < y < w < sw$.

As before, these data determine a pre-order relation $\leqslant_L$ on W and the corresponding partition of W into left cells and two-sided cells; furthermore, we obtain the characters carried by the left cells of W.

The following result establishes a link between the above two situations, where we have an abelian group Γ with a total order specified by $\Gamma_+ \subset \Gamma$ and a choice of parameters $\{v_s \mid s \in S\} \subset \Gamma_+$ on the one hand, and a weight function L on the other hand. As above,

[1] In a recent preprint, "Left cells and constructible representations" (available at http://arXiv.org/math.RT/0404510), the author has shown that (C) follows from the general conjectures of Lusztig [15, 14.2].

denote by $\mathbf{P}^*_{y,w}$ and $\mathbf{M}^s_{y,w}$ the polynomials in $\mathbb{Z}[\Gamma]$ arising in the first case, and denote by $P^*_{y,w}$ and $M^s_{y,w}$ the polynomials in $\mathbb{Z}[v, v^{-1}]$ arising in the second case.

We now define two subsets $\Gamma_+^{(a)}(W), \Gamma_+^{(b)}(W) \subseteq \Gamma_+$. First, let $\Gamma_+^{(a)}(W)$ be the set of all $\gamma \in \Gamma_+$ such that γ^{-1} occurs with non-zero coefficient in a polynomial $\mathbf{P}^*_{y,w}$ for some $y < w$ in W. Next, for any $y, w \in W$ and $s \in S$ such that $\mathbf{M}^s_{y,w} \neq 0$, we write $\mathbf{M}^s_{y,w} = n_1\gamma_1 + \cdots + n_r\gamma_r$ where $0 \neq n_i \in \mathbb{Z}$, $\gamma_i \in \Gamma$ and $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ for $2 \leqslant i \leqslant r$. We let $\Gamma_+^{(b)}(W)$ be the set of all elements $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ arising in this way, for any y, w, s such that $\mathbf{M}^s_{y,w} \neq 0$. Finally, we set $\Gamma_+(W) := \Gamma_+^{(a)}(W) \cup \Gamma_+^{(b)}(W)$.

Proposition 2.10. *Assume that we have a ring homomorphism*

$$\sigma : \mathbb{Z}[\Gamma] \to \mathbb{Z}\big[v, v^{-1}\big], \quad v_s \mapsto v^{L(s)} \quad (s \in S)$$

such that

$$\sigma\big(\Gamma_+(W)\big) \subseteq \big\{v^n \bigm| n > 0\big\}. \tag{$*$}$$

Then $\sigma(\mathbf{P}^*_{y,w}) = P^*_{y,w}$ for all $y < w$ in W and $\sigma(\mathbf{M}^s_{y,w}) = M^s_{y,w}$ for any $s \in S$ such that $sy < y < w < sw$. Furthermore, the relations $\leqslant_L$, $\sim_L$, $\leqslant_{LR}$, and $\leqslant_{LR}$ on W defined with respect to the weight function L are the same as those with respect to $\Gamma_+ \subset \Gamma$, and so are the corresponding representations of W.

Proof. The map σ induces a ring homomorphism

$$\hat{\sigma} : \mathbf{H} \to H, \quad \sum_w a_w \mathbf{T}_w \mapsto \sum_w \sigma(a_w)\, T_w.$$

We have $\overline{\hat{\sigma}(h)} = \hat{\sigma}(\bar{h})$ for all $h \in \mathbf{H}$. Thus, applying $\hat{\sigma}$ to (2.4), we obtain

$$\overline{\hat{\sigma}(\mathbf{C}_w)} = \hat{\sigma}(\mathbf{C}_w) \quad \text{and} \quad \hat{\sigma}(\mathbf{C}_w) = T_w + \sum_{\substack{y \in W\\ y<w}} \sigma\big(\mathbf{P}^*_{y,w}\big)\, T_y$$

for any $w \in W$. Now condition $(*)$ implies that $\sigma(\Gamma_-) \subseteq \{v^n \mid n < 0\}$ and so $\sigma(\mathbf{P}^*_{y,w})$ is either 0 or an integral linear combination of terms v^n with $n < 0$. Thus, the elements $\hat{\sigma}(\mathbf{C}_w)$ satisfy the defining properties for the Kazhdan–Lusztig basis of H and so we must have $\hat{\sigma}(\mathbf{C}_w) = C_w$ for all $w \in W$. This also shows that $\sigma(\mathbf{P}^*_{y,w}) = P^*_{y,w}$ for all $y < w$. Now apply $\hat{\sigma}$ to (2.6). This yields the equation

$$T_s C_w = C_{sw} - v^{L(s)} C_w + \sum_{\substack{y<w\\ sy<y}} \sigma\big(\mathbf{M}^s_{y,w}\big)\, C_y \quad \text{if } sw > w.$$

Thus, we have $M^s_{y,w} = \sigma(\mathbf{M}^s_{y,w})$ if $sy < y < w < sw$. Finally, we claim that

$$\mathbf{M}^s_{y,w} \neq 0 \quad \Longrightarrow \quad M^s_{y,w} = \sigma\big(\mathbf{M}^s_{y,w}\big) \neq 0. \tag{$\dagger$}$$

Indeed, if $\mathbf{M}^s_{y,w} \neq 0$, we write $\mathbf{M}^s_{y,w} = n_1\gamma_1 + \cdots + n_r\gamma_r$ where $0 \neq n_i \in \mathbb{Z}$ and $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+^{(b)}(W)$. By condition $(*)$, we have $\sigma(\gamma_{i-1}^{-1}\gamma_i) = v^{a_i}$ with $a_i > 0$ for all i. Consequently, $M^s_{y,w} = \sigma(\mathbf{M}^s_{y,w})$ is a combination of pairwise different powers of v and, hence, non-zero. Thus, (†) holds.

So we conclude that two elements satisfy $y \leqslant_L w$ with respect to $\Gamma_+ \subset \Gamma$ if and only if they satisfy the analogous relation with respect to the weight function L. Thus, the relations $\leqslant_L$, $\sim_L$, $\leqslant_{LR}$ and $\sim_{LR}$ are the same in the two situations, and so are the corresponding representations of W. $\square$

In order to deal with "distinguished involutions" as in Theorem 1.2, we shall need the following remark.

Remark 2.11. In the above setting, let $w \in W$ and write

(a) $\mathbf{P}^*_{1,w} = \delta_w^{-1}(n_w + \mathbb{Z}\text{-combination of } \gamma \in \Gamma_-)$,

where $\delta_w \in \Gamma_+$ and $0 \neq n_w \in \mathbb{Z}$. Thus, δ_w^{-1} is the highest monomial (with respect to the total order specified by $\Gamma_+ \subset \Gamma$) occurring in $\mathbf{P}^*_{1,w}$. Then $\delta_1 = 1$ and $\delta_w \in \Gamma_+(W)$ for $w \neq 1$.

Furthermore, given a left cell C (with respect to $\Gamma_+ \subset \Gamma$), we write

(b) $\{\delta_w \mid w \in C\} = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$, where $\gamma_{i-1}^{-1}\gamma_i \in \Gamma_+$ for $2 \leqslant i \leqslant m$.

