Journal of Pure and Applied Algebra 59 (1989) 169–177
North-Holland

MONOIDAL MORITA EQUIVALENCE

S.R. JOHNSON
Mathematics Department, Research School of Physical Sciences, Australian National University, ACT. 2601, Australia

Communicated by G.M. Kelly
Received 11 February 1988
Revised 24 June 1988

Morita equivalence has been studied for categories enriched over a monoidal category. For such enriched categories *themselves* with a monoidal structure, we define a *monoidal* Cauchy completion, and derive many of the Morita theorems in this context. Conditions under which the monoidal Cauchy completion is closed are also discussed.

1. Introduction

After the work of Morita [13], concerned with the equivalence of the categories R-Mod and S-Mod for rings R and S, many have studied Morita theory in the context of categories enriched over a monoidal category $\mathcal{V}$ or even over a bicategory. See, for example [5, 6, 10, 11, 15, 16].

A summary of results known prior to 1981 can be found in [3]. In particular, Lawvere [10] defined the Cauchy completion $\mathcal{Q}\mathcal{A}$ of a $\mathcal{V}$-category $\mathcal{A}$, generalising the Cauchy completion of a metric space (the case $\mathcal{V}=\mathbb{R}^+$) and the idempotent-splitting completion of an ordinary category ($\mathcal{V}=\mathbf{Set}$). Lindner [11] then showed that $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent ($[\mathcal{A},\mathcal{V}]\simeq[\mathcal{B},\mathcal{V}]$) precisely when their Cauchy completions are equivalent.

If we consider $\mathcal{V}$-categories with a *monoidal* structure, the questions arise whether there is a corresponding *monoidal* Cauchy completion, and whether standard Morita theorems are valid in the monoidal setting. Im and Kelly [7] have studied the free monoidal cocompletion $\mathcal{P}\mathcal{A}$ of a small monoidal $\mathcal{V}$-category $\mathcal{A}$, and much of their work extends easily to free monoidal $\mathcal{F}$-cocompletions where $\mathcal{F}$ is any set of weights for colimits. This, together with the observation of Street [14] that the Cauchy completion is just the free cocompletion under absolute colimits, gives us a monoidal structure on the Cauchy completion $\mathcal{Q}\mathcal{A}$ of any small monoidal $\mathcal{A}$.

From the principle that a monoidal functor is a monoidal equivalence if and only if it is strong (that is, preserves the monoidal structure to within isomorphism) and has an underlying functor which is an equivalence, it will follow that there is a monoidal equivalence $\mathcal{P}\mathcal{A}\simeq\mathcal{P}\mathcal{Q}\mathcal{A}$ for any monoidal $\mathcal{A}$. It can then be shown that much of Morita theory carries over to the monoidal case.

In [4], Day showed that the monoidal $\mathcal{PA}$ is biclosed for any small monoidal $\mathcal{A}$. It is far from being true that $\mathcal{QA}$ is always biclosed (in fact this implies that the tensor product of $\mathcal{A}$ preserves colimits in both variables), but if $\mathcal{A}$ is *near* closed, in a sense to be made precise, the internal hom of $\mathcal{PA}$ will restrict to $\mathcal{QA}$ making $\mathcal{QA}$ closed.

2. Preliminaries

All of the results here apply to categories enriched over a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$ with unit I and tensor product $\otimes$. A $\mathcal{V}$-functor will be called a functor if it is understood that the domain and codomain are $\mathcal{V}$-categories. Similarly a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors will be called simply a natural transformation. For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, $[\mathcal{A},\mathcal{B}]$ will denote the $\mathcal{V}$-functor category and $[\mathcal{A},\mathcal{B}]_0$ will denote its underlying ordinary category of functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them. If $\mathcal{A}$ is not small, $[\mathcal{A},\mathcal{B}]$ may only exist as a $\mathcal{V}'$ category for some extension $\mathcal{V}'$ of $\mathcal{V}$ as in [9, Section 3.11]. We use $Acc[\mathcal{A},\mathcal{V}]$ to denote the $\mathcal{V}$-category of *accessible* functors: those that are left Kan extensions of some $\mathcal{K}\to\mathcal{V}$ with $\mathcal{K}$ small (see [11], where such functors are called *small*; the term *accessible* is that used in [9] and [1]; of course every functor $\mathcal{A}\to\mathcal{V}$ is accessible when $\mathcal{A}$ is small.)

