Stokes' Formula for Lie Algebra Valued Connection and Curvature Forms

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Notations

1. \( \Phi: I^2 \to S \subset M \), an immersion of the square \( I^2 \subset \mathbb{R}^2 \) in a manifold \( M \).
2. \( A \) an \( L \)-valued connection 1-form, \( F = dA + A \wedge A \), the \( L \)-valued curvature 2-form.
3. \( P_g \) parallel transport along \( g = \{ g(s), a \leq s \leq b \} \). \( P(\sigma) \) parallel transport around a loop \( \sigma \).
4. \( A^*(t) = A(\sigma(t))(\dot{\sigma}(t)), \sigma = \{ \sigma(s), a \leq s \leq b \} \) a path.
5. \( \tilde{F}(x) = P_g^{-1}F(x)P_g \), \( g \) the right-angle path from a fixed base point to \( x \) (see Fig. 3).
6. \( \tilde{F}^*(u, v) = \tilde{F}(\Phi(u, v))(x, \beta), \alpha = \partial\Phi/\partial u, \beta = \partial\Phi/\partial v \).
7. \( L \int A^* = H(A, \sigma) \).
8. \( L \int_S \tilde{F}^* = H(\tilde{F}, S) \).
1. Introduction

Let $M$ be a smooth manifold, $I^2$ the closed unit square in the $u$-$v$ plane, $I^2 \subset U$, $U$ an open set, and $\Phi(u, v): U \rightarrow M$ a smooth function. The image of $I^2$ under such a function $\Phi$ is said to be a surface $S$ and its boundary $\partial S$ is defined to be the image under $\Phi$ of the boundary of $I^2$. Stokes' theorem for surfaces says that

$$\iint_S dA = \int_{\partial S} A$$

(1)

for any smooth 1-form defined in an open region containing $S$, where $dA$ is the exterior derivative of $A$. The orientation of $S$ is that of the parameters $(u, v)$ and that of the boundary the induced orientation so that (1) holds without minus signs.

It is the purpose of this paper to show that a version of (1) holds in the case that $A$ is a Lie algebra valued 1-form. Such forms are called connection forms, although they need not arise as the connection form of a Riemann manifold. The exterior derivative of a Lie algebra valued 1-form $A$ is denoted $\hat{d}A$, where

$$\hat{d}A = dA + A \wedge A$$

(2)

so that $\hat{d}A = dA$ if $A$ is real valued. The Lie algebra valued 2-form $\hat{d}A$ is called the curvature form of the connection $A$. The version of (1) we prove is that

$$L \iint_{S_{\alpha \beta}} P^{-1}(\hat{d}A)P = L \int_{\partial S_{\alpha \beta}}, A,$$

(3)

where $P^{-1}(\hat{d}A)P$ is an example of a twisted (Lie algebra valued) 2-form, and where $L \int$ is a new kind of integral. The point denoted by $(x)$ is a point on the boundary $\partial S$, the base point, which for convenience has been chosen to be $\Phi(0, 0)$ in the rest of the paper. This integral was defined in [1] where an explicit formula for its calculation was given for the case that the integral is over an interval. It is easy to extend the notion of $L$ integral to 1-forms along curves. In this paper we consider this extension, as well as the more difficult extension to twisted 2-forms on surfaces. After proving (3), applications are given which yield various extensions of the Gauss–Bonnet theorem.

Two papers should be mentioned in connection with our results. They are the papers of L. Gross [6] and of L. Schlesinger [7]. Gross [6], while treating certain problems in quantum field theory, obtained a result which may be shown to be equivalent to Schlesinger's theorem [7], which is
discussed later. We note that our results are formulated in a Lie algebra context where their results are formulated in the context of the corresponding Lie group. The algebra approach appears to us to be more natural. Stokes' theorem, for example, is a result which can most naturally be stated in the algebra.

For simplicity we have assumed that the algebra is a finite dimensional matrix algebra, an assumption also made in [1, 6, 7]. It is our intention to consider these problems in a more general context at a later date. Note that Gross uses the term lasso form instead of twisted form. The term twisted form seems to us to more accurately describe the nature of the forms in question.

2. Principal Definitions

Let $G \subset GL(m, \mathbb{R})$ be a finite dimensional matrix Lie group, $L$ its Lie algebra, and $\exp: L \rightarrow G$ the standard exponential map. This map is a diffeomorphism of a neighbourhood of $0 \in L$ onto a neighbourhood of $I \in G$. We assume that $L$ has a norm $|\cdot|$, and that $M$ is the positive number, which can be shown to exist, with the property that $|[X, Y]| \leq M |X| |Y|$. The symbol $[X, Y]$ is called the commutator of $X$ and $Y$ and is defined by $[X, Y] = XY - YX$. The linear operator $\text{ad}_X(Y)$ is defined by

$$\text{ad}_X(Y) = [X, Y].$$

Let $M = M^m$ be an $m$-dimensional smooth manifold where $m < \infty$. We consider, for simplicity, the tangent vector bundle $T_x M$ with the standard projection mapping $\pi: T_x M \rightarrow M$, i.e.,

$$\pi(x, \xi) = x,$$

where $x \in M$, $\xi \in T_x M$, and $T_x M$ is the tangent space of $M$ at $x$ (Fig. 1). The results hold more generally for arbitrary vector bundles with essentially unchanged proofs. See, e.g., [3] for the usual definitions.

Consider in $M^m$ a smooth 1-form with values in the Lie algebra $L$. If $\{x^i, i = 1, \ldots, m\}$ are regular coordinates in a neighbourhood of a point $x \in M$, then the 1-form may be represented as

$$A = \sum A_z(x) dx^z,$$

where the coefficients $A_z \in L$ are real (or complex) $m \times m$ matrices. As our theorems are local theorems we shall not have occasion to consider coor-
dinate transformations. The behaviour of Lie algebra valued forms under coordinate transformations is discussed, e.g., in [3].

In the remaining part of the paper we fix a coordinate system \( \{ x^i, i = 1, \ldots, m \} \) and the closure \( U \) of a ball in its coordinate neighbourhood, and represent the group \( G \) and the algebra \( L \) as matrices with respect to the standard basis \( \{ \partial / \partial x^i, i = 1, \ldots, m \} \).

**Definition 1.** The \( L \)-exterior derivative \( \bar{d}A \) is defined as

\[
\bar{d}A = dA + A \wedge A
\]

so that if \( \bar{d}A = \sum_{x^i, x^j} F_{x^i x^j} dx^i \wedge dx^j \), then

\[
-F_{x^i x^j} = \frac{\partial A_{x^i}}{\partial x^j} - \frac{\partial A_{x^j}}{\partial x^i} + [A_{x^i}, A_{x^j}]
\]
where \([A_{\alpha}, A_{\beta}]\) is the commutator of \(A_{\alpha}\) and \(A_{\beta}\) in \(L\), as is shown in [3, 4]. The \(L\)-exterior derivative of the negative of \(A\) is also called the curvature form of the connection \(A\). To simplify notation we sometimes denote it by \(F\), so that \(\hat{d}(-A) = F\).

**Definition 2.** The covariant derivative is defined as usual, so that if \(\xi\) is a covariant vector field, then its covariant derivative (relative to \(A\)) is

\[
\nabla_{\alpha}\xi = \frac{\partial \xi}{\partial x^{\alpha}} + A_{\alpha}\xi, \tag{7}
\]

and if \(B\) is a Lie algebra valued field, then

\[
\nabla_{\alpha}B = \frac{\partial B}{\partial x^{\alpha}} + [A_{\alpha}, B]. \tag{7'}
\]

**Definition 3.** If \(C = \sum_{\alpha < \beta} C_{\alpha\beta}(x) \, dx_{\alpha} \wedge dx_{\beta}\) is a differential two-form with values in \(L\), then we define

\[
\hat{d}C = \sum_{\alpha < \beta < \gamma} \left( \nabla_{\alpha}C_{\beta\gamma} + \nabla_{\beta}C_{\alpha\gamma} + \nabla_{\gamma}C_{\alpha\beta} \right) dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}, \tag{8}
\]

and call \(\hat{d}C\) the covariant differential of the two-form \(C\).

**Remark.** It is well-known that \(\hat{d}^{2}A = 0\). This identity is called the Bianchi identity.

Consider a vector field \(\xi = (\xi^{1}, ..., \xi^{m})\) in \(M^{m}\). The value \(A(x)(\xi)\) is computed in the following way:

\[
A(x)(\xi) = A(\xi) = \left( \sum_{\alpha = 1}^{m} A_{\alpha} \, dx^{\alpha} \right)(\xi)
= \sum_{\alpha = 1}^{m} A_{\alpha} \, dx^{\alpha}(\xi) = \sum_{\alpha = 1}^{m} A_{\alpha}(x) \, \xi^{\alpha}(x), \tag{9}
\]

and \(A(x)(\xi) \in L\). Consider next two vector fields \(\xi\) and \(\eta\) on \(M^{m}\). The value of the curvature form \(\hat{d}(-A) = F\) is computed as

\[
F(x)(\xi, \eta) = \left( \sum_{\alpha < \beta} F_{\alpha\beta} \, dx^{\alpha} \wedge dx^{\beta} \right)(\xi, \eta)
= \sum_{\alpha < \beta} F_{\alpha\beta}(\xi^{\alpha}\eta^{\beta} - \xi^{\beta}\eta^{\alpha}), \tag{10}
\]
and \( F(x)(xi, \eta) \in L \). The components \( \{ \xi^{a} \eta^{b} - \xi^{b} \eta^{a} \} \) are the components of the (real valued) 2-vector \( \xi \wedge \eta \).

Let \( \{ q(t), a \leq t \leq b \} \) be a piecewise smooth and compact path in \( \mathbb{R}^{m} \), where \( x_{1} = q(a) \) and \( x_{2} = q(b) \). We consider the important notion of parallel transport (with respect to the connection \( A \)) of a vector \( \xi_{0} \) at \( x_{1} \) along the curve \( q(t), a \leq t \leq b \). The result is defined to be the vector field \( \xi(t), a \leq t \leq b \), defined along the curve, which satisfies

\[
\frac{d\xi}{dt} + \left( \sum_{j=1}^{m} A_{j} \dot{q}^{j} \right) \xi = 0, \tag{11}
\]

with \( \xi(0) = \xi_{0} \).