Let $\Gamma'_+(W)$ be the union of $\Gamma_+(W)$, the set of all elements γ^{-1} where γ occurs in a $\mathbb{Z}$-combination as in (a) (for any $w \in W$), and the set of all elements $\gamma_{i-1}^{-1}\gamma_i$ $(2 \leqslant i \leqslant m)$ as in (b) (for any left cell C where $m \geqslant 2$). Assume that

$$\sigma\big(\Gamma'_+(W)\big) \subseteq \big\{v^n \bigm| n > 0\big\}. \qquad (*')$$

Then, writing $\sigma(\delta_w) = v^{\Delta(w)}$ where $\Delta(w) \in \mathbb{Z}_{\geqslant 0}$, we have

$$P^*_{1,w} = n_w v^{-\Delta(w)} + \text{strictly smaller powers of } v.$$

Furthermore, if the function $w \mapsto \delta_w$ reaches its minimum at exactly one element in a left cell C, then so does the function $w \mapsto \Delta(w)$.

Example 2.12. Let $W = \langle s, t\rangle$ be a dihedral group of order $m \geqslant 4$, where m is even. Let v_s and v_t be two independent indeterminates and consider the ring of Laurent polynomials $\mathbf{A} = \mathbb{Z}[v_s^{\pm 1}, v_t^{\pm 1}]$. Let $\Gamma = \{v_s^i v_t^j \mid i, j \in \mathbb{Z}\}$ and consider the total order specified by

$$\Gamma_+ = \big\{v_s^i v_t^j \bigm| i > 0\big\} \cup \big\{v_t^j \bigm| j > 0\big\}$$

(a lexicographic order where $v_s > v_t$). The polynomials $\mathbf{P}_{y,w}$ have been determined independently in [7, Exercise 11.4] and in [15, Chapter 7]. Let $y < w$ and write $v_w v_y^{-1} = v_s^{m_s} v_t^{m_t}$ where $m_s, m_t \geqslant 0$. Then

$$\mathbf{P}_{y,w} = \begin{cases} \sum_{i=0}^{m_t} (-1)^i v_t^{2i}, & \text{if } w < tw,\, w < wt, \text{ and } y \leqslant tsw < sw, \\ 1 + v_t^2, & \text{if } w < sw,\, w < ws, \text{ and } y \leqslant stw < tw, \\ 1, & \text{otherwise.} \end{cases}$$

The M-polynomials are given by

$$\mathbf{M}_{y,w}^s = \begin{cases} v_s v_t^{-1} + v_s^{-1} v_t, & \text{if } l(w) = l(y) + 1, sy < y < w < sw, \\ 1, & \text{if } l(w) = l(y) + 3, sy < y < w < sw. \end{cases}$$

All other M-polynomials are 0. Now consider a weight function L on W such that

$$L(s) > L(t) > 0.$$

Let v be another indeterminate; then we have a ring homomorphism

$$\sigma : \mathbb{Z}[\Gamma] \to \mathbb{Z}\big[v, v^{-1}\big], \quad v_s^i v_t^j \mapsto v^{L(s)i + L(t)j}.$$

We claim that condition $(*)$ in Proposition 2.10 is satisfied. For this purpose, we first have to determine the monomials which can occur in a polynomial $\mathbf{P}_{y,w}^*$ for $y < w$. Write $v_w v_y^{-1} = v_s^{m_s} v_t^{m_t}$ as above. Since $y < w$, we have $m_s > 0$ or $m_t > 0$. If $w < tw$, $w < wt$, and $y \leqslant tsw < sw$, then w has a reduced expression which starts and ends with s. Since y is a subexpression of w, we conclude that $m_s \geqslant m_t$. Hence $\mathbf{P}_{y,w}^*$ is a linear combination of monomials $v_s^{-m_s} v_t^j$ where $j \leqslant m_t \leqslant m_s$. On the other hand, if $w < sw$, $w < ws$, and $y \leqslant stw < tw$, then $m_s \geqslant 1$ and $m_t \geqslant 1$. So $\mathbf{P}_{y,w}^*$ is a linear combination of monomials $v_s^{-m_s} v_t^j$ where $j \leqslant 1$. Finally, in the cases where $\mathbf{P}_{y,w} = 1$, we have $\mathbf{P}_{y,w}^* = v_s^{-m_s} v_t^{-m_t}$. Thus, we find that

$$\Gamma_+(W) \subseteq \big\{ v_s^i v_t^j \;\big|\; i \geqslant 0,\ i + j \geqslant 0,\ (i, j) \neq (0, 0) \big\}.$$

Now, if $i \geqslant 0$ and $i + j \geqslant 0$, then $L(s)i + L(t)j \geqslant L(t)i + L(t)j = L(t)(i + j) \geqslant 0$. Furthermore, if $i > 0$, then the first inequality is strict and so $L(s)i + L(t)j > 0$; while if $i = 0$, then $j > 0$ and so $L(s)i + L(t)j > 0$. Next, we also see that the required condition holds for the monomials occurring in the polynomials $\mathbf{M}_{y,w}^t$. Thus, $(*)$ holds.

We conclude that $P_{y,w}^* = \sigma(\mathbf{P}_{y,w}^*)$ for all $y < w$ in W. Thus, for any weight function such that $L(s) > L(t) > 0$, the corresponding polynomials $P_{y,w}^*$ are obtained by specialisation from the polynomials $\mathbf{P}_{y,w}^*$ which have been determined for one fixed choice of $\Gamma_+ \subset \Gamma$. Furthermore, the partition of W into left cells is the same for all weight functions such that $L(s) > L(t) > 0$ (and it is given by the partition into left cells with respect to $\Gamma_+ \subset \Gamma$). An explicit description of these left cells is given in [15, Chapter 8]. The distinguished involutions are 1, s, t, tst, tw_0, w_0. For the left cell representations and constructive representations, see also [5, §6].

Definition 2.13. Let L, L' be two weight functions on W. We say that L, L' are Γ_+-*equivalent* if there exists an abelian group Γ, a total order specified by $\Gamma_+ \subset \Gamma$ and a set of parameters $\{v_s \mid s \in S\} \subset \Gamma_+$ such that the following holds:

(a) There exist ring homomorphisms $\sigma, \sigma' : \mathbb{Z}[\Gamma] \to \mathbb{Z}[v, v^{-1}]$ such that $\sigma(v_s) = v^{L(s)}$ and $\sigma'(v_s) = v^{L'(s)}$ for all $s \in S$.
(b) Condition $(*)$ in Proposition 2.10 is satisfied for both σ and σ'.

We say that L, L' are *generically equivalent* if $L = L'$ or if there exists a sequence of weight functions $L = L_0, L_1, \ldots, L_n = L'$ and abelian groups $\Gamma_1, \ldots, \Gamma_n$ such that L_{i-1}, L_i are $(\Gamma_i)_+$-equivalent with respect to a total order specified by $(\Gamma_i)_+ \subset \Gamma_i$ for $1 \leqslant i \leqslant n$. In particular, generically equivalent weight functions are *cell-equivalent*, that is, they give rise to the same partition of W into left cells.

Proposition 2.14. *Assume that W is finite and let $w_0 \in W$ be the longest element. Then there exists a constant $N \leqslant 8l(w_0)^3$ such that any weight function on W is generically equivalent to a weight function L such that $1 \leqslant L(s) \leqslant N$ for all $s \in S$.*

The proof will be given in Section 3 (see Corollary 3.6). Note that, since W is finite, there clearly exists some constant N having the above property. The point about Proposition 2.14 is that we can give an explicit bound for N. We have not tried to obtain an optimal bound theoretically. However, the proofs of Proposition 3.5 and Corollary 3.6 will show how to determine such a bound efficiently.

Remark 2.15. Let $L : W \to \mathbb{Z}$ be a weight function such that $L(s) > 0$ for all $s \in S$. Let $d > 0$ be a positive integer. Then the function $L_d : W \to \mathbb{Z}$ defined by $L_d(w) := dL(w)$ also is a weight function, and we leave it as an (easy) exercise to the reader to check that L, L_d are generically equivalent. Thus, in order to classify weight functions up to generic equivalence, it will be sufficient to consider only those weight functions L such that $\gcd(\{L(s) \mid s \in S\}) = 1$.

