Note

Characterizing $2k$-critical graphs and $n$-extendable graphs

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Abstract

Let $G$ be a graph with even order. Let $M$ be a matching in $G$ and $x_1, x_2, \ldots, x_{2r}$ be the $M$-unsaturated vertices in $G$. Then $G$ has a perfect matching if and only if there are $r$ independent $M$-augmenting paths joining the $2r$ vertices in pairs. Let $G$ be a graph with a perfect matching $M$. It is proved that $G$ is $2k$-critical if and only if for any $2k$ vertices $u_1, u_2, \ldots, u_{2k}$ in $G$, there are $k$ independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ joining the $2k$ vertices in pairs such that $P_1, P_2, \ldots, P_k$ start and end with edges in $M$. It is also proved that $G$ is $n$-extendable if and only if, for each $r$ with $0 \leq r \leq n$ and each $F \subseteq M$ with $|F| = r$, and for any $n - r$ pairs of $M$-alternating paths $x_i, x_j', y_i, y_j' \in M$ $(1 \leq i \leq n - r)$ in $G - V(F)$, there exist independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ in $G - V(F)$ joining the vertices in $Z = \{x_1, y_1, \ldots, x_{n-r}, y_{n-r}\} \cap \{x_1', y_1', \ldots, x_{n-r}', y_{n-r}'\}$, where $|Z| = 2k$, which start and end with edges in $E(G) \setminus M$.

Keywords: Perfect matching; $n$-extendable graph; $2k$-critical graph; $M$-alternating path

1. Introduction and terminology

All (D. Lou) graphs in this paper are finite, undirected, connected and simple.

Let $G$ be a graph with a perfect matching $M$ and let $n \leq (r - 2)/2$ be a positive integer. The graph $G$ is said to be $n$-extendable if any matching of size $n$ is contained in a perfect matching in $G$. Let $m \leq r - 2$ be a positive integer. A graph $G$ is said to be $m$-critical if deleting any $m$ vertices from $G$, the remaining graph has a perfect matching. If $M$ is a matching and $Q$ is a path in $G$ such that the edges on $Q$ appear in $M$ and in $E(G) \setminus M$ alternately, then $Q$ is said to be an $M$-alternating path. An $M$-alternating path $P$ that starts and ends with two $M$-unsaturated vertices is said to be an $M$-augmenting path. Let $A$ and $B$ be two sets. Then $A \Delta B$ is the symmetric difference of $A$ and $B$.

For the other terminology and notations not defined in this paper, the reader is referred to [4].

Since Plummer [13] introduced the concept of $n$-extendable graphs in 1980, an extensive research has been done (see [1,2,8]). A related topic is that on $n$-critical graphs, Zhong, Yu and Lou obtained some results on $n$-critical graphs which corresponds to results on $n$-extendable graphs (see [10–12]). For the advances in research on $n$-extendable graphs, please see [15,16].

Plummer [14] first gave the characterization of $n$-extendable bipartite graphs. This result was also obtained by Brualdi and Perfect [5]. But their result was couched in terms of matrices.

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Theorem 1 (Plummer [14] and Brualdi [5]). Let $G$ be a connected bipartite graph with bipartition $(U, W)$. Suppose $n$ is a positive integer such that $n \leq (v - 2)/2$. Then the following are equivalent:

(i) $G$ is $n$-extendable;
(ii) $|U| = |W|$ and for each non-empty subset $X$ of $U$ such that $|X| \leq |U| - n$, $|N(X)| \geq |X| + n$;
(iii) For all $u_1, u_2, \ldots, u_n \in U$ and $w_1, w_2, \ldots, w_n \in W$, $G' = G - u_1 - u_2 - \cdots - u_n - w_1 - w_2 - \cdots - w_n$ has a perfect matching.

Yu [17] gave a Holton, Little and Grant style characterization of general $n$-extendable graphs.