Written in local coordinates, Eq. (11) takes the form

\[
\frac{d\xi^{a}}{dt} + \sum_{\beta=1}^{m} \sum_{j=1}^{m} (A_{j})^{a}_{\beta} \dot{q}^{j} \xi^{\beta} = 0, \tag{12}
\]

where \( (A_{j})^{a}_{\beta} \) is the matrix corresponding to \( A_{j} \). In the case of the tangent bundle, the coefficients \( A_{j}^{a}_{\beta} \) are usually denoted \( \Gamma_{j}^{a}_{\beta} \) and called the Christoffel symbols. A frame \( (\xi_{1,0}, \ldots, \xi_{m,0}) \) is said to be parallel transported along the curve if the frame field \( (\xi_{1}(t), \ldots, \xi_{m}(t)) \), \( 0 \leq t \leq b \), satisfies \( (\xi_{1}(0), \ldots, \xi_{m}(0)) = (\xi_{1,0}, \ldots, \xi_{m,0}) \), and each frame element satisfies (11):

\[
\frac{d\xi^{i}}{dt} + \sum_{j=1}^{m} A_{j} \dot{q}^{j} \xi^{i} = 0. \tag{13}
\]

We call the expression on \( \sum_{j=1}^{m} A_{j} \dot{q}^{j} \) the action of the connection \( A \) in the direction of the vector \( \dot{q} \).

**Definition 4.** Let \( f \) be a frame at \( x_{1} \) and \( h = \{ h(t), a \leq t \leq b \} \) a piecewise smooth path from \( x_{1} \) to \( x_{2} \). Define the operator \( P_{h} \) by \( P_{h} f = P(h)(f) \) to be the operator which maps frames \( f \) at \( x_{1} = h(a) \) to the parallel transport of \( f \) along \( h \) to \( x_{2} = h(b) \) (Fig. 1).

**Definition 5.** Let \( A \) be a connection and \( F \) an arbitrary \( L \) valued two-form. Define

\[
\check{F}(x_{1}, x_{2}, h) = P_{h}^{-1}(F(x_{2})(\alpha, \beta)) \; P_{h}. \tag{14}
\]

In words, \( \check{F} \) parallel transports a frame at \( x_{1} \) to \( x_{2} \) along \( h \), there allows \( F(x_{2})(\alpha, \beta) \) to act on it, and then parallel transports it back to \( x_{1} \) by tracing the path of \( h \) backwards. The vectors \( \alpha, \beta \in T_{x_{2}}M \). We call \( \check{F} \) a
twisted two-form, and it may be seen that $\tilde{F} \in L$ by identifying the tangent spaces $T_{x_1}$ and $T_{x_2}$.

3. PARALLEL TRANSPORT SMALL LOOPS AND THE CURVATURE FORM

Let $S$ be a surface, i.e., the image of $I^2$ under $\Phi$, where $\Phi$ is a smooth map. Let $\alpha_x, \beta_x \in T_x M$ be two vectors in the span of $(\partial \Phi/\partial u) |_x$ and $(\partial \Phi/\partial v) |_x$, where $\Phi(u_0, v_0) = x \in M^m$. Let $\gamma_\epsilon = \{ \gamma_{\epsilon, t}, 0 \leq t \leq 1 \}$ be the image of the small rectangle in $I^2$ of side $\epsilon$ with initial corner at $(u_0, v_0)$ with sides in the direction of the pull-back of $\alpha_x$ and $\beta_x$, respectively. Define $P(\gamma_\epsilon)$ to be parallel transport in $M$ around the loop $\gamma_\epsilon$ which starts at $x$ initially in direction $\alpha_x$ (see Fig. 2). The following theorem is well-known (see, e.g. [3, 4]).

\begin{equation}
\text{Theorem 1.}
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (P(\gamma_\epsilon) - I) = F(x)(\alpha_x, \beta_x).
\end{equation}

Theorem 1 may be rewritten in a way which is more convenient for applications:

\begin{equation}
P(\gamma_\epsilon) = I + \epsilon^2 F(x)(\alpha_x, \beta_x) + \epsilon^2 K(x, \epsilon, \alpha_x, \beta_x).
\end{equation}
In this formula the matrix $P(\gamma_e) \in G$, the matrix $\varepsilon^2 F(x)(\alpha_x, \beta_x) \in L$, and $\bar{R}$ is a smooth function of its variables such that

$$\lim_{\varepsilon \to 0} |\bar{R}(x, \varepsilon, \alpha_x, \beta_x)| = 0$$

(17)

and this limit may be shown to be uniform with respect to $x \in U$ and $|\alpha_x| \leq Q$, $|\beta_x| \leq Q$ for any $Q > 0$, where

$$|x| = \left\{ \sum |x_i|^2 \right\}^{1/2}.$$

Another convenient way of writing (16) is

$$P(\gamma_e) = e^{\varepsilon^2 F(x)(\alpha_x, \beta_x)} + \varepsilon^2 K(x, \varepsilon, \alpha_x, \beta_x),$$

(17')

where $K$ also satisfies (17) uniformly. The matrix $K$, like $\bar{K}$, need not belong to either $G$ or $L$. Recall that the matrices of $L$ are assumed to have a norm. This is the norm used in Eq. (17).

4. Twisting 2-Forms with Respect to a Surface

Let $\Phi$ induce an immersion of $I^2$ in $M^m$ as described in the Introduction. Let $(u, v)$ be the ordinary Euclidean coordinates of $I^2$, so that they may be regarded as the parameters of the surface $S \subset M^m$ which is the image under $\Phi$ of $I^2$. The boundary $\partial S$ is defined as the image under $\Phi$ of the boundary of $I^2$. Define $g_x(t)$, $0 \leq t \leq 1$ to be the piecewise smooth arc in $S$ which is the image under $\Phi$ of the following path in $I^2$. Starting from $(0, 0)$ move
in a straight line with constant speed to \((0, v)\) and then with the same constant speed to \((u, v)\), where \(x = \Phi(u, v)\) (Fig. 3).

**Definition 6.** Given a Lie algebra valued 2-form \(F(x)\) we define \(\tilde{F}\), the twisted 2-form with respect to \(S\) and base point \(x_1 = \Phi(0, 0)\) by the formula

\[
\tilde{F}(x) = \tilde{F}(x_1, x, g_1) = P_{g_1}^{-1} F(x) P_{g_1}.
\]

Note that \(\tilde{F}\) acts on frames at \(\Phi(0, 0)\) and is defined for pairs of vectors \(\alpha, \beta\) in \(T_xM\), as is \(F\). We have chosen \(g_s\) in a certain fixed manner and there is some arbitrariness in this particular choice.

\[\text{snake-parametrization}\]
Subdivide $I^2$ into rectangles by taking the product of a division of $I$ into intervals with another division of $I$ into intervals. It is not necessary to take the lengths of all the intervals equal to $(1/n)$, but we shall do so in order to simplify the calculations. The idea in the more general case is the same. Let $\Pi$ be the image on $S$ of the small square $\pi$ on $I^2$ (see Fig. 4). Enumerate the small squares and denote them $\pi_i$, $1 \leq i \leq n^2$, as shown in Fig. 5. We call this enumeration the snake enumeration. Denote the parameters $(u_i, v_i) \in I^2$ which correspond to the lower left corner of the $i$th square under the snake enumeration. We call the pairs of parameter values $(u_i, v_i)$, $1 \leq i \leq n^2$, the discrete snake parameterization of the surface $S$. At each vertex $x_i = (u_i, v_i)$ consider the two tangent vectors $\alpha_i$, $\beta_i$ which are
tangent to the two sides of the image $\Pi_i$ of the small square $\pi_i$. The vectors $\alpha_i, \beta_i$ are the velocity vectors of the parameter curves on $S$, evaluated at $(u_i, v_i)$ (Fig. 6). Consider a square $\pi_i$ and its vertex $(u_i, v_i)$. Denote $\bar{F}(x_i)(\alpha_i, \beta_i)$ by $\bar{F}_i^*$, and the counterclockwise parallel transport around the boundary $\partial \pi_i$ by $P_i = P(\partial \pi_i)$. Then we have

$$P_i = I + \frac{1}{n^2} \bar{F}_i^* + \frac{1}{n^2} \bar{K}_i$$

$$= e^{(1/n^2) \bar{F}_i^*} + \frac{1}{n^2} \bar{K}_i$$

(18)

from (17) and (17'), where $\bar{K}_i = (x_i, (1/n), \alpha_i, \beta_i)$; $\bar{K}_i = (x_i, (1/n), \alpha_i, \beta_i)$, so that we have $|\bar{K}_i| \to 0$, $|\bar{K}_i| \to 0$, $|\bar{F}_i| \leq K$, and $\alpha_i = (\partial \Phi / \partial u)(u_i, v_i)$, $\beta_i = (\partial \Phi / \partial v)(u_i, v_i)$.

We sometimes denote $\Phi(t, v_0) = p(t)$ and $\Phi(u_0, t) = q(t)$ when the values $v_0$ and $u_0$ are clear from the context. Thus we also write $\alpha_i = \dot{p}(u_i, v_i)$, $\beta_i = \dot{q}(u_i, v_i)$. For each point $x_i$ consider the path shown in Fig. 7. We move along the image of the path in $I^2$ from $(0,0)$ to the point $(0, u_i)$, then to the point $(u_i, u_i)$ via straight lines. Denote the $\Phi$ image of this path by $g_i$. Then (see Fig. 8) we consider the path $\partial \Pi_i = d_i c_i b_i a_i$, writing the composition of paths from right to left, so that $\partial \Pi_i = \gamma_i$ (Fig. 9). This path induces a parallel transport which is denoted by $P_i$, i.e., $P_i = P(\gamma_i)$. Finally, consider the image of the path from $(u_i, v_i)$ to the point $(0,0)$ given by $g_i^{-1}$. The path $g_i^{-1} \gamma_i g_i$ is a loop with starting and ending points $x = \Phi(0,0)$. We then have that $P_i = P(\gamma_i^{-1} \gamma_i g_i) = P(g_i)^{-1} P_i P(g_i)$, where $\bar{P}_i$ is parallel transport around the loop $g_i^{-1} \gamma_i g_i$. The description of $\bar{P}_i$ is valid for all
rows, but the drawing of the path is somewhat different for even as opposed to add numbered rows. For even numbered rows see Figs. 7–9 with the rectangle $\pi_i$ in an up position. For odd numbered rows, see Figure 11 for row number one, for example, with the rectangle $\pi_i$ in a down position. In the grasshopper enumeration to be discussed later, the rectangles $\pi_i$ are in a down position for all rows. If Eqs. (18) are transformed by applying the adjoint transformation $P(g_i)^{-1}$ (*) $P(g_i)$ we obtain

$$
\tilde{P}_i = P(g_i)^{-1} P_i P(g_i) = I + \frac{1}{n^2} \tilde{F}_i + \frac{1}{n^2} \tilde{k}_i
$$

$$
= e^{(1/n^2) \tilde{F}_i + 1/n^2} \tilde{k}_i, \quad 1 \leq i \leq n^2,
$$

(19)
where \( \bar{F}_i^* = \bar{P}(g_i)^{-1} F_i^* P(g_i) \) is the value of the twisted curvature form (see Definition 5, Section 2) at \( x = \Phi(0, 0) \) evaluated at \( x_2 = x_i, \beta_i \) with respect to the path \( g_i \). In Eq. (19) we also have that

\[
|k_i| \to 0, \quad |\bar{k}_i| \to 0
\]

uniformly. Recall that \( \alpha_i = (\partial \Phi/\partial u)(u_i, v_i), \beta_i = (\partial \Phi/\partial v)(u_i, v_i) \).