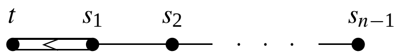
Example 2.16. In practice, the cell-equivalence classes will be determined by a set of weight functions whose values are bounded by a constant N which is much smaller than the value given in Proposition 2.14. For example, if $W = \langle s, t\rangle$ is a dihedral group of type $I_2(m)$ (with $m \geqslant 4$ even), then we may take $N = 2$. Indeed, let us specify a weight function $L : W \to \mathbb{Z}$ by the pair (a, b) such that $L(s) = a$ and $L(t) = b$. Then, by Example 2.12, there are exactly three cell-equivalence classes of weight functions:

$$\begin{aligned}
\mathcal{L}_1 &= \bigl\{(a, b) \mid a = b > 0\bigr\}, && \text{representative: } (1, 1),\\
\mathcal{L}_2 &= \bigl\{(a, b) \mid a > b > 0\bigr\}, && \text{representative: } (2, 1),\\
\mathcal{L}_3 &= \bigl\{(a, b) \mid b > a > 0\bigr\}, && \text{representative: } (1, 2).
\end{aligned}$$

In fact, the above computations show that these are even the generic equivalence classes.

If W is of type F_4, we will see in Section 4 that there are 7 cell-equivalence classes of weight functions.

Now let W be of type B_n, with diagram given as follows:

B_n 

Here, the generators s_i are all conjugate, while t and s_1 are not conjugate. Thus, a weight function $L : W \to \mathbb{Z}$ is uniquely specified by the values

$$b := L(t) > 0 \quad \text{and} \quad a := L(s_1) = L(s_2) = \cdots = L(s_{n-1}) > 0.$$

The best bound does not yet seem to be known. Recently, Bonnafé and Iancu have shown that all weight functions such that $a/b > n - 1$ are cell-equivalent. Experiments with CHEVIE lead to the following general conjecture.

Conjecture 2.17. *In type B_n with diagram and weight function as specified above, we have the following cell-equivalence classes of weight functions*:

$$\begin{aligned}
\mathcal{L}_1 &= \bigl\{(a, a, a, \ldots, a) \mid a > 0\bigr\} \quad (\textit{equal parameter case}),\\
\mathcal{L}_i &= \bigl\{(ia, a, a, \ldots, a) \mid a > 0\bigr\} \quad (\textit{where } 2 \leqslant i \leqslant n-1),\\
\mathcal{L}_{i,i-1} &= \bigl\{(b, a, a, \ldots, a) \mid ia > b > (i-1)a \geqslant 0\bigr\} \quad (\textit{where } 1 \leqslant i \leqslant n-1),\\
\mathcal{L}_{\mathrm{asymp}} &= \bigl\{(b, a, a, \ldots, a) \mid b > (n-1)a > 0\bigr\}.
\end{aligned}$$

(*The functions in $\mathcal{L}_{\mathrm{asymp}}$ correspond to the case treated by Bonnafé–Iancu* [2].)

Furthermore, if (C) in Section 2 holds, then all left cell representations with respect to L will be irreducible, unless we have $L \in \mathcal{L}_i$ for some $1 \leqslant i \leqslant n-1$ (see [15, 22.25]); if $L \in \mathcal{L}_i$ for some i, then the left cell representations will be given as in [15, 22.24].

The above conjecture is a slightly different version of a part of several conjectures that were formulated by Bonnafé (private communication). Using our CHEVIE-program, we have verified that Conjecture 2.17 holds for B_3 and B_4.

The above results are only concerned with finite Coxeter groups. It would be interesting to study equivalence classes of weight functions for affine Weyl groups.

3. On the generic equivalence classes of weight functions

We place ourselves in the general setting where W is any Coxeter group with generators S and where we are given an abelian group Γ, a total order specified by $\Gamma_+ \subset \Gamma$ and a set of parameters $\{v_s \mid s \in S\} \subset \Gamma_+$ for the corresponding Iwahori–Hecke algebra of W. One of the aims of this section is to provide a proof of Proposition 2.14. Our first task will be to get some control on the degrees of the monomials that might occur in the polynomials $\mathbf{P}^*_{y,w}$ and $\mathbf{M}^s_{y,w}$. Now, Lusztig gives some rather explicit such bounds, but only in the setting involving a weight function, and these are not entirely sufficient for our purposes. To illustrate our point, consider the following example.

Example 3.1. Let $W = \langle s, t\rangle$ be a dihedral group as in Example 2.12. Consider a weight function L where $L(s) = a > 1$ is a big number and $L(t) = 1$. Then [15, Proposition 6.4] tells us that $M^s_{y,w}$ is a $\mathbb{Z}$-linear combination of powers v^n with $-a+1 \leqslant n \leqslant a-1$ and $n \equiv L(w) - L(y) - L(s) \bmod 2$. So, a priori, $M^s_{y,w}$ could be a polynomial involving many non-zero terms. However, from the formula given in Example 2.12 and Proposition 2.10, we see that $M^s_{y,w}$ only involves very few terms:

$$M^s_{y,w} = \sigma\big(\mathbf{M}^s_{y,w}\big) = \begin{cases} v^{a-1} + v^{1-a}, & \text{if } l(w) = l(y)+1,\, sy < y < w < sw,\\ 1, & \text{if } l(w) = l(y)+3,\, sy < y < w < sw.\end{cases}$$

To explain this behaviour, we need to establish some bounds in the general framework with respect to an abelian group Γ and a total order on it.

Lemma 3.2. *Let $y, w \in W$ be such that $y \leqslant w$. Then the following hold*:

(a) $v_w v_y^{-1}\mathbf{P}^*_{y,w}$ *is a polynomial in* $\{v_s^2 \mid s \in S\}$, *with constant term* 1.
(b) $v_w v_y^{-1}\bar{\mathbf{P}}^*_{y,w}$ *is a polynomial in* $\{v_s^2 \mid s \in S\}$, *with constant term* 0.

Proof. The following proof is more or less a copy of that of [15, Proposition 5.4]. However, in [15], Lusztig exclusively considers the situation involving a weight function. Thus, in order to show that all the arguments go through in the general case, we include the details here. First, we shall need the R-polynomials in the general setting, as defined in [10]. For $y \in W$, we have

$$\bar{\mathbf{T}}_y = \mathbf{T}^{-1}_{y^{-1}} = \sum_{x\in W} \bar{\mathbf{R}}_{x,y}\mathbf{T}_x, \quad \text{where } \mathbf{R}_{x,y} \in \mathbb{Z}[\Gamma].$$

We have the following recursion formula. If $sy < y$ for some $s \in S$, then

$$\mathbf{R}_{x,y} = \mathbf{R}_{sx,sy} + \big(v_s - v_s^{-1}\big)\mathbf{R}_{x,sy}, \quad \text{if } sx > x,$$
$$\mathbf{R}_{x,y} = \mathbf{R}_{sx,sy}, \quad \text{if } sx < x.$$

(Same proof as in [15, Lemma 4.4].) Using the above recursion formula, one easily shows that $\mathbf{R}_{y,y} = 1$ and $\mathbf{R}_{x,y} = 0$ unless $x \leqslant y$. Furthermore,

$$v_y v_x^{-1}\mathbf{R}_{x,y} \in \mathbb{Z}\big[v_s^2 \bigm| s \in S\big], \quad \text{with constant term } (-1)^{l(y)-l(x)}. \tag{$*$}$$