Theorem 2 (Yu [17]). A graph $G$ is $n$-extendable ($n \geq 1$) if and only if for any $S \subseteq V(G)$

(1) $\omega(G - S) \leq |S|$ and
(2) $\omega(G - S) = |S| - 2k$ ($0 \leq k \leq n - 1$) implies that $\Phi(S) \leq k$, where $\Phi(S)$ is the size of a maximum matching in $G[S]$.

Independently, Lou [9] gave a Tutte style characterization of $n$-extendable graphs as follows:

Theorem 3 (Lon [9]). A graph $G$ is $n$-extendable if and only if, for any $S \subseteq V(G)$, $\omega(G - S) \leq |S| - 2d$, where $d = \min\{\ind(S), n\}$ and $\ind(S)$ is the maximum number of independent edges in $G[S]$.

Later Chen [6] also gave a Tutte style characterization of $n$-extendable graphs.

Theorem 4 (Chen [6]). Let $k \geq 1$. Then a graph $G$ is $k$-extendable if and only if $\omega(G - S) \leq |S| - 2k$ for every $S \subseteq V(G)$ such that $G[S]$ contains $k$ independent edges.

For $n$-critical graphs, Yu [17] and Favaron [7] independently obtained the following characterization.

Theorem 5 (Yu [17] and Favaron [7]). A graph $G$ is $k$-critical if and only if $\omega(G - S) \leq |S| - k$ for every $S \subseteq V(G)$ satisfying $|S| \geq k$.

In [3], Aldred, Holton, Lou and Saito use $M$-alternating path theory to characterize the structure of $n$-extendable bipartite graphs. Motivated by this result, in this paper, we characterize $2k$-critical graphs and the general $n$-extendable graphs using $M$-alternating path theory.

2. Preliminary results

In this section, we introduce some basic results on $n$-extendable graphs and $n$-critical graphs which will be used in the proof of the following theorems.

Lemma 6 (Plummer [14]). If $G$ is $n$-extendable for $n \geq 2$, then $G$ is also $(n - 1)$-extendable.

Lemma 7 (Favaron [7]). Every $k$-critical graph is $k$-connected.

Lemma 8 (Lou [10]). If $G$ is $n$-critical for $n \geq 3$, then $G$ is $(n - 2)$-critical.

3. A new necessary and sufficient condition for perfect matching

In this section, we propose a new necessary and sufficient condition for a graph to have a perfect matching that plays a key role in the proof of the theorems to characterize the $2k$-critical graphs and the general $n$-extendable graphs.

Theorem 9. Let $G$ be a graph with even order. Let $M$ be a matching in $G$ and $x_1, x_2, \ldots, x_2r$ be the $M$-unsaturated vertices in $G$. Then $G$ has a perfect matching if and only if there are $r$ independent $M$-augmenting paths joining the $2r$ vertices in pairs.
Theorem 10.

1. Extendable graphs: $G$ is $n$-extendable if $1$, $n+1$, $n+2$, $n+3$, ..., $n+m$ joining $m$ vertices in $\{x_1, x_2, ..., x_m\}$ so that $m$ is as large as possible, and assume $m < r$. We may assume $P_j$ joins $x_{j-1}$ and $x_j$ $(1 \leq j \leq m)$. Let $M' = M \Delta (\bigcup_{j=1}^m E(P_j))$. Then $M'$ is a matching of $G$, but since $m < r$, $M'$ is not a maximum matching of $G$.

Thus, $G$ has an $M'$-augmenting path $Q$. We may assume $Q$ joins $x_{2m+1}$ and $x_{2m+2}$. Let $H = (V(G), \bigcup_{j=1}^m E(P_j) \Delta E(Q))$.

Let $v \in V(P_i) \cap V(Q) (1 \leq i \leq m)$ and let $e$ be the unique edge of $E(P_i) \setminus M$ incident with $v$. Then $e \in M'$ and hence $e \in E(Q)$. This implies $e \notin E(H)$. From this observation, we have the following:

(i) If $v \notin \{x_{2i-1}, x_{2i}\}$, then $d_P(v) = d_Q(v) = 2$ and hence $d_H(v) \in \{0, 2\}$. Moreover, if $d_H(v) = 2$, then one of the two edges of $H$ incident with $v$ belongs to $M$.