The small squares may also be enumerated using the grasshopper enumeration, see Fig. 10. All results are valid for both enumerations.

**Definition 7.** Let \( F \) be the curvature form and \( \bar{F} \) be the twisted curvature form (see Definition 6). Define

\[
\bar{F}^*(x) = \bar{F}(x)(\alpha, \beta)
\]

where \( \alpha = \partial \Phi/\partial u \big|_{x}, \beta = \partial \Phi/\partial u \big|_{x} \).

5. PARALLEL TRANSPORT AROUND LARGE LOOPS

Consider the piecewise smooth curve \( \sigma = \partial S \) which is the image of the boundary of \( I^2 \), and denote by \( P(\sigma) \) parallel transport around that loop in a counterclockwise direction (see Fig. 11).

**Lemma 1.** Let \( \bar{P}_i = P(g_i)^{-1} P_i P(g_i) \). Then

\[
P(\sigma) = \bar{P}_n \cdots \bar{P}_2 \bar{P}_1
\]

**Proof.** Follows at once from Fig. 12 as those parts of paths which are geometrically the same have the opposite orientation when the terms on the right are factored into terms of the form \( g_i^{-1}(d_i c_i b_i a_i) g_i \).

![Figure 11](image-url)
THEOREM 2. Consider $S$ with boundary curve $\sigma$, and the snake parameterization of $S$. Then

\[ \lim_{n \to \infty} \left( e^{\frac{1}{n^2}} F^*_n \cdots e^{\frac{1}{n^2}} F^*_1 \right) = H(S) \]

exists, where $H(S) \in G$, and $H(S) = P(\sigma)$ where $P(\sigma)$ is parallel transport along $\sigma$ (Fig. 13)

\[ \lim_{n \to \infty} \left( I + \frac{1}{n^2} F^*_n \right) \cdots \left( I + \frac{1}{n^2} F^*_1 \right) = H_1(S) \]
exists, where \( H_1(S) \in G \), and

\[
P(\sigma) = H(S) = H_1(S)
\]

(c) the element \( P(\sigma) = H(S) = H_1(S) \) can be represented as the limit

\[
P(\sigma) = I + \lim_{n \to \infty} \left( \sum_{i=1}^{n^2} \frac{1}{n^2} \bar{F}^*_i \right) + \lim_{n \to \infty} \left( \sum_{i_1=1}^{n^2} \sum_{1 \leq i_2 < i_1} \frac{1}{n^4} \bar{F}_{i_1}^* \bar{F}_{i_2}^* \right) + \cdots,
\]

and all series and products converge uniformly.

Remark. Statements (a) and (b) are valid for arbitrary finite-dimensional Lie groups, but (c) only makes sense in the case of matrix valued groups as we must multiply elements of the algebra as matrices. Statement (c) is used in Theorem 3. All other results are meaningful for arbitrary finite-dimensional Lie groups.

Proof of Theorem 2. We first prove (b) starting from Lemma 1 and (19). We have

\[
P(\sigma) = \bar{P}_n \cdots \bar{P}_1 = \left( I + \frac{\bar{F}^*_1}{n^2} + \frac{\bar{k}_1}{n^2} \right) \cdots \left( I + \frac{\bar{F}^*_1}{n^2} + \frac{\bar{k}_1}{n^2} \right).
\]

Let \( A_i = \bar{F}^*_i/n^2 \), \( B_i = \bar{k}_i/n^2 \), and recall that \( \lim_{n \to \infty} \bar{k}_i = 0 \) uniformly. We then have

\[
P(\sigma) = I + \sum_{i=1}^{n^2} (A_i + B_i) + \sum_{i_1=1}^{n^2} \sum_{1 \leq i_2 < i_1} (A_{i_1} + B_{i_1}) (A_{i_2} + B_{i_2})
\]

\[
+ \sum_{i_1=1}^{n^2} \sum_{1 \leq i_2 < i_1} \sum_{1 \leq i_3 < i_2} (A_{i_1} + B_{i_1}) (A_{i_2} + B_{i_2}) (A_{i_3} + B_{i_3})
\]

\[
+ \cdots + (A_{n^2} + B_{n^2}) \cdots (A_1 + B_1).
\]

A typical element of the sum may be written

\[
S = \sum_{i_1=1}^{n^2} \sum_{1 \leq i_2 < i_1} \sum_{1 \leq i_3 < i_2} (A_{i_1} + B_{i_1}) \cdots (A_{i_n} + B_{i_n})
\]

\[
= \sum \sum A_{i_1} \cdots A_{i_n}
\]

\[
+ (C'_1 \text{ identical sum with one } A \text{ replaced by one } B)
\]

\[
+ (C'_2 \text{ identical sums with two } A's \text{ replaced by two } B's) + \cdots
\]

\[
+ (C'_n \text{ identical sum with all } A's \text{ replaced by all } B's).
\]
Let \( M_r = \sum \cdots \sum A_{i_1} \cdots A_{i_r} \). Then

\[
|M_r| \leq \left( \frac{k}{n^2} \right)^r \sum_{i_1=1}^{n^2} \cdots \sum_{1 \leq i_r < i_{r-1} \leq n-1} 1
\]

\[
= \left( \frac{k}{n^2} \right)^r C_r^n = \left( \frac{k}{n^2} \right)^r \frac{n^2!}{(n^2-r)! \cdots 1}
\]

\[
\leq \frac{k^r}{(n^2)^r} \frac{(r^2)^r}{r!} = \frac{k^r}{r!}.
\]

(21)

To see that \( \sum_{N \geq i_1 > i_2 \cdots > i_r \geq 1} 1 = C_r^N \), note that \( C_r^N \) is the number of ways of choosing \( r \) distinct integers from \( N \), and each such choice may be put in increasing order in just one way. Formula (21) shows that

\[
\sum |M_r| \leq \sum \frac{k^r}{r!} = e^k.
\]

and each term is bounded by the term of an absolutely convergent series. Next, note that for \( x > 0 \)

\[
\sum_{j=1}^{r} C_j^r x^j = r x + \frac{r \cdot (r-1)}{2!} x^2 + \cdots
\]

\[
\leq r x \left[ 1 + x + \frac{(rx)^2}{2!} + \cdots \right]
\]

\[
= r x e^{r x}.
\]

This implies that if we let \( N_r = S - M_r \), then

\[
|N_r| \leq |M_r| \sum_{j=1}^{r} C_j^r [o(1/n)]^j
\]

\[
\leq |M_r| r o(1/n) e^{o(1/n)}
\]

\[
\leq \left[ k e^{o(1/n)} \right]^r (r-1)! o(1/n).
\]

It therefore follows that

\[
\sum_{r=1}^{\infty} |N_r| \leq o(1/n) k e^{o(1/n)} e^{k e^{o(1/n)}}
\]

and thus that \( \lim_{n \to \infty} \sum_{r=1}^{\infty} |N_r| = 0 \). This is sufficient to prove part (b) since for each \( r \), \( \lim_{n \to \infty} M_r \) exists (and equals an integral, see Theorem 3).
More exactly, we have shown that $P(\sigma) = H_1(S) \in G$. Next, to prove (a), write Eq. (19),

$$\bar{P}_i = e^{(1/n^2)} F_i^* + \frac{1}{n^2} k_i = I + \frac{1}{n^2} \bar{F}_i^* + \frac{1}{n^2} \bar{k}_i,$$

so that

$$e^{(1/n^2)} F_i^* = I + \frac{1}{n^2} \bar{F}_i^* + \frac{1}{n^2} (\bar{k}_i - k_i)$$

$$= I + A_i + B_i.$$ 

It follows as in the proof of part (b) that

$$\lim_{n \to \infty} \prod_{i=1}^{n^2} e^{(1/n^2)} F_i^* = H_1(S)$$

and as $H_1(S) = P(\sigma)$, part (a) is proved. Part (c) follows from the preceding calculations, and the theorem is proved.

**Theorem 3.**

$$P(\sigma) = H_1(S) = H(S) = I + \int_S \bar{F}^* dS$$

$$+ \int_0^1 \int_0^{v_2} \left( \int_0^{v_1} \int_0^{v_2} \bar{F}^*(u_2, v_2) \bar{F}^*(u_1, v_1) \, du_1 \, dv_1 \right) \, du_2 \, dv_2$$

**Proof.** The proof follows from the proof of Theorem 2. It is shown there that

$$P(\sigma) = \lim_{n \to \infty} \sum_{r=1}^{n^2} \left( \sum_{i_1 \geq i_2 \geq \cdots \geq i_r \geq 1} A_{i_1} \cdots A_{i_r} \right), \quad (20')$$

where the terms are bounded by the terms of an absolutely convergent series. Also (see Fig. 10)

$$\lim_{n \to \infty} \sum_{i_1 \geq i_2 \geq \cdots \geq i_r \geq 1} A_{i_1} \cdots A_{i_r}$$

$$= \int_0^1 \int_0^{v_r} \cdots \int_0^{v_2} \int_0^{v_1} \bar{F}^*(u_r, v_r) \cdots \bar{F}^*(u_1, v_1) \, du_1 \, dv_1 \cdots du_r \, dv_r,$$

and the theorem is proved.
Lemma 2. Let \( P = \{0 = a_0 < a_1 < \cdots < a_n = 1\} \) be a partition of the unit interval, and let \( \Lambda_i = (a_i - a_{i-1}) \). Then

\[
\sum_{n > i_r > \cdots > i_1 \geq 1} A_{i_r} \cdots A_{i_1} \leq \frac{1}{r!}.
\] (22)

Proof. Consider \( I' \), the \( r \)-dimensional unit cube in Euclidean space \( \mathbb{R}^r \), and the simplex \( S = \{1 \geq t_r \geq t_{r-1} \geq \cdots \geq t_1 \geq 0\} \). Each term

\[
\Lambda_{i_r} \cdots \Lambda_{i_1} = (a_i - a_{i-1}) \cdots (a_{i_1} - a_{i_1 - 1})
\]

may be regarded as the volume of the \( r \)-rectangle

\[
\{a_i \geq t_r > a_{i_r - 1}, a_{i_r - 1} \geq t_{r-1} > a_{i_{r-1} - 1}, \ldots, a_{i_1} \geq t_1 > a_{i_1 - 1}\},
\]

and this rectangle is contained in the simplex \( S \). Also, if \((n \geq i_r > \cdots > i_1 \geq 1) \neq (n \geq i'_r > \cdots > i'_1 \geq 1)\), then it follows easily that the corresponding \( r \)-rectangles are disjoint. As the \( r \)-volume of \( S \) is \((1/r!)\), this proves the lemma.