If $\mathcal{F}$ is any set of accessible $\mathcal{V}$-functors which have codomain $\mathcal{V}$, *$\mathcal{F}$-colimits* are colimits weighted (or indexed) by elements of $\mathcal{F}$, and an $\mathcal{F}$-cocomplete $\mathcal{V}$-category is a $\mathcal{V}$-category admitting $\mathcal{F}$-colimits. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{F}$-cocomplete, then a functor from $\mathcal{A}$ to $\mathcal{B}$ is called *$\mathcal{F}$-cocontinuous* if its preserves all $\mathcal{F}$-colimits, and a functor $F:\mathcal{A}\otimes\mathcal{B}\to\mathcal{C}$ is called *separately $\mathcal{F}$-cocontinuous* if $F(A,-):\mathcal{B}\to\mathcal{C}$ and $\mathcal{F}(-,B):\mathcal{A}\to\mathcal{C}$ are $\mathcal{F}$-concontinuous for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$. We write $\mathcal{F}$-Coc$[\mathcal{A},\mathcal{B}]$ for the full subcategory of $[\mathcal{A},\mathcal{B}]$ determined by the $\mathcal{F}$-cocontinuous functors, and $\mathcal{F}$-Coc$(\mathcal{A},\mathcal{B})$ for its underlying ordinary category. Similarly *$S\mathcal{F}$-Coc*$[\mathcal{A}\otimes\mathcal{B},\mathcal{C}]$ will denote the full subcategory of $[\mathcal{A}\otimes\mathcal{B},\mathcal{C}]$ determined by the separately $\mathcal{F}$-cocontinuous functors and *$S\mathcal{F}$-Coc*$(\mathcal{A}\otimes\mathcal{B},\mathcal{C})$ will denote its underlying ordinary category. All of this notation follows that of Kelly [9].

For a $\mathcal{V}$-category $\mathcal{A}$ let $\mathcal{FA}$ denote the closure in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ of $\mathcal{A}$ under $\mathcal{F}$-colimits and let $y=y_{\mathcal{A}}:\mathcal{A}\to\mathcal{FA}$ denote the Yoneda embedding seen as landing in $\mathcal{FA}$. Thus, letting $\mathcal{P}$ be the set of all accessible weights, we get by [9, Section 5.7] that $\mathcal{PA}=Acc[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$, the free cocompletion of $\mathcal{A}$. Kelly there gives a construction of $\mathcal{FA}$ by transfinite induction and shows that $\mathcal{FA}$ is the free $\mathcal{F}$-cocompletion of $\mathcal{A}$ in the sense that for any $\mathcal{F}$-cocomplete $\mathcal{B}$, composition with y is an equivalence $\mathcal{F}$-*Coc*$[\mathcal{FA},\mathcal{B}]\simeq[\mathcal{A},\mathcal{B}]$, with inverse Lan_y (= left Kan extension along y). Furthermore, by [9, Theorem 5.56] (see also [2, Section 2]) we have:

Proposition 2.1. *If $\mathcal{B}$ is $\mathcal{F}$-cocomplete and if $F:\mathcal{FA}\to\mathcal{B}$, then F has a right adjoint iff F is $\mathcal{F}$-cocontinuous and $\mathcal{B}(F\cdot y_{\mathcal{A}}-,B)\in\mathcal{FA}$ for all $B\in\mathcal{B}$.*

An accessible functor $F:\mathscr{K}^{\mathrm{op}}\to\mathscr{V}$ is called an *absolute weight* if all colimits weighted by F are absolute (i.e. preserved by any functor). Let $\mathscr{Q}$ be the set of all absolute weights. For small $\mathscr{A}$, Street shows in [14] that $\mathscr{Q}\mathscr{A}$ is just the Cauchy completion of $\mathscr{A}$ (defined in [9, Section 5.5] as the full subcategory of $\mathscr{P}\mathscr{A}$ determined by the small projectives). In this case, since every functor preserves all absolute colimits, $\mathscr{Q}\text{-}Coc[\mathscr{A},\mathscr{C}]=[\mathscr{A},\mathscr{C}]$ and $S\mathscr{Q}\text{-}Coc[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]=[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]$ for any Cauchy-complete $\mathscr{C}$.