(ii) If $v = x_{2i-1}$ or $v = x_{2i}$, then $d_H(v) = 1$, and since $v$ is $M$-unsaturated, the unique edge of $H$ incident with $v$ belongs to $E(H) \setminus M$.

Therefore, $H$ consists of $m+1$ independent $M$-augmenting paths joining vertices in $\{x_1, x_2, ..., x_{2m+2}\}$ in pairs, and possibly $M$-alternating cycles. This contradicts the maximality of $m$. □

4. Characterizing 2k-critical graphs and n-extendable graphs

Using Theorem 9, we prove the following theorems to show the role of $M$-alternating paths in 2k-critical graphs and n-extendable graphs:

Theorem 10. Let $G$ be a graph with a perfect matching $M$. Then $G$ is 2k-critical ($k \geq 1$) if and only if for any 2k vertices $v_1, v_2, \ldots, v_{2k}$ in $G$, there are $k$ independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ joining the 2k vertices in pairs such that $P_1, P_2, \ldots, P_k$ start and end with edges in $M$.

Proof. To prove necessity, let $v_1, v_2, \ldots, v_{2k}$ be any 2k vertices in $G$. Then there are $k$ independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ joining the 2k vertices in pairs such that $P_1, P_2, \ldots, P_k$ start and end with edges in $M$. Then, in $G' = G - \{v_1, v_2, \ldots, v_{2k}\}$, $P_1, P_2, \ldots, P_k$ become $r (r \leq k)$ independent $M$-augmenting paths, where $r$ - paths of $P_1, P_2, \ldots, P_k$ are single matching edges. Then we can easily find a perfect matching in $G'$.

To prove necessity, let $G$ be a 2k-critical graph and $M'$ be a perfect matching in $G$. Let $v_1, v_2, \ldots, v_{2k}$ be any 2k vertices of $G$. Suppose there are $r$ pairs of vertices in $\{v_1, v_2, \ldots, v_{2k}\}$ that are joined by an edge in $M$, respectively. Without loss of generality, assume $v_{2i-1}v_{2i} \in E, i = 1, 2, \ldots, r$. Then $G' = G - \{v_i | i = 1, 2, \ldots, r\}$ has a perfect matching. But $M'' = M' \setminus (v_1v_2) = G - \{v_1, v_2, \ldots, v_{2k}\}$ is a matching in $G''$. By Theorem 9, there are $(k - r)$ independent $M''$-augmenting paths in $G''$ joining the vertices in $v_1v_2$. These $(k - r)$ paths plus $\{v_i | i = 2r, 2r+2, \ldots, 2k\}$ form $(k - r)$ $M$-augmenting paths in $G''$ joining the vertices in $v_1v_2$. Considering $\{v_2v_3 \ldots, v_{2k}\}$, we obtain the $k$ required paths. Hence we complete the proof. □

Theorem 11. Let $G$ be a graph with a perfect matching $M$. Then the following statements are equivalent:

1. $G$ is n-extendable;
2. For each $r$ with $0 \leq r \leq n$ and each $F \subseteq M$ with $|F| = r$, and for any $n - r$ pairs of $M$-alternating paths $x_iy_j$ in $G - V(F)$ such that $x_i^r, y_j^r \in M (1 \leq i \leq n - r)$ and $x_1, x_2, y_1, y_2$, ..., $x_{n-r}, y_{n-r}$ are $2(r - n)$ different vertices, there exist independent $M$-augmenting paths $P_1, P_2, \ldots, P_m$ in $G - V(F)$ joining the vertices in $Z = \{x_1, y_1, \ldots, x_{n-r}, y_{n-r}\} \setminus \{x_1^r, y_1^r, \ldots, x_{n-r}^r, y_{n-r}^r\}$, where $|Z| = 2m$, which start and end with edges in $E(G) \setminus M$.