Theorem 4. Let \( A(t), 0 \leq t \leq 1 \), be a bounded and Riemann integrable \( G \)-valued matrix function, and let \( P = \{0 = a_0 < a_1 < \cdots < a_n = 1\} \) be a partition of the unit interval. Then the limit

\[
\lim_{A \to 0} (I + A_n \Lambda_n) \cdots (I + A_1 \Lambda_1) = \Pi(I + A(t)) \ dt
\] (23)
exists, where \( \Lambda_i = a_i - a_{i-1} \). In addition

\[
\Pi(I + A(t)) \ dt = I + \int_0^1 A(t_1) \ dt_1 + \int_0^1 \int_0^{t_2} A(t_2) A(t_1) \ dt_1 \ dt_2 + \cdots, \tag{24}
\]
and the series is absolutely convergent.

Proof. The product may be written

\[
\prod_{i=1}^n (I + A_i \Lambda_i) = I + \sum_{r=1}^n \sum_{n \geq i_r > \cdots > i_1 \geq 1} A_{i_r} \cdots A_{i_1} \Lambda_{i_r} \cdots \Lambda_{i_1}
\]
and by Lemma 2,

\[
\sum_{n \geq i_r > \cdots > i_1 \geq 1} |A_{i_r} \cdots A_{i_1}| \Lambda_{i_r} \cdots \Lambda_{i_1} \leq (K)^r/r!.
\]
In addition,
\[
\lim_{n \to \infty} \prod_{n \geq i > \cdots > i_n \geq 1} A_{i_1} \cdots A_{i_n} A_{i_n} \cdots A_{i_1} = \int_{1 \geq t_n > \cdots > t_1 \geq 1} A(t_n) \cdots A(t_1) \, dt_n \cdots dt_1
\]
and Theorem 4 is proved. Expansion (29) is known as Peano's expansion.

**Theorem 5.** Under the same assumptions as in Theorem 4,
\[
\lim_{\Delta \to 0} \prod_{i=1}^{n} e^{A_{i_1} \cdots A_{i_n} \cdots A_{i_1}} = \Pi(I + A(t)) \, dt.
\]

**Proof.** That the limit exists is proved in [2]; see also [1]. Once the limits in (23) and (25) have been shown to exist, it is sufficient to prove equality for the partitions \( \Delta = 1/n \). For this case, a proof similar to the proof of Theorem 2 is sufficient, and thus Theorem 5 is proved.

**Theorem 6 (Schlesinger).** If \( P(\sigma) \) is parallel transport around the boundary \( \partial S \) of \( S \) (the image of the boundary of \( I^2 \) under the immersion \( \Phi \)), then
\[
P(\sigma) = \Pi \left( I + \int_0^1 \tilde{F}(s, t) \, ds \right) \, dt.
\]

**Proof.** It follows from Theorem 4 that the integral on the right exists for arbitrary partitions as \( \Delta \to 0 \). We show that it equals \( P(\sigma) \) by considering partitions with \( \Delta_i = (1/n) \).

We now change from the discrete snake enumeration to the grasshopper enumeration (see Fig. 10) for clarity although strictly speaking it is unnecessary. As noted earlier, all previous results are valid with this enumeration also, and so in particular Lemma 1. We reorder the indices \( i_r, r = 1, ..., n^2 \), by row and column (see Fig. 14a) so that \((i, j)\) is the index of the small square in the \( i \)th row and \( j \)th column. We may then write
\[
P(\sigma) = \tilde{P}_{r(n)} \cdot \tilde{P}_{r(n-1)} \cdots \tilde{P}_{r(1)},
\]
where \( \tilde{P}_{r(k)} \) is parallel transport from the origin to the start of the \( k \)th row, counterclockwise around that row and back along the left side to the origin (see Fig. 14b). It follows from Theorem 2 that
\[
\tilde{P}_{r(k)} = \left( I + \int_{0}^{1} \tilde{F}^*(u, v_k) \, du \right) (1/n) + o(1/n^2), \tag{a}
\]
and using the same arguments as in the proof of Theorem 2, we obtain that

\[ P(\sigma) = \Pi \left( 1 + \int_0^1 \tilde{F}^*(s, t) \, ds \right) \, dt \]

\[ = e^L \int_0^{\alpha} \left( \int_0^1 \tilde{F}(s, t) \, ds \right) \, dt, \]

where we have assumed that the boundary is traced as the parameter ranges from 0 to 4. This concludes the proof of Theorem 6.

**Remark.** Using the $L$ integral of [1], it may be shown that

\[ (l/\pi) + o(1/n^3) \quad (b) \]

which implies

\[ L \int_0^4 \left( \int_0^1 \tilde{F}(s, t) \, ds \right) \, dt = L \int_0^4 \left( \int_0^1 \tilde{F}^*(s, t) \, ds \right) \, dt. \]

6. **STOKES' THEOREM FOR CONNECTION AND CURVATURE FORMS:**

**THE MAIN THEOREM**

Let $I^2$ be a square of side $\alpha$ and let $\Phi: I^2 \to M$ be the immersion. Let $S$ be the image of $I^2$ and $\partial S$ be the image of the boundary of $I^2$. Let $\sigma(t)$, $0 \leq t \leq 4\alpha$, be the image of the boundary of $I^2$ traced at unit speed, for example, in a counter-clockwise direction with base $(0, 0)$, i.e., starting and ending at the origin. Consider the connection 1-form $A$ and the loop $\sigma$ and construct $A^*(t)$, a Lie algebra valued function of $t$, by setting

\[ A^*(t) = A(\sigma(t))(\dot{\sigma}(t)). \]
It is clear that $A^*(t), 0 \leq t \leq 4\alpha$, may be regarded as a curve in the Lie algebra. We have that

$$A(\sigma(t))(\dot{\sigma}(t)) = \sum_{j=1}^{m} A_j(\sigma(t)) \frac{dx_j}{dt},$$

where $\sigma(t) = (x^1(t), \ldots, x^m(t)) \in \mathcal{M}^m$ and $A_j(\sigma(t)) \in L$, that is, $A_j(\sigma(t))$ are matrices in $L$.

Suppose that for each $t \in [a, b]$, $K(t) \in L$, where $L$ is a fixed matrix Lie algebra. Suppose also that

$$\int_{a}^{b} |K(t)| \, dt < \infty$$

and that $K(t), t \in [a, b]$ is Riemann integrable in the sense that each entry of the matrix function is Riemann integrable.

**Definition 8.** We define the $H$-series of $\{K(t), t \in [a, b]\}$ (as above) by

$$H(K, [a, b]) = \sum_{n=0}^{\infty} H_n(K, [a, b]),$$

where the $H_n$ are defined inductively by

$$(n + 1) H_{n+1} - T_n + \sum_{r=1}^{n} \left\{ \frac{1}{2} \left[ H_r, T_{n-r} \right] + \sum_{\substack{p \geq 1 \\ 2p \leq r \\ m_j > 0 \\ m_1 + \cdots + m_{2p} = r}} [H_{m_1}, \ldots, [H_{m_{2p}}, T_{n-r}] \cdots] \right\},$$

where $k_{2p}(2p)!$ are Bernoulli's numbers,

$$H_1 = T_0 = \int_{a}^{b} K(t) \, dt,$$

and, for $k \geq 1$,

$$T_k = \int_{t_1 = a}^{b} \cdots \int_{t_{k+1} = a}^{t_k} \left[ ..., [K(t_1), K(t_2)], ..., K(t_{k+1}) \right] dt_1 \cdots dt_k \, dt_2 \, dt_1,$$

or equivalently,

$$T_k = \int_{b \geq t_1 \geq \cdots \geq t_{k+1} \geq a} \left[ ..., [K(t_1), K(t_2)], ..., K(t_{k+1}) \right] dt_1 \cdots dt_k \, dt_2 \, dt_1.$$
DEFINITION 9. If the $H$-series of $K(t), t \in [a, b]$, converges to a matrix $H \in L$, then we write

$$L \int_a^b K(t) \, dt = H$$

and say that the $L$-integral of $K$ over $[a, b]$ exists and equals $H$.

We now apply Theorem 5 of [1] to $K = -A^*$ and obtain

STATEMENT 1. Let $\sigma(t), 0 \leq t \leq 4\alpha$, be the boundary curve of $S$, and suppose that

$$\int_0^{4\alpha} |A^*(t)| \, dt < \delta,$$

$\delta > 0$ from Theorem 5 of [1]. Then the $H$-series of $\{-A^*(t), 0 \leq t \leq 4\alpha\}$ converges, and writing this sum as $L \int_0^{4\alpha} -A^*(t) \, dt$ we have that

$$P(\sigma) = e^{L \int_0^{4\alpha} A^*(t) \, dt}.$$

Remark. The condition that $\int_0^{4\alpha} |A^*(t)| \, dt < \delta$ is satisfied either if the surface has a sufficiently small boundary, in which case $|A^*|$ can be large, or if $|A^*|$ is small and the surface is large. The norm $|L \int_0^{4\alpha} -A^*(t) \, dt|$ was computed in [1], where it was shown that (Corollary 4 of Theorem 6)

$$\left| L \int_0^{4\alpha} -A^*(t) \, dt \right| \leq (1/M) h(z_0),$$

where $z_0 = M \int_0^{4\alpha} |A^*(t)| \, dt$ and $h$ is the solution of a certain differential equation.

DEFINITION 10. If

$$\int_0^2 |A^*(t)| \, dt < \delta$$

and

$$\int_0^2 \int_0^2 |\hat{F}^*(u, v)| \, du \, dv < \delta, \text{ where } \delta > 0,$$

is the number given in statement 1, then we say that the surface $S$ is regular with respect to the connection and curvature forms.