We let $\mathscr{A}=(\mathscr{A},\circ,K)$ denote a monoidal $\mathscr{V}$-category where $\circ:\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ is the tensor product of $\mathscr{A}$ and where K is the unit. As in [7], $\Phi=(\phi,\tilde{\phi},\phi^0):\mathscr{A}\to\mathscr{A}'$ denotes a monoidal functor where $\phi:\mathscr{A}\to\mathscr{A}'$ is a $\mathscr{V}$-functor, $\tilde{\phi}:\phi A\circ'\phi B\to\phi(A\circ B)$ is a natural transformation and $\phi^0:K'\to\phi K$ is an arrow in $\mathscr{A}'$ satisfying the usual coherence conditions. Recall that Φ is *strong* if $\tilde{\phi}$ and ϕ^0 are isomorphisms. Also recall that a monoidal natural transformation is just a natural transformation subject to two coherence conditions. We denote the resulting 2-category of (strong) monoidal categories by [*Str*]*Mon*. We will sometimes combine these prefixes with the prefix $\mathscr{F}$-*Coc* so that, for instance, $StrMon\mathscr{F}\text{-}Coc(\mathscr{A},\mathscr{C})$ will denote the (ordinary) category of strong monoidal $\mathscr{F}$-cocontinuous functors from $\mathscr{A}$ to $\mathscr{C}$ and monoidal natural transformations between them. The results here will be proved for monoidal $\mathscr{V}$-categories, but the corresponding results will also hold for symmetric monoidal ones, with essentially unchanged proofs.

We recall [7, Proposition 2.2] due to Kelly in [8]:

Proposition 2.2. *Let $\Phi=(\phi,\tilde{\phi},\phi^0)$ be a monoidal functor. In order that Φ be a left adjoint in Mon, it is necessary and sufficient that ϕ be a left adjoint in $\mathscr{V}$-Cat and that Φ be strong. In fact, if $\eta,\varepsilon:\phi\dashv\psi$ is an adjunction in $\mathscr{V}$-Cat and Φ is strong, there is a unique enrichment of ψ to a monoidal Ψ (not in general strong) that renders η and ε monoidal; so that $\eta,\varepsilon:\Phi\dashv\Psi$ in Mon. Hence the monoidal Φ is an equivalence in Mon if and only if Φ is strong and ϕ is an equivalence in $\mathscr{V}$-Cat. The same results hold in the symmetric monoidal case.*

3. The free monoidal $\mathscr{F}$-cocompletion

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{V}$-categories and that $\mathscr{C}$ is an $\mathscr{F}$-cocomplete $\mathscr{V}$-category for some set $\mathscr{F}$ of weights. Letting $R:S\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C})\to[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]_0$ be the functor derived by composition with $y\otimes y:\mathscr{A}\otimes\mathscr{B}\to\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B}$, we have a generalisation of [7, Proposition 3.1]:

Proposition 3.1. *The functor R is an equivalence*

$$S\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C})\simeq[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]_0.$$

Proof. The inverse of R is the underlying functor of $L:[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]\simeq[\mathscr{A},[\mathscr{B},\mathscr{C}]]\simeq\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{A},\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{B},\mathscr{C}]]\simeq S\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C}]$ which takes a functor $T:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$ to its left Kan extension along $y\otimes y$. □

Now let $\mathscr{A}=(\mathscr{A},\circ,K)$ be monoidal. From [4] there is a monoidal structure on $\mathscr{P}\mathscr{A}$ with unit $J=\mathscr{A}(-,K)$, and tensor product $*$ defined by the convolution formula

$$(f*g)A = \int^{B,C\in\mathscr{A}} fB\otimes gC\otimes\mathscr{A}(A,B\circ C)$$

$$\cong \operatorname{colim}(f-,\operatorname{colim}(g?,\mathscr{A}(A,-\circ ?)))$$

for $A\in\mathscr{A}$ and $f,g\in\mathscr{P}\mathscr{A}$. Although this result is only stated for small $\mathscr{A}$ in [4], it is clearly valid even when $\mathscr{A}$ is not small (though the monoidal $\mathscr{P}\mathscr{A}$ may not be closed in that case). From [1], we know that $\mathscr{F}\mathscr{A}$ is closed in $\mathscr{P}\mathscr{A}$ under f-weighted colimits whenever $f:\mathscr{K}^{\mathrm{op}}\to\mathscr{V}$ is an object of $\mathscr{F}\mathscr{K}$. Thus, $*$ restricts to a tensor product of $\mathscr{F}\mathscr{A}$ which we will denote by $*:\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{A}\to\mathscr{F}\mathscr{A}$. The unit $J=\mathscr{A}(-,K)$ is in $\mathscr{F}\mathscr{A}$ as this always contains the representables. The associativity and unit isomorphisms of $\mathscr{P}\mathscr{A}$ and the strong monoidal enrichment $y:\mathscr{A}\to\mathscr{P}\mathscr{A}$ of the Yoneda embedding from [7] all restrict to $\mathscr{F}\mathscr{A}$. Thus we have the following generalisation of Day's result [4], as re-formulated in part of [7, Proposition 4.1]:

Proposition 3.2. *If $\mathscr{A}=(\mathscr{A},\circ,K)$ is (symmetric) monoidal, then $\mathscr{F}\mathscr{A}$ has a (symmetric) monoidal structure $(\mathscr{F}\mathscr{A},*,J)$ and there is a strong monoidal inclusion $y=y_{\mathscr{A}}:\mathscr{A}\to\mathscr{F}\mathscr{A}$.*

Of course, we do not in general have (as in the case when $\mathscr{F}=\mathscr{P}$ and $\mathscr{A}$ is small) that $\mathscr{F}\mathscr{A}$ is biclosed even if $\mathscr{A}$ is. For instance, the countable colimit closure of the ordinary Cartesian closed category **1** is not Cartesian closed.

A monoidal $\mathscr{C}=(\mathscr{C},*',J')$ is called *monoidally $\mathscr{F}$-cocomplete* if $\mathscr{C}$ is $\mathscr{F}$-cocomplete and $*':\mathscr{C}\otimes\mathscr{C}\to\mathscr{C}$ is separately $\mathscr{F}$-cocontinuous. The proof of [7, Theorem 5.1] now generalises (using Proposition 2.1 above) easily to give

Theorem 3.3. *For a monoidal $\mathscr{A}$ and a monoidally $\mathscr{F}$-cocomplete $\mathscr{C}$, the functor $R:Mon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A},\mathscr{C})\to Mon(\mathscr{A},\mathscr{C})$ given by composition with y is an equivalence of categories which restricts to an equivalence $StrMon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A},\mathscr{C})\simeq StrMon(\mathscr{A},\mathscr{C})$. Moreover, the monoidal $F:\mathscr{F}\mathscr{A}\to\mathscr{C}$ has a right adjoint in Mon iff $F\in StrMon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A},\mathscr{C})$ and $\mathscr{C}(F\circ y_{\mathscr{A}}-,C)\in\mathscr{F}\mathscr{A}$ for all $C\in\mathscr{C}$. The corresponding results are true in the symmetric monoidal case.*

4. Monoidal Morita equivalence

Of special interest is the case $\mathscr{F}=\mathscr{Q}$ where $\mathscr{Q}\mathscr{A}$ is the Cauchy completion of $\mathscr{A}$. We shall use $q=q_{\mathscr{A}}:\mathscr{A}\to\mathscr{Q}\mathscr{A}$ to denote the Yoneda embedding in this case. A monoidal $\mathscr{V}$-category $\mathscr{C}$ is called *monoidally Cauchy complete* if it is monoidally $\mathscr{Q}$-cocomplete. Since any tensor product is separately $\mathscr{Q}$-cocontinuous, $\mathscr{C}$ is monoidally Cauchy complete iff it is monoidal and Cauchy complete as a $\mathscr{V}$-category. The results of the previous section give:

Corollary 4.1. *For any monoidal $\mathscr{A}$, Day's monoidal structure on $\mathscr{P}\mathscr{A}$ restricts to $\mathscr{Q}\mathscr{A}$ so that there is a strong monoidal enrichment $\mathbf{q} : \mathscr{A} \to \mathscr{Q}\mathscr{A}$ of the functor q. If $\mathscr{C}$ is any monoidally Cauchy complete $\mathscr{V}$-category, the functor $R : Mon(\mathscr{Q}\mathscr{A}, \mathscr{C}) \to Mon(\mathscr{A}, \mathscr{C})$ given by composition with $\mathbf{q}$ is an equivalence of categories which restricts to an equivalence $StrMon(\mathscr{Q}\mathscr{A}, \mathscr{C}) \simeq StrMon(\mathscr{A}, \mathscr{C})$. We call $\mathscr{Q}\mathscr{A}$ the monoidal Cauchy completion of $\mathscr{A}$.*

By [1, Section 3] or Section 2 above, the cocontinuous functor L_q, unique to within isomorphism, for which we have