Proof. To prove that (2) implies (1), let $x_i^r, y_j^r (i = 1, 2, \ldots, n)$ be any $n$ independent edges in $G$. Assume that $r$ of the $n$ edges are in $M$. Without loss of generality, assume $F = \{x_i^r, y_j^r | i = n - r + 1, n - r + 2, \ldots, n\} \subseteq M$ with $|F| = r$. Then we have $x_i^r, y_j^r \in M (i = 1, 2, \ldots, n - r)$. By statement (2), there are $m$ independent $M$-augmenting paths $P_1, P_2, \ldots, P_m$ in $G - V(F)$ joining the $2m$ vertices in $Z = \{x_i, y_j | i = 1, 2, \ldots, n - r\} \setminus \{x_i^r, y_j^r | i = 1, 2, \ldots, n\}$ in pairs, where $|Z| = 2m$, such that $P_1, P_2, \ldots, P_m$ start and end with edges in $E(G) \setminus M$. Then, in $G'' = G - \{x_i^r, y_j^r | i = 1, 2, \ldots, n\}$, $P_1, P_2, \ldots, P_m$ are independent $M$-augmenting paths joining the $2m$ unsaturated vertices in pairs. So we can easily find a perfect matching in $G''$. Hence $G$ is $n$-extendable.

To prove that (1) implies (2), suppose $G$ is $n$-extendable. Let $F = \{x_iy_j | i = n - r + 1, n - r + 2, \ldots, n\} \subseteq M$ with $|F| = r$ and $G'' = G - V(F)$. Then $G''$ is $(n - r)$-extendable and $M'' = M' \setminus F$ is a perfect matching in $G''$. Consider any $2(r - n)$ different
vertices $x_1, y_1, x_2, y_2, \ldots, x_{n-r}, y_{n-r}$ such that there is a path $x_i x'_i y'_i y_i$ with $x_i x'_i, y'_i y_i \in M'$ ($i = 1, 2, \ldots, n-r$). Since $G'$ is $(n-r)$-extendable, $G'' = G' - \{x'_i, y'_i \mid i = 1, 2, \ldots, n-r\}$ has a perfect matching. Note that $\{x_i, y_i \mid i = 1, 2, \ldots, n-r\}$ and $\{x'_i, y'_i \mid i = 1, 2, \ldots, n-r\}$ may intersect. Assume $Z = \{x_i, y_i \mid i = 1, 2, \ldots, n-r\} \setminus \{x'_i, y'_i \mid i = 1, 2, \ldots, n-r\}$ with $|Z| = 2m$. But $M'' = M'' \setminus \{x_i x'_i, y'_i y_i \mid i = 1, 2, \ldots, n-r\}$ is a matching in $G''$. By Theorem 9, there are $m$ independent $M''$-augmenting paths $P_1, P_2, \ldots, P_m$ joining the $2m$ unsaturated vertices in $Z$ in pairs. Then $P_1, P_2, \ldots, P_m$ are the required paths in statement (2). The proof of this theorem is complete. □

Corollary 12. Let $G$ be a graph with a perfect matching $M$. If $G$ is $n$-extendable, then for any pair of vertices $u$ and $v$ such that there is a $(u, v)$-$M$-alternating path $P$ of length $2m + 1$ ($1 \leq m \leq n$) starting and ending with edges in $M$, there is an independent $(u, v)$-$M$-alternating path $P$ with respect to $Q$ starting and ending with edges in $E(G) \setminus M$.

Proof. The proof is the same as the necessity proof of Theorem 11. Let the $(u, v)$-$M$-alternating path $Q = x_0 y_0 x_1 y_1 \cdots x_m y_m$, where $x_0 = u, y_m = v, x_i y_i \in M$ ($i = 0, 1, 2, \ldots, m$) and $y_i x_{i+1} \in E(G) \setminus M$ ($i = 0, 1, 2, \ldots, m - 1$). Then we have $2m$ pair of different vertices $u_j, v_j \mid i = 1, 2, \ldots, m$ such that $u_j = x_{i-1}, v_j = y_i$ ($i = 1, 2, \ldots, m$), and there is an $M$-alternating path $Q_j$ of length $3$ joining $u_j$ and $v_j$ ($i = 1, 2, \ldots, m$) starting and ending with edges in $M$. Since $G$ is $n$-extendable, $G$ is also $m$-extendable. By Theorem 11, there is an $M$-alternating path $P$ joining the two vertices $u = x_0$ and $v = y_m$ that are not internal vertices of $Q_j$ ($i = 1, 2, \ldots, m$). Hence the corollary is proved. □

Theorem 13. Let $G$ be a $2k$-critical graph. Then for every $F \subset E(G)$ with $|F| = k$, $G - F$ has a perfect matching.