THEOREM 7. Let $A$ be a connection 1-form and $F = -(dA + A \wedge A)$ its curvature form. Let $S$ be a surface and $\partial S$ its boundary (immersion of $I^2$ and the boundary of $I^2$ under the immersion). Let $\hat{F}^*$ be the twisted curvature form with base at the image of the origin, and $P_\sigma$ counterclockwise parallel
transport around the boundaries \( \partial S \) from the image of the origin back to the image of the origin. Then

(a) \( P(\sigma) = \Pi(I - A^*(t)) \, dt \)

\[ = \Pi \left( I + \int_0^\alpha \tilde{F}^*(s, t) \, ds \right) \, dt \]

(b) \( \Pi(I - A^*(t)) \, dt = I - \int_0^\alpha A^*(t_1) \, dt_1 \)

\[ + \int_0^\alpha \int_0^{t_2} A^*(t_2) \, A^*(t_1) \, dt_1 \, dt_2 - \cdots \]

(c) \( \Pi \left( I + \int_0^\alpha \tilde{F}^*(s, t) \, ds \right) \, dt \)

\[ = \text{a similar expansion as in part (b) but with} \]

\[ - A^*(t_k) \text{ replaced by } \int_0^\alpha F^*(s, v_k) \, ds. \]

**Proof of Theorem 7.** Theorem 7 is just a rewriting of Theorems 2, 3, and 6.

**Definition 11.** Let \( G(t), 0 \leq t \leq \alpha, \) be defined by

\[ G(t) = \int_0^\alpha \tilde{F}^*(u, t) \, du, \]

where \( \tilde{F}^* \) is given in Definition 7. We define the \( H \)-series of \( \tilde{F}^*(u, v), (u, v) \in I^2(\alpha) \), as the \( H \)-series (Def. 8) of \( \{ G(t), 0 \leq t \leq \alpha \} \), and if this series converges we denote it by

\[ L \int_0^\alpha \int_0^\alpha \tilde{F}^*(u, v) \, du \, dv. \]

**Theorem 8 (Main Theorem).** If \( S \) is regular with respect to the connections and curvature forms, then the \( H \)-series of \( A^* \) and \( \tilde{F}^* \) converge and

\[ P(\sigma) = e^{L \int_0^\alpha - A^*(t) \, dt} = e^{L \int_0^\alpha \int_0^\alpha \tilde{F}^*(u, v) \, du \, dv} \tag{27} \]

and there exists \( \varepsilon > 0 \) such that if

\[ \left| L \int_0^{4\alpha} - A^*(t) \, dt \right| < \varepsilon \quad \text{and} \quad \left| L \int_0^\alpha \int_0^\alpha \tilde{F}^*(u, v) \, du \, dv \right| < \varepsilon \tag{28} \]
then

\[ L \int_0^{4\pi} -A^*(t) \, dt = L \int_0^{2\pi} \int_0^2 \bar{F}^*(u, v) \, du \, dv. \quad (29) \]

Remark. The \( \varepsilon > 0 \) of Theorem 8 is called the injective radius of the algebra \( L \). In the case of classical compact Lie groups \( \varepsilon = \pi \) (see [4, 5]). It follows from standard results in differential geometry that both Theorems 7 and 8 are invariant under coordinate changes. The integrals

\[ L \int_0^{4\pi} -A^*(t) \, dt \quad \text{and} \quad L \int_0^{2\pi} \int_0^2 \bar{F}^*(u, v) \, du \, dv \]

which exist if

\[ \int_0^{4\pi} |A^*(t)| \, dt < \delta \quad \text{and} \quad \int_0^{2\pi} \int_0^2 |\bar{F}^*(u, v)| \, du \, dv < \delta \]

do not depend on the parameterization.

Furthermore, the integrals \( L \int_0^{2\pi} \int_0^2 \bar{F}^*(u, v) \, du \, dv \) and \( L \int_0^{4\pi} -A^*(t) \, dt \) are invariant under surface deformations which leave the boundary fixed.

Note also that the operation of twisting is meaningful for arbitrary \( k \)-forms \( B \) with respect to the connection form \( A \), and thus in particular for \( B = A \). None of the integrals involving \( A \) requires that \( A \) be twisted, however.

Corollary 1.

\[ L \int_0^{4\pi} -A^*(t) \, dt = \int_0^{4\pi} -A^*(t_1) \, dt_1 \]

\[ + \frac{1}{2} \int_{t_2 = 0}^{4\pi} \left[ A^*(t_2), \int_{t_1 = 0}^{t_2} A^*(t_1) \, dt_1 \right] \, dt_2 - \cdots \]

\[ L \int_0^{2\pi} \int_0^{2\pi} \bar{F}^*(u, v) \, du \, dv = \int_S \bar{F}^* \, dS + \frac{1}{2} \int_S \left[ \bar{F}^*(u_2, v_2), \left( \int_{v_1 = 0}^{v_2} \left( \int_{u_1 = 0}^{u_2} F^*(u_1, v_1) \, du_1 \, dv_1 \right) \, du_2 \, dv_2 \right) \right] + \cdots \]

and the series converge, provided that

\[ \int_0^{4\pi} |A^*(t)| \, dt < \delta, \quad \int_0^{2\pi} \int_0^{2\pi} |\bar{F}^*(u, v)| \, du \, dv < \delta \]
and also that 

$$\left| L \int_0^{\alpha} A^*(t) \, dt \right| < \varepsilon, \quad \left| L \int_0^{\alpha} \tilde{F}^*(u, v) \, du \, dv \right| < \varepsilon,$$

where $\alpha$ is the length of the sides of $I^2(\pi)$.

**Remark.** The second term of (30) can be written in the form

$$\int_S \int_S \tilde{F}^*(u_i, v) \, du \, dv,$$

where for $t^2(\pi)$ see Fig. 15a. The second term of (31) can be written in the form

$$\int_{S_{u_1}} \int_{S_{v_1}} \tilde{F}^*(u_1, v_1) \, du_1 \, dv_1,$$

where the surface $S_2$ is represented in Fig. 15b. This surface is that part of the surface $S$ which is formed by all points $x$, where the second coordinate is no more than $v$. The formulas can thus be written in the form

$$\int_\sigma -A^*(t_1) \, dt_1 + \frac{1}{2} \int_\sigma \left[ A^*(t_2), \int_{\sigma_1(t_1)} A^*(t_1) \, dt_1 \right] dt_2 - \cdots$$

$$= \int_S \tilde{F}^*(u_1, v_1) \, du_1 \, dv_1 + \frac{1}{2} \int_S \left[ \tilde{F}^*(u_2, v_2) \right]$$

$$\left( \int_{S_{v_2}} \tilde{F}^*(u_1, v_1) \, du_1 \, dv_1 \right) \, du_2 \, dv_2 + \cdots. \quad (32)$$

or:

$$\int_\sigma -A^* + \frac{1}{2} \int_\sigma \left[ A^*, \int_{\sigma_1} A^* \right] - \cdots = \int_S \tilde{F}^* + \frac{1}{2} \int_S \left[ \tilde{F}^*, \int_S \tilde{F}^* \right] + \cdots. \quad (33)$$

Formula (33) is the most compact expression of the corollary.

**Remarks.** To prove Theorem 8, it is necessary to prove convergence in the group. After this step one proves convergence in the algebra, and finally the uniqueness of the exponential sufficiently close to the unit to get the final result. In the case of a compact Lie group with the standard invariant Riemann metric, for example $SO(3)$ or $U(n)$, the injective radius $\varepsilon$ is $\pi$, as
mentioned earlier. It is in fact possible to prove the result for a larger set. This set is the union of the rotations about zero of a certain parallelepiped in the Cartan subalgebra of $L$.

Note also that we may consider figures in the plane, for example, disks, whose interiors are diffeomorphic to $I^2$.

Proof of Theorem 8. We outline two proofs of Theorem 8. Consider the discrete snake parameterization and the limit

$$\lim_{n \to \infty} B_{n^2} \left( \frac{x^2}{n^2} \vec{F}_{x^*}, \ldots, \frac{x^2}{n^2} \vec{F}_{x^*}, \frac{x^2}{n^2} \vec{F}_{x^*} \right).$$

According to Theorem 3 of [1], the infinite series which represents $B_{n^2}$ converges uniformly and each term of the series converges to the corresponding integral. More precisely,

$$B_{n^2} = \sum_{k=1}^{\infty} H_k n^2,$$

where the series converges uniformly in $n$. It follows from the recursion formulas (28) and (29) of [1] that, for example,

$$H_1 n^2 = \sum_{j=1}^{n^2} \frac{x^2}{n^2} \vec{F}_{j^*},$$

so that

$$\lim_{n \to \infty} H_1 n^2 = \int_0^\infty \int_0^\infty \vec{F}^*(u, v) \, du \, dv,$$

and more complicated but similar arguments yield the rest of the proof of Theorem 8.

An alternative approach is to apply Theorem 5 to

$$G(t) = \int_0^2 \vec{F}^*(u, t) \, du$$

and this together with part (a) of Theorem 7 also yields a proof of Theorem 8.

7. Connections with the Classical Case

We first want to explore the connection between the usual form of Stokes' theorem and its form for connection and curvature forms. The following is a simple corollary of Theorem 8.
Corollary 2. Under the assumptions of Theorem 8, if $G$ is commutative, then formula (40),
\[
\int_{\sigma} -A^* + \frac{1}{2} \int_{\sigma} \left[ A^*, \int_{\sigma} A^* \right] - \cdots = \int_S \overline{F}^* + \frac{1}{2} \int_S \left[ \overline{F}^*, \int_{S'} \overline{F}^* \right] + \cdots,
\]
becomes
\[
\int_{\sigma} A^* = \int_S dA^*
\]
as all commutators in the algebra are zero, so that in particular
\[
F = -dA + A \wedge A = -dA \quad \text{and} \quad \overline{F} = g^{-1} F g = F.
\]

To see a deeper connection, consider the connection 1-form $A$ and multiply it by a real or a complex scalar $\mu$ to obtain another 1-form $\mu A$. Its curvature 2-form $F(\mu A)$ is easily seen to be
\[
F(\mu A) = -\mu dA + \mu^2 A \wedge A.
\]
For $\mu$ sufficiently small, $|\mu A^*| \cdot 4\alpha < \delta$ and $|\overline{F}^*(\mu A)| \alpha^2 < \delta$ and so the conditions of Theorem 8 are satisfied and thus
\[
L \int_0^{4\alpha} - (\mu A)^* (t) \, dt = L \int_0^{\alpha} \int_0^{\alpha} (\mu \overline{F})^* (u, v) \, du \, dv,
\]
or what is the same thing
\[
\sum_{n=1}^{\infty} H_n(-\mu A^*, \sigma) = \sum_{n=1}^{\infty} H_n(\mu, A, S).
\]

Lemma 3. For $n \geq 1$,
\[
H_n(-\mu A^*, \sigma) = \mu^n H_n(-A, \sigma).
\]