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{y_{\mathscr{A}}} & \mathscr{P}\mathscr{A} \\ {\scriptstyle q}\downarrow & \cong & \downarrow{\scriptstyle L_q} \\ \mathscr{Q}\mathscr{A} & \xrightarrow[y_{\mathscr{Q}\mathscr{A}}]{} & \mathscr{P}\mathscr{Q}\mathscr{A} \end{array}$$

is in fact the left Kan extension along $y_{\mathscr{A}}$ of $y_{\mathscr{Q}\mathscr{A}} \cdot q$. Indeed, by [1, Section 3] $L_q F = Lan_{q^{\mathrm{op}}} F$. Because $\mathscr{A}$ is small, L_q has by [9, Theorem 4.51] a right adjoint which is easily seen to be $\mathscr{P}q$ given by composition with q^{op}. From Lindner's result [11, Proposition 3.4] or [9, Theorem 5.27] we know that this adjunction $L_q \dashv \mathscr{P}q$ is an adjoint equivalence. If $\mathscr{A}$ is monoidal this equivalence enriches to a monoidal equivalence.

Proposition 4.2. *Let $\mathscr{A}$ be small monoidal and let $\mathscr{P}\mathscr{A}$, $\mathscr{Q}\mathscr{A}$ and $\mathscr{P}\mathscr{Q}\mathscr{A}$ have the monoidal structures derived as above from that of $\mathscr{A}$. Then there are monoidal enrichments $\mathbf{L}_q : \mathscr{P}\mathscr{A} \to \mathscr{P}\mathscr{Q}\mathscr{A}$ and $\mathscr{P}\mathbf{q} : \mathscr{P}\mathscr{Q}\mathscr{A} \to \mathscr{P}\mathscr{A}$ of L_q and $\mathscr{P}_q$ respectively, such that $\mathbf{L}_q \dashv \mathscr{P}\mathbf{q}$ is an adjoint equivalence $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{Q}\mathscr{A}$ in Mon.*

Proof. Since $\mathbf{y}_{\mathscr{Q}\mathscr{A}}$ and $\mathbf{q}$ are strong monoidal, so is their composite $\mathbf{y}_{\mathscr{Q}\mathscr{A}} \cdot \mathbf{q}$. Thus by Theorem 3.3 there is a unique (up to isomorphism) strong monoidal cocontinuous functor $\mathbf{L}_q : \mathscr{P}\mathscr{A} \to \mathscr{P}\mathscr{Q}\mathscr{A}$ such that

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mathbf{y}_{\mathscr{A}}} & \mathscr{P}\mathscr{A} \\ {\scriptstyle \mathbf{q}}\downarrow & \cong & \downarrow{\scriptstyle \mathbf{L}_q} \\ \mathscr{Q}\mathscr{A} & \xrightarrow[\mathbf{y}_{\mathscr{Q}\mathscr{A}}]{} & \mathscr{P}\mathscr{Q}\mathscr{A} \end{array}$$

where the isomorphism is monoidal. Clearly $\mathbf{L}_q$ is a monoidal enrichment of (some choice for) L_q.

By Proposition 2.2 there is a unique monoidal enrichment $\mathscr{P}\mathbf{q}$ of $\mathscr{P}q$ giving a monoidal adjoint equivalence $\mathbf{L}_q \dashv \mathscr{P}\mathbf{q}$. □

We know from Lindner's [11, Proposition 3.9] that the opposite of any Cauchy complete $\mathcal{V}$-category is Cauchy complete and that if $\mathcal{A}$ is small the left Kan extension of $q^{op}_{\mathcal{A}^{op}} : \mathcal{A} \to \mathcal{Q}(\mathcal{A}^{op})^{op}$ along $q_{\mathcal{A}}$ gives an equivalence of $\mathcal{V}$-categories $\widehat{q^{op}_{\mathcal{A}^{op}}} : \mathcal{Q}\mathcal{A} \to \mathcal{Q}(\mathcal{A}^{op})^{op}$. Thus, since q preserves limits, it also preserves colimits, unlike most of the embeddings $y_{\mathcal{A}} : \mathcal{A} \to \mathcal{F}\mathcal{A}$. Again, this equivalence $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}(\mathcal{A}^{op})^{op}$ enriches to a monoidal equivalence if $\mathcal{A}$ is monoidal.