(Same proof as in [15, Lemma 4.7].) The Kazhdan–Lusztig polynomials and the R-polynomials are related by the following identity (see [10, Proposition 2]). We have

$$\bar{\mathbf{P}}^*_{x,w} - \mathbf{P}^*_{x,w} = \sum_{x<y\leqslant w} \mathbf{R}_{x,y}\mathbf{P}^*_{y,w} \quad \text{for all } x < w \text{ in } W.$$

Now, for the proof of (a) and (b), we proceed by induction on $l(w) - l(y)$. If $y = w$, then $\mathbf{P}^*_{w,w} = 1$ and there is nothing to prove. Now assume that $y < w$. Multiplying both sides of the identity relating Kazhdan–Lusztig polynomials and R-polynomials with $v_w v_y^{-1}$ yields

$$v_w v_y^{-1}\bar{\mathbf{P}}^*_{y,w} - v_w v_y^{-1}\mathbf{P}^*_{y,w} = \sum_{y<x\leqslant w} \big(v_x v_y^{-1}\mathbf{R}_{y,x}\big)\big(v_w v_x^{-1}\mathbf{P}^*_{x,w}\big).$$

By induction and $(*)$, all terms on the right-hand side are polynomials in $\{v_s^2 \mid s \in S\}$. Hence so is the left-hand side. Since $\mathbf{P}^*_{y,w}$ and $\bar{\mathbf{P}}^*_{y,w}$ have no terms in common, we conclude that both $v_w v_y^{-1}\mathbf{P}^*_{y,w}$ and $v_w v_y^{-1}\bar{\mathbf{P}}^*_{y,w}$ are polynomials in the variables v_s^2 ($s \in S$). Now consider the constant terms on both sides of the above equation. We begin with the right-hand side. By induction and $(*)$, it has constant term

$$\sum_{y<x\leqslant w} (-1)^{l(x)-l(y)} \cdot 1 = -1 + (-1)^{l(y)} \sum_{y\leqslant x\leqslant w} (-1)^{l(x)} = -1,$$

where the last equality holds by [15, Proposition 4.8] (an identity due to D.N. Verma). It remains to observe that $v_w v_y^{-1}\bar{\mathbf{P}}^*_{y,w} \in \mathbb{Z}[\Gamma_+]$ and so the constant term is 0. Hence the constant term of $-v_w v_y^{-1}\mathbf{P}^*_{y,w}$ equals -1, as required. □

Lemma 3.3. *Let $y, w \in W$ and $s \in S$ be such that $sy < y < w < sw$. Then $v_s v_w v_y^{-1}\mathbf{M}^s_{y,w}$ is a polynomial in $\{v_t^2 \mid t \in S\}$, with constant term* 0.

Proof. As in the proof of [10, Proposition 4], one considers the identity (arising from (2.6)):

$$\mathbf{T}_s\mathbf{C}_w - \mathbf{C}_{sw} + v_s\mathbf{C}_w - \sum_{\substack{y<w\\ sy<y}} \mathbf{M}^s_{y,w}\,\mathbf{C}_y = 0.$$

Expressing all terms in the basis $\{\mathbf{T}_y \mid y \in W\}$ of $\mathbf{H}$, the coefficient of every $\mathbf{T}_y$ must be zero. That coefficient is given by

$$f_y = v_s\mathbf{P}^*_{y,w} + \mathbf{P}^*_{sy,w} - \mathbf{P}^*_{y,sw} - \sum_{\substack{y\leqslant z<w\\ sz<z}} \mathbf{P}^*_{y,z}\mathbf{M}^s_{z,w}.$$

Hence, given that $f_y = 0$, we obtain

$$\mathbf{M}^s_{y,w} = \mathbf{P}^*_{sy,w} - \mathbf{P}^*_{y,sw} + v_s\mathbf{P}^*_{y,w} - \sum_{\substack{y<z<w\\ sz<z}} \mathbf{P}^*_{y,z}\mathbf{M}^s_{z,w}.$$

Since $sy < y$ and $sw > w$, we have $v_y = v_s v_{sy}$ and $v_{sw} = v_s v_w$. Thus, multiplying the above equation by $v_s v_w v_y^{-1}$ yields that

$$v_s v_w v_y^{-1}\mathbf{M}^s_{y,w} = \mathbf{P}_{sy,w} - \mathbf{P}_{y,sw} + v_s^2\mathbf{P}_{y,w} - \sum_{\substack{y<z<w\\ sz<z}} \mathbf{P}_{y,z}\big(v_s v_w v_z^{-1}\mathbf{M}^s_{z,w}\big).$$

Hence, the assertion follows by induction on $l(w) - l(y)$ and using Lemma 3.2. $\square$

From now on, we assume that W is finite and let $w_0 \in W$ be the longest element. Then, by the classification of finite Coxeter groups, unequal parameters can only occur for W of type $I_2(m)$ (with m even), B_n (any $n \geqslant 3$) or F_4. Furthermore, in these cases, a weight function on W may take at most 2 different values on the generators of W. Thus, we will now consider an abelian group $\Gamma = \{x^i y^j \mid i, j \in \mathbb{Z}\}$ where x and y are independent indeterminates and where $\Gamma_+ \subset \Gamma$ is any total order. Furthermore, let $S = S_x \amalg S_y$ be a partition (where $S_x, S_y \neq \emptyset$) such that no generator in S_x is conjugate to any generator in S_y. The parameters of the corresponding Iwahori–Hecke algebra will be assumed to be given by

$$v_s = x \quad (\text{if } s \in S_x) \qquad \text{and} \qquad v_t = y \quad (\text{if } t \in S_y).$$

Lemma 3.4. *The monomials involved in any polynomial $\mathbf{P}^*_{y,w}$ or in any polynomial $\mathbf{M}^s_{y,w}$ are of the form $x^i y^j$ where $-l(w_0) < i, j < l(w_0)$. In particular, we have $\Gamma_+(W) \subseteq \{x^i y^j \mid -l(w_0) < i, j < l(w_0)\}$.*

Proof. Let $y, w \in W$, $y \leqslant w$. Thus, since y is a subexpression of w, we have $v_w v_y^{-1} = x^a y^b$ where $a, b \geqslant 0$. Furthermore, let us write $\mathbf{P}^*_{y,w} = \sum_{(i,j)\in I} n_{ij} x^i y^j$ where $I \subseteq \mathbb{Z} \times \mathbb{Z}$ is a finite subset and $n_{ij} \in \mathbb{Z}$. Thus, using Lemma 3.2, we have

$$v_w v_y^{-1}\mathbf{P}^*_{y,w} = \textstyle\sum_{(i,j)\in I} n_{ij} x^{a+i} y^{b+j} \in \mathbb{Z}\big[x^2, y^2\big],$$
$$v_w v_y^{-1}\overline{\mathbf{P}}^*_{y,w} = \textstyle\sum_{(i,j)\in I} n_{ij} x^{a-i} y^{b-j} \in \mathbb{Z}\big[x^2, y^2\big].$$

Now let $(i, j) \in I$. We certainly have $0 \leqslant a, b < l(w_0)$. This yields $0 \leqslant a + i < l(w_0) + i$ and $0 \leqslant a - i < l(w_0) - i$. Consequently, we have $-l(w_0) < i < l(w_0)$. A similar argument shows that we also have $-l(w_0) < j < l(w_0)$.