Proof. It follows from the remarks in the introduction of Section 2 of [1] that
\[
c^n_n(\mu X_1, \ldots, \mu X_N) = \mu^n c^n_n(X_1, \ldots, X_N),
\]
and the lemma follows from this as
\[
H_n(\mu A^*, \sigma) = \lim_{N \to \infty} c^n_n.
\]
We next examine the right hand side of (42). Since

$$F(\mu A) = -\mu \, dA + \mu^2 A \wedge A,$$

it follows that

$$\tilde{F}(\mu A) = \mu \, \tilde{d}A + \mu^2 \tilde{A} \wedge \tilde{A},$$

where the tilde signifies the twisting operation with respect to the right angle path defined earlier. In general, \( \tilde{d}A \neq d\tilde{A} \) but \( \tilde{A} \wedge \tilde{A} = \tilde{A} \wedge \tilde{A} \). To see this, recall that

$$(A \wedge A)_{x_2, x_1} (X) = [A_\alpha(X), A_\beta(X)]$$

and since

$$P^{-1}[X, Y]P = [P^{-1}XP, P^{-1}YP]$$

it follows that \( \tilde{A} \wedge \tilde{A} = \tilde{A} \wedge \tilde{A} \). We have therefore shown that

$$\tilde{F}(\mu A) = -\mu \, \tilde{d}A + \mu^2 \tilde{A} \wedge \tilde{A},$$

where

$$\tilde{d}A = P_{\mu A}^{-1} \, dA P_{\mu A}$$

and

$$\tilde{A} = P_{\mu A}^{-1} \, A P_{\mu A}. \tag{35}$$

We may also write that

$$\tilde{F}(\mu A) = P_{\mu A}^{-1} \, F(\mu A) \, P_{\mu A}. \tag{36}$$

Either (35) or (36) implies that

$$H_n(\tilde{F}(\mu A)) = \sum_{k=1}^{\infty} \varpi_k(A, dA) \mu^k, \tag{37}$$

where \( \varpi_k \) does not depend on \( \mu \), and where \( P_{\mu A} \) is parallel transport along the right angle path used in the twisting operation with respect to the connection \( \mu A \). Substituting (37) in (34) and collecting like powers of \( \mu \) on the right, we obtain

$$\sum_{n=1}^{\infty} \mu^n H_n(A, \sigma) = \sum_{n=1}^{\infty} \mu^n D_n(A, dA, S)$$

and as both sides converge for sufficiently small \( \mu \), we have
THEOREM 9. For \( n \geq 1 \)

\[
H_n(A, \sigma) = D_n(A, dA, S). \tag{38}
\]

We next find explicit expressions for the first few \( D_n \). We have from (32) that

\[
\int_\sigma - A^* + \frac{1}{2} \int_\sigma \left[ A^*, \int_\partial A^* \right] - \cdots = \int_S \tilde{F}^* + \frac{1}{2} \int_S \left[ \tilde{F}^*, \int_{\partial S} \tilde{F}^* \right] + \cdots,
\]

where \( \tilde{F}^* = P^{-1}F^*P, \ P = P_g \), and \( g = g_x \) is the path shown on Fig. 3, namely, the path which is the right angle path required by the twisting operation. Consider the transformation \( A \rightarrow \mu A \). Then \( A^* \rightarrow \mu A^* \) and

\[
\tilde{F}^* \rightarrow P^{-1}_\mu A(g_x) F^*(-\mu A) P^{-1}_\mu A(g_x) = P^{-1}_\mu A(g_x)(-\mu dA^* + \mu^2 A^* \wedge A^*) P^{-1}_\mu A(g_x).
\]

From Theorem 5 of [1] it follows that

\[
P_{\mu A}(g_x) = \exp(H_1(-A, g_x) + H_2(-A, g_x) + \cdots).
\]

This implies that

\[
P_{-\mu A}(g_x) = \exp(H_1(-\mu A, g_x) + H_2(-\mu A, g_x) + \cdots) = \exp(\mu H_1(-A, g_x) + \mu^2 H_2(-A, g_x) + \cdots) \quad \text{(by Lemma 3)}
\]
\[
= I + (\mu H_1 + \mu^2 H_2 + \cdots) + \frac{1}{2}(\mu H_1 + \mu^2 H_2 + \cdots)^2 + \cdots \quad \text{(using the usual expansion for the exponential)}
\]
\[
= I + (\mu H_1 + \mu^2 (H_2 + \frac{1}{2}H_1^2) + \cdots).
\]

This implies that

\[
P^{-1}_{-\mu A}(g_x) F^*(-\mu A) P^{-1}_{-\mu A}(g_x)
\]
\[
= (I - \mu H_1 - \mu^2 (H_2 - \frac{1}{2}H_1^2) + \cdots)(-\mu dA^* + \mu^2 A^* \wedge A^* + \cdots)
\]
\[
\times (I + \mu H_1 + \mu^2 (H_2 + \frac{1}{2}H_1^2) + \cdots)
\]
\[
= (-\mu dA^* + \mu^2 A^* \wedge A^* - \mu^2 H_1 dA^* + \cdots)
\]
\[
\times (I + \mu H_1 + \mu^2 (H_2 + \frac{1}{2}H_1^2) + \cdots)
\]
\[
= -\mu dA^* + \mu^2 (A^* \wedge A^* + [-dA^*, H_1]) + \mu^3(\cdots) + \cdots.
\]
Finally,

\[ \mu \int_{\sigma} -A^* + \frac{\mu^2}{2} \int_{\sigma} \left[ A^*, \int_{\sigma} A^* \right] + \cdots \]

\[ = \iint_S \left\{ -\mu \, dA^* + \mu^2 (A^* \wedge A^* + [ -dA^*, H_1(g_\tau) ]) + \cdots \right\} \]

\[ + \frac{1}{2} \iiint_S \left[ -\mu \, dA^* + \mu^2 (\cdots) + \cdots, \iint_{S_\tau} -\mu \, dA^* \right. \]

\[ + \mu^2 (\cdots) + \cdots \left. \right] + \cdots, \]

and so

\[ \mu \int_{\sigma} -A^* + \frac{1}{2} \mu^2 \int_{\sigma} \left[ A^*, \int_{\sigma} A^* \right] + \cdots \]

\[ = \mu \iint_S -dA^* + \mu^2 \iiint_S \left\{ A^* \wedge A^* + [ -dA^*, H_1(g_\tau) ] \right\} \]

\[ + \frac{1}{2} \left[ -dA^* \cdot \iint_{S_\tau} -dA^* \right\} + \cdots. \]

Comparing coefficients of \( \mu \) and \( \mu^2 \) we obtain

**Theorem 10.**

\[ \int_{\sigma} A^* = \int_S dA^*, \]

\[ \int_{\sigma} \left[ A^*, \int_{\sigma} A^* \right] = \iiint_S \left\{ \left[ dA^*, \iint_{S_\tau} dA^* \right] + 2A^* \wedge A^* \right. \]

\[ + 2[ -dA^*, H_1(g_\tau) ] \right\}. \tag{39} \]

The first equation of (39) is \( H_1 = D_1 \) and is the usual form of Stokes' formula for connection 1-forms \( A \). The second equation, \( H_2 = D_2 \), is a new equation. Of course, the usual Stokes' formula may be proved much more efficiently by the usual methods. The result is obtained in order to show that Theorem 8 does indeed contain the usual form of Stokes' theorem.

We now investigate to what extent the second equation of (39) is independent of the first. We shall show that it is in general independent of the first equation. Consider a small square \( AS \) in \( \mathbb{R}^n = M^n \) and consider a
non-zero but constant connection 1-form (i.e., one whose coefficients are
constant matrices in \( L \)) such that \( A \land A \neq 0 \). Then the first equation
becomes \( 0 = 0 \) since \( dA \equiv 0 \). On the other hand, the second equation
becomes

\[
\int_\sigma \left[ A^*, \int_{\sigma_i} A^* \right] = \iint_{\Delta S} A^* \land A^*.
\]

Since \( A \land A \neq 0 \) there exists a position for \( \Delta S \) such that

\[
\iint_{\Delta S} A^* \land A^* \approx (A^* \land A^*)(\Delta S) \neq 0
\]

and thus the second equation is not of the form \( 0 = 0 \). This means that the
second equation is independent of the first equation in general.

In certain special cases there is a very close connection between the usual
form of Stokes' theorem and the second equation. We shall now consider
an example for which this is valid. Consider a surface \( S \) in \( M^n \) and a con-
nection form \( A \). By restricting \( A \) to \( S \) it is possible to assume that \( S = M^n \).
As we assume that \( S \) is the immersion of a square, we may assume that
\( S = I^2 \subset \mathbb{R}^2 \), by considering the pull-back of \( A \) to \( I^2 \). Let \((u, v) \in I^2 \) be the
coordinates of a point in \( I^2 \). We may then write

\[
A = A_1(u, v) \, du + A_2(u, v) \, dv,
\]

where \( A_i \in L, \, i = 1, 2 \). The second equation of (39) may be written in coor-
dinates as

\[
\int_u \left[ A^*, \int_{\nu_i} (A^*_1 \, du + A^*_2 \, dv) \right] \, du
+ \left[ A^*_2, \int_{\nu_i} (A^*_1 \, du + A^*_2 \, dv) \right] \, dv
\]

\[
= \iint_S \left[ (A^*_2, u) - A^*_1, v \right]
+ \iint_{S_i} (A^*_2, u) - A^*_1, v) \, du \land dv
\]

\[
+ 2[ A^*_1, A^*_2] + 2[ -dA^*, H_1(g_1)] \, du \land dv.
\]

Where \( A_{1,v} = \partial A_1/\partial u \) and \( A_{2,v} = \partial A_2/\partial v \).

**Theorem 11.** If \( S = I^2 \) and if the connection 1-form is closed (i.e.,
\( dA = 0 \)), then the second equation of (47) may be written in the form

\[
\int_\sigma B^* = \iint_S dB^*
\]

for another 1-form \( B \) (and in this case, the first equation of (47) becomes
\( 0 = 0 \)), where \( B = [A, R] = [A_1, R] \, du + [A_2, R] \, dv \), and where \( dR = A \).
Proof. Let \((u, v)\) be the coordinates of \(I^2 \subset R^2\) again and write

\[ A = A_1(u, v) \, du + A_2(u, v) \, dv. \]

The functions \(A_1\) and \(A_2\) take their values in the Lie algebra \(L\). As \(A\) has been assumed closed, \(dA = 0\), and so \(\partial A_1/\partial v = \partial A_2/\partial u\). Since \(S = I^2\), there exists a smooth \(L\)-valued function \(R = R(u, v)\) on \(S\) such that \(A = dR\). As in the scalar case, the condition \(A_1 \, du + A_2 \, dv = dR\) is equivalent to the system of equations \(A_1 = R_u\) and \(A_2 = R_v\), and \(R_{uv} = R_{vu}\) implies \(A_{1,v} = A_{2,u}\), i.e., \(dA = 0\). As in the scalar case, the \(L\)-valued function \(R\) is defined uniquely except for all additive constant elements from the algebra \(L\). Consequently we can assume that the value of the function \(R\) in at the point \((0, 0) \in \sigma = \partial S\) is equal to zero. If \(t\) is the parameter along \(\sigma\), then we can consider the restriction \(r(t) = R(\sigma(t))\) of the function \(R(x, y)\) from the surface \(S\) to the boundary \(\sigma\), and \(r(4\alpha) = 0\).