Proposition 4.3. *If $\mathcal{A}$ is small monoidal and $\mathcal{Q}\mathcal{A}$ is given the monoidal structure derived from that of $\mathcal{A}$, then $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}(\mathcal{A}^{op})^{op}$ monoidally.*

Proof. Since $q_{\mathcal{A}^{op}}$ is a strong monoidal functor, there is a strong monoidal $q^{op}_{\mathcal{A}^{op}} : \mathcal{A} \to \mathcal{Q}(\mathcal{A}^{op})^{op}$ obtained by taking the inverse of the isomorphism $\mathcal{A}(\mathcal{Q}, A) * \mathcal{A}(-, B) \cong \mathcal{A}(-, A \circ B)$. Since $\mathcal{Q}(\mathcal{A}^{op})^{op}$ is Cauchy complete, it is monoidally Cauchy complete. Hence by Corollary 4.1 there is a unique (up to isomorphism) strong monoidal functor $\widehat{q^{op}_{\mathcal{A}^{op}}}$ such that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{q_{\mathcal{A}}} & \mathcal{Q}\mathcal{A} \\ & q^{op}_{\mathcal{A}^{op}} \searrow \quad \cong \quad \swarrow q^{op}_{\mathcal{A}^{op}} & \\ & \mathcal{Q}(\mathcal{A}^{op})^{op}. & \end{array}$$

By the Cauchy completion property $\widehat{q^{op}_{\mathcal{A}^{op}}}$ is an enrichment of (some choice for) $\widehat{q^{op}_{\mathcal{A}^{op}}}$ which is an equivalence. Hence by Proposition 2.2, $\widehat{q^{op}_{\mathcal{A}^{op}}}$ which is an equivalence. Hence by Proposition 2.2, $q^{op}_{\mathcal{A}^{op}}$ is a monoidal equivalence. □

The last two propositions give us the following, which was proved in the non-monoidal context in [11, Corollary 3.7].

Theorem 4.4. *If $\mathcal{A}$ and $\mathcal{B}$ are small monoidal $\mathcal{V}$-categories, then the following are equivalent (where all equivalences shown are monoidal).*

(i) $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}\mathcal{B}$.

(ii) $\mathcal{P}\mathcal{A} \simeq \mathcal{P}\mathcal{B}$.

(iii) $\mathcal{P}(\mathcal{A}^{op}) = [\mathcal{A}, \mathcal{V}] \simeq [\mathcal{B}, \mathcal{V}] = \mathcal{P}(\mathcal{B}^{op})$.

(iv) $\mathcal{Q}(\mathcal{A}^{op}) \simeq \mathcal{Q}(\mathcal{B}^{op})$.

Proof. (i) $\Rightarrow$ (ii). $\mathcal{P}\mathcal{A} \simeq \mathcal{P}\mathcal{Q}\mathcal{A}$ (by Proposition 4.2) $\simeq \mathcal{P}\mathcal{Q}\mathcal{B}$ (since $\mathcal{P} : MON^{coop} \to MON$ is a 2-functor) $\simeq \mathcal{P}\mathcal{B}$.

(ii) $\Rightarrow$ (i). The equivalence of underlying $\mathcal{V}$-categories $\mathcal{P}\mathcal{A} \simeq \mathcal{P}\mathcal{B}$ restrict to an equivalence $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}\mathcal{B}$ which clearly enriches to a monoidal equivalence.

(iii) $\Leftrightarrow$ (iv). The dual of the above.

(i) $\Leftrightarrow$ (iv). Follows immediately from Proposition 4.3. □

Of course the corresponding results for the symmetric monoidal case hold.

We end this section with the observation that the one-object case of *monoidal* Morita equivalence is (unlike the non-monoidal case) trivial.

Proposition 4.5. *If $\mathscr{A}$ and $\mathscr{B}$ are one-object monoidal $\mathscr{V}$-categories, then there is a monoidal equivalence $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$ iff there is a monoidal isomorphism $\mathscr{A} \cong \mathscr{B}$.*

Proof. Let $*$ denote the one object of $\mathscr{A}$ or $\mathscr{B}$. Then if there is a monoidal equivalence $\Phi : \mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$ we have by Yoneda that $\mathscr{A} = \mathscr{A}(*,*) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-,*), \mathscr{A}(-,*)) = \mathscr{P}\mathscr{A}(J,J) \cong \mathscr{P}\mathscr{B}(\phi J, \phi J) \cong \mathscr{P}\mathscr{B}(\mathscr{B}(-,*), \mathscr{B}(-,*)) \cong \mathscr{B}$, where these isomorphisms are all monoidal. □

Thus, for example, if R is a commutative ring with unit, ring multiplication is a monoidal tensor product on R giving rise (via the above convolution formula from [4]) to $\otimes_R$ on R-Mod. In this case any monoidal (or even unit-preserving) equivalence R-Mod $\simeq$ S-Mod must come from an isomorphism $R \cong S$. (This fact was pointed out to me by Dr. Martin Ward.)