Now assume that $sy < y < w < sw$ and write $\mathbf{M}^s_{y,w} = f + c + \overline{f}$ where $c \in \mathbb{Z}$ and $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. Let $f = \sum_{(i,j)\in J} f_{ij} x^i y^j$ where $J \subseteq \mathbb{Z} \times \mathbb{Z}$ is a finite subset and $f_{ij} \in \mathbb{Z}$. As above, we see that $v_s v_w v_y^{-1} = x^a y^b$ where $0 \leqslant a, b < l(w_0)$. (Note that $y < w < w_0$.) Using Lemma 3.4, this yields

$$\begin{aligned} v_s v_w v_y^{-1}\mathbf{M}^s_{y,w} &= x^a y^b f + c x^a y^b + x^a y^b \overline{f} \\ &= c x^a y^b + \sum_{(i,j)\in J} f_{ij}\big(x^{a+i} y^{b+j} + x^{a-i} y^{b-j}\big) \in \mathbb{Z}\big[x^2, y^2\big]. \end{aligned}$$

Arguing as above, we see that $-l(w_0) < i, j < l(w_0)$ for all $(i, j) \in J$. $\square$

Now, a weight function $L : W \to \mathbb{Z}$ is uniquely specified by the values

$$a := L(s) > 0 \quad (\text{where } s \in S_x) \qquad \text{and} \qquad b := L(t) > 0 \quad (\text{where } t \in S_y).$$

We shall write $L = L_{a,b}$. Let us consider the set

$$\mathcal{E} := \bigl\{x \in \mathbb{Q}_{>0} \mid x = \pm i/j \text{ where } i, j \neq 0 \text{ and } -2l(w_0) < i, j < 2l(w_0)\bigr\}$$

and write $\mathcal{E} = \{x_1, \ldots, x_n\}$ where $0 < x_1 < x_2 < \cdots < x_n$. By convention, we set $x_0 = 0$ and $x_{n+1} = \infty$. For any $0 \leqslant k \leqslant n$, we consider the set of weight functions

$$\mathcal{L}_k := \{L_{a,b} \mid a, b > 0 \text{ such that } x_k < b/a < x_{k+1}\}.$$

Let us fix $0 \leqslant k \leqslant n$ and write $x_k = d/c$ where c, d are integers such that $0 \leqslant c, d < 2l(w_0)$ and $c \neq 0$. Then we consider the total order in Γ specified by

$$\Gamma_+^{(k)} = \bigl\{x^i y^j \bigm| ci + dj > 0\bigr\} \cup \bigl\{x^i y^j \bigm| ci + dj = 0 \text{ and } i > 0\bigr\} \quad \text{if } d \geqslant c, \quad \text{or}$$
$$\Gamma_+^{(k)} = \bigl\{x^i y^j \bigm| ci + dj > 0\bigr\} \cup \bigl\{x^i y^j \bigm| ci + dj = 0 \text{ and } j > 0\bigr\} \quad \text{if } d < c$$

(a weighted lexicographic order). Note that, if $k = d = 0$, then

$$\Gamma_+^{(0)} = \bigl\{x^i y^j \bigm| i > 0, j \in \mathbb{Z}\bigr\} \cup \bigl\{y^j \bigm| j > 0\bigr\}$$

(a pure lexicographic order).

Proposition 3.5. *In the above setting, all the weight functions in $\mathcal{L}_k$ are $\Gamma_+^{(k)}$-equivalent.*

Proof. Let $a, b > 0$ be such that $x_k < b/a < x_{k+1}$. The idea is to get some control on the set $\Gamma_+(W) \subseteq \Gamma_+$ and to show that condition $(*)$ in Proposition 2.10 is satisfied for the ring homomorphism

$$\sigma_{a,b} : \mathbb{Z}[\Gamma] \to \mathbb{Z}\bigl[v, v^{-1}\bigr], \quad x^i y^j \mapsto v^{ai+bj}$$

and the total order $\Gamma_+^{(k)} \subset \Gamma$ specified above. Now, by Lemma 3.4, we have

$$\Gamma_+(W) \subseteq \bigl\{x^i y^j \bigm| x^i y^j \in \Gamma_+ \text{ and } -2l(w_0) < i, j < 2l(w_0)\bigr\}.$$

To check condition $(*)$, assume first that $c < d$. Let $x^i y^j \in \Gamma_+^{(k)}(W)$. In particular, this means that $ci + dj \geqslant 0$. Furthermore, we have $-2l(w_0) < i, j < 2l(w_0)$ and so $\pm i/j \in \mathcal{E}$. Now, we must show that $ai + bj > 0$. If $i = 0$ or $j = 0$, this is clear. If $j > 0$, then we have

$$ai + bj = a(i + jb/a) > a(i + x_k j) = a(i + jd/c) = (a/c)(ci + dj) \geqslant 0,$$

as required. Next assume that $j < 0$. Then, by the definition of $\Gamma_+^{(k)}$ (recall that we are assuming $c < d$), we must have $ci + dj > 0$ and so $-i/j > d/c = x_k$. Now, if we had $ai + bj \leqslant 0$, then we would obtain

$$x_k < -i/j \leqslant b/a < x_{k+1}$$

and so $-i/j \notin \mathcal{E}$, a contradiction. Thus, condition $(*)$ holds. The argument for the case where $d \leqslant c$ is completely analogous. □

Corollary 3.6. *Let $\mathcal{E} = \{x_1, \ldots, x_n\}$ as above. Let $L = L_{a,b}$ be any weight function on W where $a, b > 0$.*

(1) *If $b/a = x_k$ for some $1 \leqslant k \leqslant n$, then $L_{a,b}$ is generically equivalent to $L_{c,d}$ where $0 < c, d < 2l(w_0)$ are such that $b/a = d/c$.*
(2) *If $b/a \notin \mathcal{E}$, then there exist integers $1 \leqslant a', b' \leqslant 8l(w_0)^3$ such that $L_{a,b}$ is generically equivalent to $L_{a',b'}$.*

Proof. Recall that $x_0 = 0$ and $x_{n+1} = \infty$. Hence there exists some $k \in \{0, 1, \ldots, n\}$ such that $x_k \leqslant b/a < x_{k+1}$. We write $x_k = d/c$ where $0 \leqslant c, d < 2l(w_0)$ and $c \neq 0$. If $x_k = b/a$, then $L_{a,b}, L_{c,d}$ are equivalent by Remark 2.15. Thus, (1) is proved. Now assume that $x_k < b/a < x_{k+1}$. Since both x_k and x_{k+1} are rational numbers where the numerator and the denominator are strictly bounded by $2l(w_0)$, we certainly have $1/4l(w_0)^2 < x_{k+1} - x_k$. Furthermore, note that $x_n < 2l(w_0)$. Thus, we can find some integers a', b' such that $1 \leqslant a', b' \leqslant 8l(w_0)^3$ and $x_k < b'/a' < x_{k+1}$. Then $L_{a,b}$ and $L_{a',b'}$ are equivalent by Proposition 3.5. Thus, (2) is proved. □

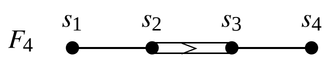
Example 3.7. Assume that $a, b > 0$ are such that $a/b \geqslant 2l(w_0)$. Then $L_{a,b}$ is generically equivalent to the weight function $L_{2l(w_0),1}$.

To see this, note that $1/2l(w_0) < x_1$. Hence, we are in the case where $b/a \leqslant 1/2l(w_0) < x_1$. Thus, we have $L_{a,b} \in \mathcal{L}_0$. By Proposition 3.5, all weight functions in $\mathcal{L}_0$ are generically equivalent. It remains to note that $L_{2l(w_0),1}$ also belongs to $\mathcal{L}_0$.

This example provides a more formal justification for [2, Remark 6.1].