Because the form \(A\) is closed, Eq. (40) may be written as

\[
\int_{\sigma} \left[ A^\ast_1, dR^\ast \right] \, dx + \left[ A^\ast_2, dR^\ast \right] \, dv = 2 \int_{S} \left[ A^\ast_1, A^\ast_2 \right] \, du \wedge dv. \tag{41}
\]

Consider the new 1-form \(B = [A_1 \, du + A_2 \, dv, R] = [A_1, R] \, du + [A_2, R] \, dv\). This form is well-defined on \(S = I^2\). Equation (41) implies that

\[
-\int_{\sigma} \left[ A^\ast_1, R^\ast \right] \, du + \left[ A^\ast_2, R^\ast \right] \, dv = 2 \int_{S} \left[ A^\ast_1, A^\ast_2 \right] \, du \wedge dv, \tag{42}
\]

because \(\int_{\sigma} dR^\ast = R(\sigma(4\alpha)) - R(\sigma(t)) = -R(\sigma(t))\) since \(R(\sigma(4\alpha)) = 0\).

Equation (42) has the form

\[
-\int_{\sigma} B = 2 \int_{S} \left[ A_1, A_2 \right] \, du \wedge dv. \tag{43}
\]

We now show that

\[
dB = -2\left[ A_1, A_2 \right] \, du \wedge dv. \tag{44}
\]

We have

\[
dB - \left[ A_{1,v}, R \right] \, dv \wedge du + \left[ A_1, R_v \right] \, du \wedge dv \\
+ \left[ A_{2,u}, R \right] \, du \wedge dv + \left[ A_2, R_u \right] \, du \wedge dv \\
= \left[ A_{2,u}, R \right] \, dv \wedge du + \left[ A_1, A_2 \right] \, dv \wedge du + \left[ A_{2,u}, R \right] \, du \wedge dv \\
+ \left[ A_2, A_1 \right] \, du \wedge dv \quad \text{(because} \ A_{1,v} = A_{2,u}, R_u = A_1, R_v = A_2) \\
= -2\left[ A_1, A_2 \right] \, du \wedge dv.
\]
Thus \( \int_\sigma B^* = \iint_S dB^* \) and the theorem is proved.

**Remark.** In the case that \( dA \neq 0 \), the second equation does not in general have the form

\[
\int B^* = \iint dB^*,
\]

\( B^* = [A^*, S, A^*] \), as this expression is only defined along the boundary. It is therefore natural to suppose that the system of equations \( H_n = D_n, n \geq 1 \), is independent.

8. **GENERALIZATIONS OF THE GAUSS-BONNET THEOREM**

Suppose that \( S \) is a compact orientable surface without a boundary, a closed surface. The well-known Gauss–Bonnet Theorem says that

\[
\iint_S K dS = 4\pi(1 - g),
\]

where \( g \) is the genus of the surface (i.e., the number of handles), and \( K \) the Gauss curvature of the surface. In the case of the sphere, for example, \( g = 0 \) and \( \iint K dS = 4\pi \). We now consider natural generalizations of this result.

The sphere can be represented as \( I^2 \) with the four sides identified in the following way. Consider an orientation of \( I^2 \) and the induced orientation along the boundary. Starting with the bottom side, label the sides \( a_1 b_1 b_1^{-1} a_1^{-1} \) to indicate that the bottom side is oriented with the induced orientation, the right hand side also, and the top with opposite orientation, as well as the left side. The sides with the same letters are glued together respecting the orientation. The torus can similarly be represented but with the labelling \( a_1 b_1 a_1^{-1} b_1^{-1} \). It is well known that except for the sphere, i.e., when \( g = 0 \), every compact and orientable surface can be represented by the square \( I^2 \) with a certain identification of the boundary. The boundary is divided into \( 4g \) consecutive intervals (possibly containing a corner) and they are labelled \( a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \) starting with a certain interval. The letters with a superscript of \((-1)\) indicate that the corresponding interval is taken with orientation opposite to the induced orientation. The letters without a superscript indicate that the corresponding interval is taken with the induced orientation. Finally the boundary is glued so as to glue the same letters with the same index together, respecting the orientation. The word

\[
W = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}
\]
uniquely defines the identification. We see that $W$ is the product of commutators $a_i b_i a_i^{-1} b_i^{-1}$. It is easy to see that all vertices in Fig. 16 identify to one vertex $P(0,0)$ on $S$. This implies that each separate side on the boundary of $I^2$ is identified with a loop on $S$. We therefore obtain $g$ loops of kind $a_i$ on $S$ and $g$ loops of kind $b_i$ on $S$ (see Fig. 17). For the case $g=0$ (i.e., for the sphere $S^2$) we consider the square $I^*$ with the identification represented in Fig. 18. In this case $W = a b b^{-1} a^{-1}$. After identifying the sides of $I^*$ with the word $W$, we obtain the sphere $S^2$. We may also simply contract the boundary to a point; see Fig. 18 also.

Consider a surface $S$ embedded in the manifold $M^n$ with an $L$-valued connection $A$. Applying the non-Abelian extension of Stokes’ formula we obtain

$$P(\sigma) = e^{H(-A,S)} = e^{H(\overline{F},S)}$$

and

$$H(-A, S) = H(\overline{F}, S).$$
$g = 3$

**Figure 17**

---

$S^2$

$P(0,0)$

$\sim S^2$

sphere

$T^2$

$P(0,0)$

$g = 1$

**Figure 18**
We can calculate the element $P(a) \in L$, $\sigma = a_1 b_1 b_1^{-1} a_1^{-1}$ or $\sigma =$ "point" for $g = 0$ and $\sigma = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ for $g > 0$ (Fig. 19). Consequently,

(a) if $g = 0$, then $P(\sigma) = P(a_1) P(b_1) P^{-1}(b_1) P^{-1}(a_1) = I$, or $P(\sigma =$ point$) = I$; or

(b) if $g > 0$, then

$$P(\sigma) = P(a_1) P(b_1) P^{-1}(a_1) P^{-1}(b_1) \cdots P(a_g) P(b_g) P^{-1}(a_g) P^{-1}(b_g).$$

Thus, $P(\sigma) = I$ for $g = 0$ and $P(\sigma) = e^{-A_1} e^{-B_1} e^{B_1} e^{A_1} \cdots e^{-A_g} e^{-B_g} e^{B_g} e^{A_g}$ for $g > 0$.

Let us suppose again (as usual in this paper) that the surface is regular with respect to the connection and curvature forms (see Definition 10). Recall that it is nevertheless possible, under this assumption, for the norms of $H(-A, \sigma)$ and $H(\vec{F}, S)$ to be large.

**THEOREM 12** (Non-abelian extension of the classical Gauss–Bonnet theorem). Let $S$ be a compact closed 2-dimensional orientable surface in $M^n$ with connection $A$ and curvature $F$. Then

(a) if $S$ is the sphere ($g = 0$), then

$$\exp \left( \int_S \vec{F} + \cdots \right) = I,$$

and therefore $H(\vec{F}, S)$ (i.e., the $L$ integral of the twisted curvature form over $S$) belongs to the distinguished set $D = \{ \exp^{-1}(I) \} \subset L$. 

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(b) If $g > 0$ then
\[
e^{H(F, S)} = e^{H(-A_1, -B_1, A_1, B_1, ..., -A_g, -B_g, A_g, B_g)}
\]
\[
= e^{\sum_{i=1}^{2g} [A_i, B_i] + \text{commutators of degree } \geq 3}
\]  
(47)

if $|H(-A, \sigma)| < \varepsilon$, and $|H(F, S)| < \varepsilon$, where $\varepsilon$ is the injective radius of $L$; then it follows from (47) that

\[
H(F, S) = \sum_{i=1}^{g} [A_i, B_i] + \text{commutators of degree } \geq 3.
\]

Further, the integral $\int_L \mathcal{F}^* = H(F, S)$ does not change if the surface is deformed with a fixed boundary.

Proof of Theorem 12. For $g = 0$, the sphere (46) follows from the calculations made earlier and Theorem 8. For $g > 0$ we see that $H(A_1, ..., -B_g)$ is equal to $B_{4g}(A_1, ..., -B_g)$. From the discrete recurrence formulas of [1] it follows that $c_1^{4g} = -A_1 - B_1 + A_1 + B_1 + \cdots - A_g - B_g + A_g + B_g = 0$. It also follows easily from these formulas that $c_2^{4g} = [A_1, B_1] + \cdots + [A_g, B_g]$. The proof is finished.

It is interesting that in the case $g > 0$ the expansion $H(F, S) = B_{4g}(A_1, ..., -B_g)$ begins with a term of degree 2. We see that in the case of a closed surface $S$ the non-commutative $L$ integral of $\mathcal{F}$ along the surface $S$ is calculated in "discrete terms" of $2g$ elements $A_1, B_1, ..., A_g, B_g$ from the algebra.

The formulas (46), (47), and (48) can be considered to be the non-Abelian extension of the Gauss–Bonnet formula. Before demonstrating this fact we must discuss the structure of the distinguished set $D = \{\exp^{-1}(I)\} \subset L$, where $\exp: L \to G$ and $I$ is the unit in $G$. The set $D = \exp^{-1}(I)$ is closed in $L$ and can consist, in the general case, of an infinite number of closed non-intersecting components. In the case of many classes of Lie groups the set $\exp^{-1}(I)$ can be completely described. We consider here only two of the most important examples: (a) $G$ is a connected compact commutative Lie group, and (b) $G$ is a connected compact Lie group with compact universal covering (the universal covering is a simply-connected group). In the case (b) $G$ is automatically non-commutative, because a simply-connected commutative connected Lie group is $\mathbb{R}^n$, i.e., not compact.

In the case (a) the group $G$ is isomorphic to the matrix group

\[
g = \begin{pmatrix}
e^{i\phi_1} & 0 \\
\cdot & \\
0 & e^{i\phi_n}
\end{pmatrix}
\]
for some $n$, where $\phi_1, ..., \phi_n$ are arbitrary real numbers. Consequently, the group $G$ is homeomorphic to the direct product of $n$ copies of the circle $S^1$; i.e., $G$ is the $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$ ($n$ times). The corresponding Lie algebra $L$ is isomorphic to the linear space of all matrices of the form

$$X = \begin{pmatrix} i\phi_1 & 0 \\ \vdots & \ddots \\ 0 & & i\phi_n \end{pmatrix}.$$ 

The exponential map acts as follows: $\exp \text{diag}(i\phi_1, ..., i\phi_n) = \text{diag}(e^{i\phi_1}, ..., e^{i\phi_n})$.