Proof. Assume $G - F$ has no perfect matching. Note that $G$ has even order since $G$ is $2k$-critical. Then $o((G - F) - S) \geq |S| + 2$ by Tutte’s theorem and the parity of $|V(G)|$. Since $o(G - S) \leq o((G - F) - S) - 2k$, we have $o(G - S) \geq |S| - 2k + 2$. If $|S| \geq 2k$, this contradicts Theorem 5. Hence, $|S| \leq 2k - 1$. Since $o(G - F - S) \geq |S| + 2$, if $|S| \geq k$, $G - S$ is disconnected and $G$ is not $2k$-connected. This contradicts Lemma 7. Hence, we have $|S| \leq k - 1$. However, since $G$ is $2k$-connected by Lemma 7, $G - S$ is $(k + 1)$-connected, and hence $(G - F) - S$ is connected, which contradicts $o((G - F) - S) \geq |S| + 2$. Therefore, the theorem follows. □

Remark 1. Since adding new edges to a graph $G$ with a perfect matching, the resulting graph still has a perfect matching. Theorem 13 can be reformulated as Let $G$ be a $2k$-critical graph. Then for every $F \subset E(G)$ with $|F| \leq k$, $G - F$ has a perfect matching.

Corollary 14. Let $G$ be a $2k$-critical graph and $M$ be a perfect matching in $G$. Then for any $2k$ vertices $v_1, v_2, \ldots, v_{2k}$, there are $k$ independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ joining the $2k$ vertices in pairs such that $P_1, P_2, \ldots, P_k$ start and end with edges in $E(G) \setminus M$.

Proof. Suppose there are $r$ pairs of vertices in $\{v_1, v_2, \ldots, v_{2k}\}$ that are joined by a matching edge, respectively. Without loss of generality, assume that $v_{2j-1} v_{2j} \in M$, $j = 1, 2, \ldots, r$, and $v_i v'_i \in M$, $i = 2r + 1, 2r + 2, \ldots, 2k$.

By Theorem 10, there are $k$ independent $M$-alternating paths $P_1, P_2, \ldots, P_k$ joining the $2k$ vertices in $\{v_1, v_2, v_2', v_2 + 1, v_2' + 2, \ldots, v_{2k}\}$ in pairs such that $P_1, P_2, \ldots, P_k$ start and end with edges in $M$. Without loss of generality, assume that $P_i = v_{2j-1} v_{2j}$, $j = 1, 2, \ldots, r$. Then there is a perfect matching $M' = M \triangle \bigcup_{i=r+1}^{k} E(P_i)$ in $G' = G - \{v'_i \mid i = 2r + 1, 2r + 2, \ldots, 2k\}$ which contains $\{v_{2j-1} v_{2j} \mid i = 1, 2, \ldots, r\}$ but $G'$ is $2r$-critical since $G$ is $2k$-critical. By Theorem 13, $G'' = G - \{v'_i \mid i = 2r + 1, 2r + 2, \ldots, 2k\}$ has a perfect matching. However, $M'' = M \setminus \{v_{2j-1} v_{2j} \mid i = 1, 2, \ldots, r\}$ is a matching in $G''$ and $v_1, v_2, \ldots, v_{2k}$ are the $M''$-unsaturated vertices in $G''$. By Theorem 9, there are $k$ independent $M''$-augmenting paths $Q_1, Q_2, \ldots, Q_k$ joining the $2k$ vertices $v_1, v_2, \ldots, v_{2k}$ in pairs. Then $Q_1, Q_2, \ldots, Q_k$ are the required paths in $G$. The proof is complete. □

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