4. Kazhdan–Lusztig polynomials and left cells in type F_4

Our aim is to work out the cell-equivalence classes of weight functions on a Coxeter group of type F_4. Throughout this section, let W be a Coxeter group of type F_4, with generating set $S = \{s_1, s_2, s_3, s_4\}$ and Dynkin diagram given as follows:



There are 25 irreducible representations of W, denoted by

$$1_1, 1_2, 1_3, 1_4, 2_1, 2_2, 2_3, 2_4, 4_1, 4_2, 4_3, 4_4, 4_5, 6_1, 6_2, 8_1, 8_2, 8_3, 8_4, 9_1, 9_2, 9_3, 9_4, 12_1, 16_1;$$

see [11, 4.10] or [7, 5.3.6 and Table C.3]. The generators s_1, s_2 are conjugate in W, and so are the generators s_3, s_4 (while s_2 and s_3 are not conjugate). Thus, a weight function $L: W \to \mathbb{Z}$ is uniquely determined by

$$L(s_1) = L(s_2) = a > 0 \quad \text{and} \quad L(s_3) = L(s_4) = b > 0.$$

We shall denote such a weight function by $L = L_{a,b}$. By the symmetry of the above diagram, we may assume throughout that $a \leqslant b$.

Let x, y be independent indeterminates over $\mathbb{Z}$ and consider the abelian group

$$\Gamma = \{x^i y^j \mid i, j \in \mathbb{Z}\}.$$

Let v be another indeterminate. Then we have a ring homomorphism

$$\sigma_{a,b} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[v, v^{-1}], \quad x^i y^j \mapsto v^{ai+bj}.$$

Now, in type F_4, we have $l(w_0) = 24$ and so, by Corollary 3.6, we know that $L_{a,b}$ is generically equivalent to a weight function $L_{c,d}$ where $1 \leqslant c \leqslant d \leqslant 48^3 = 110592$. In principle, we could just go through all these possibilities, determine the corresponding left cell representations and so on—but these are far too many cases! However, now we can use our CHEVIE-program to compute explicitly all the polynomials $\mathbf{P}^*_{y,w}$ and $\mathbf{M}^s_{y,w}$ for any total order on Γ. The explicit knowledge of these polynomials will yield much sharper bounds than the general bounds obtained in Lemma 3.4.

As a first illustration of this idea, we consider the following case.

Lemma 4.1. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid j > 0,\ i \in \mathbb{Z}\} \cup \{x^i \mid i > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that $b/a > 4$. In particular, all these weight functions are Γ_+-equivalent.*

Proof. The idea is basically the same as in the proof of Proposition 3.5. In fact, the general strategy in Corollary 3.6 shows that all $L_{a,b}$ are Γ_+-equivalent, provided that $b/a > 2l(w_0) = 48$. But now we use our CHEVIE-program to compute explicitly all the polynomials $\mathbf{P}^*_{y,w}$ and $\mathbf{M}^*_{y,w}$ (with respect to $\Gamma_+ \subset \Gamma$). By inspection of all these polynomials, we find that

$$\Gamma_+(W) \subseteq \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + 4j \geqslant 0\}.$$

Now let us check that condition $(*)$ in Proposition 2.10 holds for $\sigma_{a,b}$ provided that $b > 4a$. Let $i, j \in \mathbb{Z}$ be such that $x^i y^j \in \Gamma_+(W)$. We must show that $ai + bj > 0$. If $j = 0$, then

$i > 0$ and so $ai + bj = ai > 0$. On the other hand, if $j > 0$ and $i + 4j \geqslant 0$, then $ai + bj = a(i + jb/a) > a(i + 4j) \geqslant 0$, as required.

We can now apply Proposition 2.10 and conclude that all weight functions $L_{a,b}$ such that $b/a > 4$ are Γ_+-equivalent. $\square$

In order to deal with weight functions $L_{a,b}$ such that $b/a < 4$, we now proceed as follows. We look again at the elements in $\Gamma_+(W)$ computed in the proof of Lemma 4.1. Let

$$\mathcal{E} = \{x \in \mathbb{Q}_{>0} \mid x = \pm i/j \text{ where } j \neq 0,\ x^i y^j \in \Gamma_+(W)\}.$$

Then we note that the largest element of $\mathcal{E}$ below 4 is 3. This leads us to consider weight functions $L_{a',b'}$ where $b'/a' > 3$.

Lemma 4.2. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid i + 3j > 0\} \cup \{x^{-3j} y^j \mid j > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $4 > b/a > 3$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. This is completely analogous to that of Lemma 4.1. Now we find that

$$\begin{aligned}\Gamma_+(W) \subseteq {} & \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\} \\ & \cup \{x^i y^j \mid i > -j > 0,\ -i/j \geqslant 4\} \cup \{x^i y^j \mid -i > j > 0,\ -i/j \leqslant 3\}.\end{aligned}$$

As before, we see that condition $(*)$ in Proposition 2.10 holds, provided that $4a > b > 3a$. Indeed, let i, j be such that $x^i y^j \in \Gamma_+(W)$. If $j = 0$, then $i > 0$ and so $ai + bj = ai > 0$. If $j > 0$ and $i + j \geqslant 0$, then $ai + bj > ai + 3aj > a(i + j) \geqslant 0$. If $i > -j > 0$ and $-i/j \geqslant 4$, then $ai + bj = i(a + bj/i) > ia(1 + 4j/i) \geqslant 0$. Finally, if $-i > j > 0$ and $-i/j > 3$, then $ai + bj = j(ai/j + b) > aj(i/j + 3) \geqslant 0$, as required. $\square$

As before, we now look again at the elements in $\Gamma_+(W)$ computed in the proof of Lemma 4.2. Define $\mathcal{E}$ in a similar way as above. Then we note that the largest element of $\mathcal{E}$ below 3 is $5/2$. This leads us to the following case.

Lemma 4.3. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid 2i + 5j > 0\} \cup \{x^{-5j} y^{2j} \mid j > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $3 > b/a > 5/2$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. Again, this is completely analogous to that of Lemma 4.1. Now we find that

$$\begin{aligned}\Gamma_+(W) \subseteq {} & \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\} \\ & \cup \{x^i y^j \mid i > 0,\ i + 3j \geqslant 0\} \cup \{x^i y^j \mid -i > j > 0,\ -i/j \leqslant 5/2\}.\end{aligned}$$

We omit further details. □

We now continue the above procedure. This yields the following cases.

Lemma 4.4. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid i + 2j > 0\} \cup \{x^{-2j} y^j \mid j > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $5/2 > b/a > 2$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. Again, this is completely analogous to that of Lemma 4.1. Now we find that

$$\begin{aligned}\Gamma_+(W) \subseteq {} & \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\} \\ & \cup \{x^i y^j \mid i > -j > 0,\ -i/j \geqslant 5/2\} \cup \{x^i y^j \mid -i > j > 0,\ -i/j \leqslant 2\}.\end{aligned}$$

We omit further details. □

Lemma 4.5. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid 2i + 3j > 0\} \cup \{x^{-3j} y^{2j} \mid j > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $2 > b/a > 3/2$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. Again, this is completely analogous to that of Lemma 4.1. Now we find that

$$\begin{aligned}\Gamma_+(W) \subseteq {} & \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\} \\ & \cup \{x^i y^j \mid i > -j > 0,\ -i/j \geqslant 2\} \cup \{x^i y^j \mid -i > j > 0,\ -i/j \leqslant 3/2\}.\end{aligned}$$

We omit further details. □

Lemma 4.6. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid 3i + 4j > 0\} \cup \{x^{-4j} y^{3j} \mid j > 0\}.$$

Then condition $(*)$ *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $3/2 > b/a > 4/3$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. Again, this is completely analogous to that of Lemma 4.1. Now we find that

$$\begin{aligned}\Gamma_+(W) \subseteq \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\}\\ \cup \{x^i y^j \mid i > -j > 0,\ -i/j \geqslant 3/2\} \cup \{x^{-4j} y^{3j} \mid j > 0\}.\end{aligned}$$

We omit further details. □

Lemma 4.7. *Consider the total order on Γ specified by*

$$\Gamma_+ = \{x^i y^j \mid i + j > 0\} \cup \{x^{-j} y^j \mid j > 0\}.$$

Then condition (∗) *in Proposition* 2.10 *is satisfied for all weight functions $L_{a,b}$ such that* $4/3 > b/a > 1$. *In particular, all these weight functions are Γ_+-equivalent.*

Proof. Again, this is completely analogous to that of Lemma 4.1. Now we find that

$$\Gamma_+(W) \subseteq \{x^i \mid i > 0\} \cup \{x^i y^j \mid j > 0,\ i + j \geqslant 0\} \cup \{x^i y^j \mid i > 0,\ 3i + 4j \geqslant 0\}.$$

We omit further details. □

Thus, we have finally covered all cases of unequal parameters. A detailed analysis of the partition of left cells obtained in each case leads us to the following result.