Consequently, all elements $X$ in the set $\exp^{-1}(I)$ have the form

$$\text{diag}(2\pi i m_1, 2\pi i m_2, ..., 2\pi i m_n),$$

(49)

$n = 1$

\[ \begin{array}{cccccc}
-6\pi & -4\pi & -2\pi & 0 & 2\pi & 4\pi & 6\pi \\
\end{array} \]

FIGURE 20
where $m_1, m_2, ..., m_n$ are arbitrary integers. This set is isomorphic to the additive group $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ($n$ times), which forms a lattice in the additive group (= linear space) $\mathbb{R}^n = L$. This lattice consists of all points with coordinates $(2\pi m_1, ..., 2\pi m_n)$ in $\mathbb{R}^n$. Two examples for $n = 1$ and $n = 2$ are given in Fig. 20.

Next, consider the standard invariant Riemannian metric on the commutative Lie group $T^n$. This metric is locally Euclidean and consequently the corresponding Lie algebra $L$ has an invariant norm which is the usual Euclidean norm. The injective radius $\varepsilon = R$ for the commutative group $T^n$ is clearly equal to $\pi$, because $e^{\pi i} = -1$. The ball of radius $R = \pi$ is shown in Fig. 21. For each two points $X \neq Y$ in this ball we have that $e^X \neq e^Y$.

In case (a) it is well-known (for details, see e.g. [3, 5]) that each compact group $G$ with a compact universal covering contains the maximal commutative subgroup $T^n$, where $n$ is called the rank of the group $G$, and $T^n$ is called its Cartan subgroup. Each element $g \in G$ is conjugate to some element $\tau$ of the subgroup; i.e., for each $g$ there exists some element $a \in G$ such that $a^{-1}ga = \tau \in T^n$ and consequently, $g = ata^{-1}$.

The set of all elements $ata^{-1}$, where $\tau$ is fixed but $a$ runs over the total group $G$ is called the orbit of the element $\tau$ and can be denoted by $\mathcal{O}(\tau) \subset G$.

This means that we can choose in the Lie algebra $L$ (where $G = \exp L$) the maximal commutative subalgebra $C^n$ (the so-called Cartan subalgebra), which is isomorphic to $\mathbb{R}^n$ and $T^n = \exp C^n$. For each element $X \in C^n$ we consider its orbit $\mathcal{O}(X) = aXa^{-1}$, where $a$ runs over the whole group $G$ (recall that we are considering the matrix group case).

The union of all orbits $\mathcal{O}(X)$ (for all $X \in C^n$) gives us the whole Lie algebra $L$. In other words, each element in $L$ is conjugate to some element in the Cartan subalgebra. The topological structure of each orbit $\mathcal{O}(X)$ is

![Figure 21](image-url)
well-known, namely \( \mathcal{O}(X) = G/Z(X) \) is a homogeneous manifold, where \( Z(X) \) is a subgroup in \( G \) such that \( Z(X) = \{ a \in G : aX = Xa \} \). See details in [5].

Consider in the Cartan subalgebra \( C'' \) the lattice \( \exp_{C''}(Z) \), where \( \exp_{C''} : C'' \to T'' \). Then the set \( \exp_{C''}^{-1}(I) \) (where \( \exp_{C''} : L \to G \) of all pre-images of the unit \( I \) in the Lie algebra \( L \) is the disjoint union of the orbits \( \mathcal{O}(l) \), where \( l \) runs over all elements of the lattice \( \exp_{C''}^{-1}(I) \) in \( C'' \).

Consider for example \( G = SO(3) \). Then \( L = SO(3) \) is the 3-dimensional space of all skew-symmetric matrices and can be identified with the usual \( \mathbb{R}^3 \) with standard Euclidean metric (which is adjoint-invariant). The Cartan subalgebra is one-dimensional (see Fig. 21). The lattice \( \exp_{C''}^{-1}(I) \) was described above (see Fig. 20). Each point \( X \) in this lattice generates in the Lie algebra \( SO(3) \) an orbit \( \mathcal{O}(X) \), which is isometric to the usual standard sphere of radius \( |X| \). Consequently, in the case of \( G = SO(3) \), the set \( \exp^{-1}(I) \) is the union of Euclidean spheres of radii \( 0, 2\pi, 4\pi, 6\pi, \ldots \) in \( SO(3) = \mathbb{R}^3 \) (the sphere of radius \( 0 \) is a point). The set \( \exp^{-1}(I) \) in the Lie algebra \( su(2) \) has the same structure, because the Lie algebra \( su(2) \) is isomorphic to the Lie algebra \( so(3) \).

Now we can return to the non-Abelian generalization of the Gauss-Bonnet formulae (46) (= the case of the sphere) and (47), (48) (= the case of the 2-surfaces of genus \( g \geq 1 \)).

Consider the case of the sphere \( S = S^2 \). According to Theorem 12, \( \exp(\int_{S^2} \tilde{F} + \cdots) = I \). Consequently, the element \( \int_{S^2} \tilde{F} + \cdots \) belongs to the set \( \exp^{-1}(I) \), which is the discrete union of the special orbits \( \mathcal{O} \) in \( L \). Assume that \( L \) is a commutative Lie algebra. In this case we know (see above) that the set \( \exp^{-1}(I) \) is the lattice \( \mathbb{Z}^n \) in \( L \cong \mathbb{R}^n \). Moreover, the size of the unit cube in this lattice is equal to \( 2\pi \). In the commutative case the element \( \int_{S^2} \tilde{F} + \cdots \) (commutators of degree \( \geq 2 \)) is reduced to the element \( \int_{S^2} \tilde{F} \).

**Corollary 3.** Let \( S = S^2 \) be the sphere (i.e. a closed 2-surface with boundary \( \sigma = a \) single point, see Fig. 18) in some \( M^m \) with connection \( A \) and curvature \( F \) with values in a commutative Lie algebra \( L \cong \mathbb{R}^n \). Then

\[
\int_{S^2} \tilde{F} = (\text{some element of the lattice } \mathbb{Z}^n \text{ in } \mathbb{R}^n, \text{ where the side of the lattice is equal to } 2\pi).
\]

If \( M^m = S^2 \) and the connection \( A \) is induced by a Riemannian metric on \( S^2 \), then \( L = \mathbb{R}^1 \) is a one-dimensional Lie algebra, \( \tilde{F} = F \) is the usual Gaussian curvature, \( \int_{S^2} F = 4\pi \) (usual Gauss-Bonnet formula).

Here \( 4\pi = 2 \cdot 2\pi \) is the second nontrivial element of the lattice \( 2\pi \cdot n = (..., 0, 2\pi, 4\pi, ...) \) in \( \mathbb{R}^1 \).
Corollary 3 gives us another interpretation of the number $4\pi$ in the classical Gauss-Bonnet formula: the number $4\pi = 2 \cdot 2\pi$ is the second element in the lattice $2\pi \cdot \mathbb{Z}$ on the real line $\mathbb{R}^1 = \text{the Cartan subalgebra}.$

**Proof.** The first statement clearly follows from Theorem 12 and the description of the set $\exp^{-1}(I)$, which was given above. Consider the second statement. If $M = S^2$, then curvature $F$ is reduced to the usual Gaussian curvature. From the usual variational theory (see, for example, [3]) it follows that the integral $\int_S F$ does not depend on the choice of Riemannian metric and it is therefore possible to assume that the metric is the invariant metric. This metric is induced by the standard embedding of $S^2$ in $\mathbb{R}^3$. Because the connection $A$ is induced (by assumption) by the metric, the vector bundle $\pi: E \to M$ is the tangent bundle $T_{\ast} S^2 \to S^2$, the Lie group $G$ is $SO(2)$, and $L$ is the commutative Lie algebra $\mathbb{R}^1$. Consider the family $S_t$ of surfaces, parameterized by the parameter $t$, where $S_t$ is a spherical sector of radius $r$ (which is measured along the meridian from the north pole $N$, see Fig. 22). It is clear that $0 \leq t \leq \pi$ and $S_{t=0}$ is the point $N$, the surface $S_{\pi/2}$ is the hemisphere, and $S_{\pi} = \mathbb{S}^2$ is the entire sphere $S^2$. Let us consider the element $g(t) = \exp \int_{S_t} F$ in the group $G = SO(2) \approx S^1$. The element $S_t g(t)$ as a function of $t$ (when $0 \leq t \leq \pi$) is shown in Fig. 23. When $t = 0$ $g(0) = I$ and $\int_{S_0} F = 0$. As $t$ increases, the element $g(t)$ moves along $S^1$ and the element $\int_{S_t} F = \exp^{-1}(g(t))$ moves along $\mathbb{R}^1$ (Fig. 23). When $t = \pi/2$, the element $g(t)$ equals $I$, because the boundary curve $\sigma_{\pi/2}$ is the equator on the standard sphere $S^2$ and consequently parallel transport along the curve $\sigma_{\pi/2}$ is the identity and $\int_{S_{\pi/2}} F = 2\pi$. We continue to

![Figure 22](image-url)
Figure 23

increase \( \tau \) and when \( \tau \) equals \( \pi \), we obtain that \( g(\pi) = I \), because here the path \( \sigma_{\pi} \) is a single point of \( S \) (the south pole of the sphere). Consequently, the element \( g(\tau) \) equals \( I \) the second time. The corresponding element \( \int_{S} F \) moves from \( 2\pi \) to \( 4\pi \) when \( \tau \) changes from \( \pi/2 \) to \( \pi \) and thus \( \int_{S} F = 4\pi \). Corollary 3 is proved.

**Corollary 4.** Let \( S = M_2^2 \) be a closed compact 2-manifold of genus \( g \geq 1 \) with boundary \( \sigma = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \) (Fig. 17) in some \( M^m \) with connection \( A \) and curvature \( F \) with values in a commutative Lie algebra \( L \cong \mathbb{R}^n \). Then \( \bar{F} = F \) and

\[
\int_{M_2^2} F = (\text{some element of the lattice } \mathbb{Z}^n \text{ in } \mathbb{R}^n, \text{ where the side of the lattice is equal to } 2\pi).
\]

The proof is similar to the proof of Corollary 3.

Let us note that in the general case of an arbitrary connection form \( A \) only formula (47) of Theorem 12 is valid and not (48), because in the general case the condition that \( |H(A, \sigma)| < R \) and \( |H(\bar{F}, S)| < R \) (where \( R \) is the injective radius) does not hold. For example, if \( G = SO(2) \) (a commutative group) then \( R = \pi \) and \( \exp(\int_{S} F) = I \). But, for example, in the case of the connection 1-form \( A \) induced by a Riemannian metric on
\( S = M_s^2 = M'' \), we have that \( \int_S F = 4\pi (1 - g) \) and consequently \( \int F = 4\pi |1 - g| > \pi \), if \( g \neq 1 \). This means that in general we can conclude that the inequality \( e^A = e^B \) only implies \( A \equiv B \mod \exp^{-1}(I) \).

REFERENCES