5. Closed monoidal Cauchy completions

If $\mathscr{A}$ is a small closed monoidal $\mathscr{V}$-category, we can use the equivalence $(\mathscr{Q}\mathscr{A})^{\mathrm{op}} \simeq \mathscr{Q}(\mathscr{A}^{\mathrm{op}})$ and the equivalence $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{Q}\mathscr{A}] \simeq [\mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A}, \mathscr{Q}\mathscr{A}]$ (from Proposition 3.1) to see that $\mathscr{Q}\mathscr{A}$ is closed. More generally, however, $\mathscr{Q}\mathscr{A}$ may be closed even when $\mathscr{A}$ is only *near* closed in a sense we will make precise.

Definition. A $\mathscr{V}$-functor $G : \mathscr{B} \to \mathscr{A}$ is a *near right adjoint* to $F : A \to \mathscr{B}$ if there are natural transformations $\eta : 1 \to GF$ and $\varepsilon : FG \to 1$ such that

$$\begin{array}{ccc} F & \xrightarrow{\;1\;} & F \\ & {\scriptstyle F\eta}\searrow \quad \nearrow{\scriptstyle \varepsilon F} & \\ & FGF & \end{array}$$

or equivalently if $\mathscr{B}(F-,B)$ is a retract of $\mathscr{A}(-,GB)$ in $\mathscr{P}\mathscr{A}$, naturally in B.

If $F : \mathscr{A} \to \mathscr{B}$, we let $\mathscr{Q}F$ denote the unique functor such that

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\;q_{\mathscr{A}}\;} & \mathscr{Q}\mathscr{A} \\ {\scriptstyle F}\downarrow & \cong & \downarrow{\scriptstyle \mathscr{Q}F} \\ \mathscr{B} & \xrightarrow[\;q_{\mathscr{B}}\;]{} & \mathscr{Q}\mathscr{B} \end{array}$$

From the case $\mathscr{F} = \mathscr{Q}$ of Proposition 2.1, $\mathscr{Q}F$ has a right adjoint iff $\mathscr{B}(F-,B) \in \mathscr{Q}\mathscr{A}$ for all $B \in \mathscr{B}$. As $\mathscr{Q}\mathscr{A}$ always contains the retracts of the representables (and consists solely of them when $\mathscr{V} = \mathbf{Set}$) we get:

Proposition 5.1. *For any $\mathscr{A}$ we have the following chain of implications*:

(i) *$F: \mathscr{A} \to \mathscr{B}$ has a near right adjoint.*

$\Rightarrow$ (ii) *$\mathscr{Q}F: \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{B}$ has a right adjoint.*

$\Rightarrow$ (iii) *$\mathscr{Q}F$ preserves any colimits which exist in $\mathscr{Q}\mathscr{A}$.*

$\Rightarrow$ (iv) *F preserves any colimits which exist in $\mathscr{A}$.*

If $\mathscr{V}=$ **Set**, *then we also have* (ii) $\Rightarrow$ (i), *though none of the implications can be reserved in general.*

Proof. For (iii) $\Rightarrow$ (iv) note that $q_{\mathscr{A}}$ preserves colimits and $q_{\mathscr{B}}$ is fully faithful. The rest follows from the above remarks. Alternatively, see Paré's result [12, Exercise 4 of Section IV.1] $\square$

Definition. If $\mathscr{A}$ is monoidal, we say that $\mathscr{A}$ is *near closed* if each $-\circ A: \mathscr{A} \to \mathscr{A}$ has a near right adjoint.

For any small monoidal $\mathscr{A}$, the tensor $*$ of $\mathscr{P}\mathscr{A}$ derived from that of $\mathscr{A}$ is always separately cocontinuous and therefore $\mathscr{P}\mathscr{A}$ is biclosed. Let $[F,-]$ denote the right adjoint in $\mathscr{P}\mathscr{A}$ to $-*F$ for $F \in \mathscr{P}\mathscr{A}$.