Corollary 4.8. *Let $L = L_{a,b}$ and $L' = L_{a',b'}$ be two weight functions on W such that $b \geqslant a > 0$ and $b' \geqslant a' > 0$. Then L, L' are cell-equivalent if and only if $L, L' \in \mathcal{L}_i$ for $i \in \{0, 1, 2, 3\}$, where $\mathcal{L}_i$ are defined as follows*:

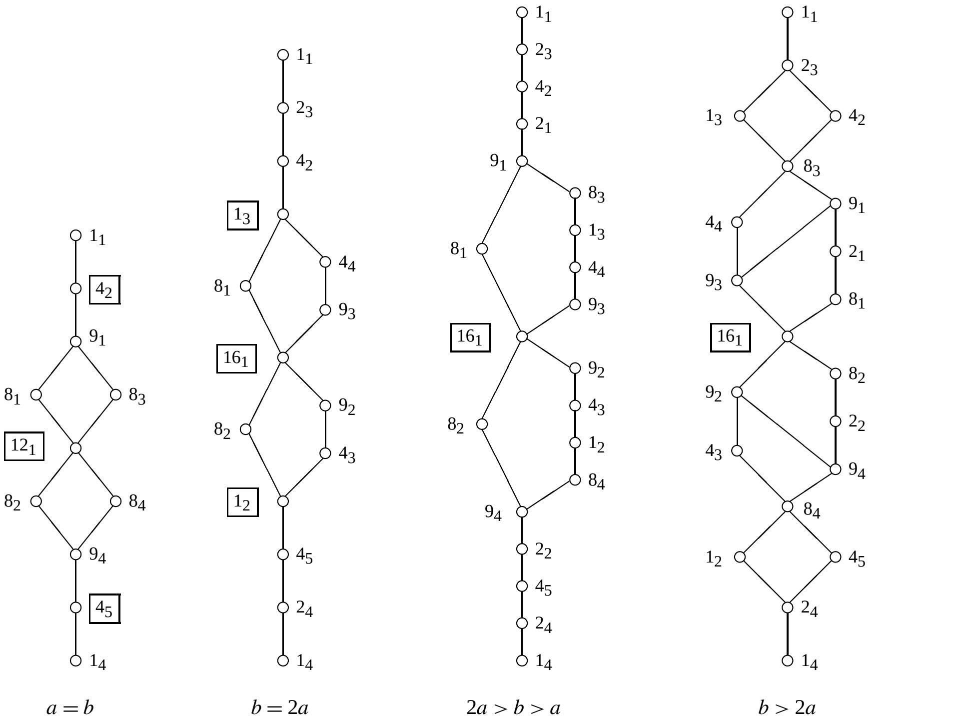
$$\begin{aligned}\mathcal{L}_0 &= \{(c, c, c, c) \mid c > 0\},\\ \mathcal{L}_1 &= \{(c, c, 2c, 2c) \mid c > 0\},\\ \mathcal{L}_2 &= \{(c, c, d, d) \mid 2c > d > c > 0\},\\ \mathcal{L}_3 &= \{(c, c, d, d) \mid d > 2c > 0\}.\end{aligned}$$

In all cases, the left cell representations are precisely the constructible representations, as defined in [15, Chapter 22]; *in particular, if two weight functions define the same partition of W into left cells, then they also give rise to the same set of left cell representations. The partial order relation $\leqslant_{LR}$ on two-sided cells and the left cell representations are given in Tables* 1 *and* 2. *Furthermore, the statements in Theorems* 1.2 *and* 1.3 *hold for any weight function L.*

Note that the list of constructible representations given in [15, §22.27, Case 1], has to be corrected as specified in Table 2; see Remark 4.10 below.

Proof. Let $L = L_{c,d}$ be any weight function on W where $d \geqslant c > 0$. In addition to the results obtained in Lemmas 4.1–4.7, we use our CHEVIE program to compute all the required data in the cases where

Table 1
Partial order on two-sided cells in type F_4



A box indicates a two-sided cell with several constructible representations, see Table 2. Otherwise, the two-sided cell has only one irreducible, constructible res presentation.

Table 2
Left cell representations in type F_4

$a=b$		$b=2a$		$b \notin \{a, 2a\}$	
$\boxed{4_2}$:	2_3+4_2,	$\boxed{1_3}$:	1_3+8_3,	$\boxed{16_1}$:	$6_1+12_1+16_1$,
	2_1+4_2		2_1+9_1,		$6_2+12_1+16_1$,
$\boxed{12_1}$:	$9_3+6_1+12_1+4_4+16_1$,		9_1+8_3		4_1+16_1
	$9_2+6_1+12_1+4_3+16_1$,	$\boxed{16_1}$:	$6_1+12_1+16_1$		
	$4_1+9_2+9_3+6_2+12_1+2 \cdot 16_1$,		$6_2+12_1+16_1$		
	$1_3+2 \cdot 9_3+6_2+12_1+4_4+16_1$,		4_1+16_1		
	$1_2+2 \cdot 9_2+6_2+12_1+4_3+16_1$,	$\boxed{1_2}$:	1_2+8_4,		
$\boxed{4_4}$:	2_4+4_5,		2_2+9_4,		
	2_2+4_5		9_4+8_4		

(1) $\{c,d\} \in \{(1,4),(1,3),(2,5),(1,2),(2,3),(3,4),(1,1)\}$.

Then, by Remark 2.15, we have covered all generic equivalence classes of weight functions on W. In each of the above cases, our CHEVIE program has automatically computed the preorder relations $\leqslant_L$ and $\leqslant_{LR}$ and checked that Theorem 1.3 holds. Furthermore, by inspection of the partitions into left cells obtained in the various cases, we find the above four cell-equivalence classes of weight functions $\mathcal{L}_i$ ($0 \leqslant i \leqslant 3$). The decompositions of the left cell representations are determined by explicit computations using the character table of W. By inspection, we see that the left cell representations are precisely the constructible representations as determined by Lusztig [15, §22.27] (modulo the error in Case 1 in Lusztig's list).

It remains to prove the statements in Theorem 1.2, concerning the distinguished involutions. For this purpose, we use a similar procedure as before, beginning with a total order $\Gamma_+ \subset \Gamma$ as specified in Lemma 4.1. But now we have to work with the larger set $\Gamma'_+(W)$ defined in Remark 2.11 in each step and make sure that $(*')$ holds. For example, the analogue of Lemma 4.1 now reads:

Let $\Gamma_+ \subset \Gamma$ be a pure lexicographic order as in Lemma 4.1. *Then condition* $(*')$ *in Remark* 2.11 *holds provided that $b/a > 9$.*

Then we continue with an analogue of Lemma 4.2 and so on. Thus, there will be more cases to be considered, but the whole argument is basically the same. We omit the details. Once this is done, one can argue as follows. Let C be a left cell of W (with respect to a total order $\Gamma_+ \subset \Gamma$ similar to one of the cases in Lemmas 4.1–4.7). By inspection, one checks that the following holds:

There exists a (*unique*) *$d_0 \in C$ such that $\delta_{d_0}^{-1}\delta_w \in \Gamma_+$ for every $w \in C \setminus \{d_0\}$.*

(Here, δ_w is defined as in Remark 2.11.) Thus, we may regard d_0 as a *distinguished involution* in C. Now, the fact that condition $(*')$ in Remark 2.11 holds in these cases shows that the function $w \mapsto \Delta(w)$ restricted to C also reaches its minimum at $d_0 \in C$ and that $\Delta(w) > \Delta(d_0)$ for all $w \in C \setminus \{d_0\}$. □

Remark 4.9. Let L, L' be two weight functions such that $L(w) > 0$ and $L'(w) > 0$ for all $1 \neq w \in W$. Assume that L, L' give rise to the same partition of W into left cells. By inspection of the results obtained in Corollary 4.8 and its proof, we find the following:

(a) Let $\mathcal{D}$ be the set of distinguished involutions with respect to L and $\mathcal{D}'$ the analogous set with respect to L' (see Theorem 1.2). Then, quite remarkably, we have $\mathcal{D} = \mathcal{D}'$. In fact, we even have that $\mathcal{D} = \mathcal{D}'$ if we just assume that L and L' give rise to the same set of left cell representations. (For example, $L_{2,3}$ and $L_{1,3}$ define the same set of left cell representations, but the partitions into left cells are different.)

(b) As already implicitly stated in Corollary 4.8, the pre-order relation $\leqslant_{LR}$ defined with respect to L is the same as that defined with respect to L'. (However, this is not necessarily the case for the left pre-order relation $\leqslant_L$; for example, the weight functions $L_{1,3}$ and $L_{1,4}$ give rise to different pre-order relations $\leqslant_L$.)

Remark 4.10. Consider the case where $b = 2a > 0$. Lusztig states in [15, 22.27, Case 1] that $1_3 \oplus 2_1$ and $1_2 \oplus 2_2$ are constructible. However, these representations are not constructible. (In fact, we just have to omit them from the list given by Lusztig.) Let us add some details about this. The **a**-invariants of the irreducible representations of W are given by

$\mathbf{a}$	0	a	$2a$	$3a$	$5a$	$6a$	$7a$	$10a$	$11a$	$12a$	$15a$	$20a$	$25a$	$36a$
ρ	1_1	2_3	4_2	1_3	4_4	9_3	4_1	9_2	4_3	8_2	1_2	4_5	2_4	1_4
				2_1		8_1	6_1				2_2			
				9_1			6_2				9_4			
				8_3			12_1				8_4			
							16_1							

For $i \in \{1, 2, 3, 4\}$, let W_i be the parabolic subgroup of W generated by $S \setminus \{s_i\}$. The maximal **a**-invariant of a representation of W_i (for $i = 1, 2, 3, 4$) is given by $15a$, $7a$, $6a$, or $12a$, respectively. Furthermore, that maximal value is reached only at the sign representation. Thus, since the restriction of 1_2 to W_i is not the sign representation, we conclude that 1_2 cannot occur in the J-induction of any representation of any W_i. Hence 1_3 (obtained from 1_2 by tensoring with sign) must occur in the J-induction from some proper parabolic subgroup. Now, the restriction of 1_3 to W_1 (type C_3) is given by $(\emptyset, 3)$. Furthermore, this representation is constructible. The restriction of 1_3 to W_2 (type $A_1 \times A_2$) is given by $(11) \boxtimes (3)$. Furthermore, this representation is constructible. The restriction of 1_3 to W_3 (type $A_2 \times A_1$) is given by $(111) \boxtimes (2)$. Furthermore, this representation is constructible. The restriction of 1_3 to W_4 (type B_3) is given by $(111, \emptyset)$. Furthermore, the representation $(111, \emptyset) + (11, 1)$ is constructible, and this is the only constructible representation in which $(111, \emptyset)$ occurs; see [15, Chapter 22]. We have

$$\mathrm{J}_{W_1}^{W}\big((\emptyset, 3)\big) = 2_3, \qquad \mathrm{J}_{W_3}^{W}\big((111) \boxtimes (2)\big) = 1_3 \oplus 8_3,$$
$$\mathrm{J}_{W_2}^{W}\big((11) \boxtimes (3)\big) = 2_3, \qquad \mathrm{J}_{W_4}^{W}\big((111, \emptyset) + (11, 1)\big) = 1_3 \oplus 8_3.$$

Thus, $1_3 \oplus 8_3$ is the only constructible representation of W in which 1_3 occurs.

Remark 4.11. The case $b = 2a$ in type F_4 also shows that, in general, there no longer exist representations which would have similar properties as the "special" representations in the equal parameter case (see [9, §12]). Indeed, consider the two-sided cell containing 1_3. Then the three constructible representations belonging to that two-sided cell do not have an irreducible constituent in common.

References

[1] D. Alvis, The left cells of the Coxeter group of type H_4, J. Algebra 107 (1987) 160–168.

[2] C. Bonnafé, L. Iancu, Left cells in type B_n with unequal parameters, Represent. Theory 7 (2003) 587–609.

[3] K. Bremke, Hecke—Algebren mit ungleichen Parametern, Diploma thesis, Lehrstuhl D für Mathematik, RTWH Aachen, 1992.

[4] F. Ducloux, Computing Kazhdan–Lusztig polynomials for arbitrary Coxeter groups, Experiment. Math. 11 (2002) 387–397.
[5] M. Geck, Constructible characters, leading coefficients and left cells for finite Coxeter groups with unequal parameters, Represent. Theory 6 (2002) 1–30 (electronic).
[6] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer, CHEVIE—A system for computing and processing generic character tables, Appl. Algebra Engrg. Comm. Comput. 7 (1996) 175–210; available at http://www.math.rwth-aachen.de/~CHEVIE.
[7] M. Geck, G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras, London Math. Soc. Monogr. (N.S.), vol. 21, Oxford Univ. Press, New York, 2000.
[8] G. Lusztig, Representations of Finite Chevalley Groups, CBMS Regional Conf. Ser. in Math., vol. 39, Amer. Math. Soc., Providence, RI, 1977.
[9] G. Lusztig, A class of irreducible representations of a finite Weyl group II, Indag. Math. 44 (1982) 219–226.
[10] G. Lusztig, Left cells in Weyl groups, in: R.L.R. Herb, J. Rosenberg (Eds.), Lie Group Representations, I, in: Lecture Notes in Math., vol. 1024, Springer-Verlag, Berlin, 1983, pp. 99–111.
[11] G. Lusztig, Characters of reductive groups over a finite field, Ann. of Math. Stud., vol. 107, Princeton Univ. Press, Princeton, NJ, 1984.
[12] G. Lusztig, Cells in affine Weyl groups, in: Algebraic Groups and Related Topics, in: Adv. Stud. Pure Math., vol. 6, Kinokuniya and North–Holland, 1985, pp. 255–287.
[13] G. Lusztig, Sur les cellules gauches des groupes de Weyl, C. R. Acad. Sci. Paris 302 (1986) 5–8.
[14] G. Lusztig, Cells in affine Weyl groups II, J. Algebra 109 (1987) 536–548.
[15] G. Lusztig, Hecke Algebras with Unequal Parameters, CRM Monog. Ser., vol. 18, Amer. Math. Soc., Providence, RI, 2003.