Corollary 5.2. *If $\mathscr{A}$ is small monoidal, then we have the following chain of implications*:

(i) *$\mathscr{A}$ is near closed.*

$\Rightarrow$ (ii) *$\mathscr{Q}\mathscr{A}$ is closed* (*with the restriction of* $[-,-]$ *as the internal hom functor*).

$\Rightarrow$ (iii) *For all $A \in \mathscr{A}$, $-\circ A: \mathscr{A} \to \mathscr{A}$ preserves any colimits which exist in $\mathscr{A}$.*

If $\mathscr{V}=$ **Set**, *then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). As $\mathscr{A}$ is near closed each $\mathscr{Q}(-\circ\mathscr{A}) \cong -*\mathscr{A}(-,A): \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ has a right adjoint $[\![\mathscr{A}(-,A),-]\!]$ by Proposition 5.1. Let $[\![-,-]\!]: \mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ be the unique functor (from Proposition 3.1) such that

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} & \xrightarrow{q^{\mathrm{op}} \otimes 1} & \mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} \\ & {\scriptstyle [\![q-,-]\!]} \searrow \quad \cong \quad \swarrow {\scriptstyle [\![-,-]\!]} & \\ & \mathscr{Q}\mathscr{A}. & \end{array}$$

To check that $[\![F,-]\!]: \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ is a right adjoint to $-*F: \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ we need to check the isomorphism $\mathscr{Q}\mathscr{A}(G*F,H) \cong \mathscr{Q}\mathscr{A}(G,[\![F,H]\!])$, but by Proposition 3.1 we need only check this for representable F and G for which we know it is true.

Finally, to see that $[\![-,-]\!]$ agrees with $[-,-]$ on $\mathscr{Q}\mathscr{A}$ note that for $A \in \mathscr{A}$, $[\![F,G]\!]A \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-,A),[\![F,G]\!]) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-,A)*F,G) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-,A),[F,G]) \cong [F,G]A$ naturally in A.

(ii) $\Rightarrow$ (iii). Follows immediately from Proposition 5.1 as does (ii) $\Rightarrow$ (i) in the case $\mathscr{V}=$ **Set**. $\square$

Acknowledgment

I would like to thank Ross Street and G.M. Kelly for suggesting this problem to me, for informing me about the relevant literature, and for their useful comments on presentation.

References

[1] M.H. Albert and G.M. Kelly, The closure of a class of colimits, J. Pure Appl. Algebra 51 (1988) 1–17.
[2] R. Betti, Cocompleteness over coverings, J. Austral. Math. Soc. Ser. A 39 (1985) 169–177.
[3] G.J. Bird, Morita theory for enriched categories, M. Sc. Thesis, University of Sydney, 1981.
[4] B. Day, On closed categories of functors, Lecture Notes in Mathematics 137 (Springer, Berlin, 1970) 1–38.
[5] J. Fisher-Palmquist and P.H. Palmquist, Morita contexts and equivalences of enriched functor categories, unpublished.
[6] J. Fisher-Palmquist and P.H. Palmquist, Morita contexts of enriched categories, Proc. Amer. Math. Soc. 50 (1975) 55–60.
[7] G.B. Im and G.M. Kelly, A universal property of the convolution monoidal structure, J. Pure Appl. Algebra 43 (1986) 75–88.
[8] G.M. Kelly, Doctrinal adjunction, Lecture Notes in Mathematics 420 (Springer, Berlin, 1974) 257–280.
[9] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Soc. Lecture Notes Series 64 (Cambridge University Press, Cambridge, 1982).
[10] F.H. Lawvere, Metric spaces, generalized logic and closed categories, Rend. Sem. Mat. Fis. Milano 43 (1974) 135–166.
[11] H. Lindner, Morita equivalence of enriched categories, Cahiers Topologie Géom. Différentielle 15 (1974) 377–397.
[12] S. MacLane, Categories for the Working Mathematician, (Springer, Berlin, 1971).
[13] K. Morita, Duality for modules and its application to the theory of rings with minimum condition, Science Reports of the Tokyo Kyoiku Daigaku 150 (1958) 83–142.
[14] R. Street, Absolute colimits in enriched categories, Cahiers Topologie Géom. Différentielle 24 (1983) 377–379.
[15] R. Street, Enriched categories and cohomology, Quaestiones Mathematicae 6 (1983) 265–283.
[16] R.F.C. Walters, Sheaves on sites as Cauchy-complete categories, J. Pure Appl. Algebra 24 (1982) 95–102.