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Traveling Wavefronts for Delayed Reaction-Diffusion Systems via a Fixed Point Theorem

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TO MY MOTHER, MME. CHANGMING WANG

By using Schauder's fixed point theorem, we prove some existence results for

an admissible pair of supersolution and subsolution which are easy to construct in practice. Finally, to illustrate our main results, we study the existence of traveling wavefronts for a delayed predator-prey model with diffusion as well as the reaction-diffusion system with the well-known Belousov–Zhabotinskii reaction, and the obtained results improve the existing ones. © 2001 Academic Press

Key Words: fixed point theorem; traveling wavefront; delayed reaction-diffusion system; quasimonotonicity; supersolution; subsolution; predator-prey model; Belousov–Zhabotinskii reaction.

1. INTRODUCTION

For a long time, the systems of nonlinear parabolic partial differential equations have attracted much attention due to their significant nature in sciences and engineering. In those systems, some special translation invariant solutions, such as traveling wave solutions, are studied as a paradigm for behavior exhibited in many model problems.

One of the most frequently encountered class of systems of parabolic partial differential equations is the reaction-diffusion systems

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \tag{1.1}$$

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where $u \in \mathbb{R}^n$, D is an appropriate matrix, and Δ is the Laplacian operator. The first instances in which traveling wave solutions were investigated were in the celebrated papers of Kolmogorov *et al.* [1] and Fisher [2]. Since then, a large number of research papers have been devoted to the study of wave solutions of various parabolic systems and the number has been continuously increasing.

On the traveling wave problem for scalar reaction-diffusion equations, much has been done by using the phase plane technique. On those problems for reaction-diffusion systems, many papers have also been published [3, 4, 6, 9, 12–19]. Since the classical phase plane technique is not applicable any more for reaction-diffusion systems, some distinct methods, such as degree theory method and the Conley index method, have been developed [7, 12–14].

It is well known that time delay should be and has been incorporated into many realistic models in applications. However, to the best of our knowledge, it seems that little has been done for traveling waves of scalar reaction-diffusion equations with delay, not to mention the study of delayed systems of reaction-diffusion equations. One exception is the pioneering work of Schaaf [5], where two scalar reaction-diffusion equations with a discrete delay for the so-called Huxley nonlinearity as well as Ficher nonlinearity were systematically studied, using the phase plane technique, the maximum principle for parabolic functional differential equations, and the general theory of ordinary functional differential equations. The other is the recent work by Zou and Wu [8, 10, 11], where some existence results for traveling wavefronts of delayed reaction-diffusion systems with quasimonotonicity reactions were obtained by developing a technique of monotone iteration for parabolic systems. More precisely, the authors employed the idea of upper-lower solutions and an iteration scheme to construct a monotone sequence of upper solutions which is proved to converge to a solution of the corresponding wave equation of the reaction-diffusion system under consideration. It is worth mentioning that the initial iteration is an upper solution of the wave equation, which converge to two distinct trivial solutions of the wave equation when $t \to -\infty$ and $t \to +\infty$, respectively.

The present paper is motivated by the work of Zou and Wu [10, 11]. The purpose of this paper is to tackle the existence of traveling wavefront solutions of delayed reaction-diffusion systems by using some fixed point theorems. As far as traveling wavefront solutions are concerned, the corresponding wave equation of the delayed reaction-diffusion system under consideration must have two trivial solutions. In order to obtain a non-trivial traveling wavefront, we also use the idea of lower-upper solutions to construct in an appropriate Banach space a closed bounded covex set in which there are no trivial solutions of the corresponding wave equation. One important feature of our method, which is different from the work of Zou and Wu [10, 11], is that the upper solution of the wave equation is not necessary to converge to two distinct trivial solutions when $t \to -\infty$ and $t \to +\infty$ respectively. In particular, some components of the upper solution may be constants. Thus, whenever systems are concerned, our method used in this paper is significant.

The rest of this paper is organized as follows. In Section 2, we reduce the existence of traveling wavefront solutions of delayed reaction-diffusion systems in which reaction terms are monotone with respect to delayed arguments to the existence of an admissible pair of supersolution and subsolution of the corresponding wave equation, which are used by Schaaf [5] and are easy to construct in practice. In Section 3, as examples, we study the existence of traveling wavefront solutions for a delayed predator-prey model with diffusion as well as the reaction-diffusion system with the Belousov–Zhabotinskii reaction and a descrete delay.

2. MAIN RESULTS

In the present paper, we will consider the following system of reaction-diffusion equations with time delay

$$\frac{\partial}{\partial t} u(t, x) = D \frac{\partial^2}{\partial x^2} u(t, x) + f(u_t(x)), \tag{2.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $u \in \mathbb{R}^n$, $D = \operatorname{diag}(d_1, d_2, ..., d_n)$ with $d_i > 0$, $i = 1, ..., n$, f: $C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and for any fixed $x \in \mathbb{R}$, $u_t(x) \in C([-\tau, 0], \mathbb{R}^n)$ is defined by $u_t(x)(\theta) = u(t+\theta, x)$, $\theta \in [-\tau, 0]$.

A *traveling wave solution* of (2.1) is a special translation invariant solution of the form $u(t, x) = \varphi(x+ct)$, where $\varphi \in C^2(\mathbb{R}, \mathbb{R}^n)$ is the profile of the wave that propagates through the one-dimensional spatial domain at a constant velocity $c > 0$. Substituting $u(t, x) = \varphi(x+ct)$ into (2.1) and letting $s = x + ct$, we obtain the corresponding wave equation

$$D\varphi''(s) - c\varphi'(s) + f^c(\varphi_s) = 0, \qquad s \in \mathbb{R}, \tag{2.2}$$

where f^c: $C([-c\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is given by

$$f^c(\psi) = f(\psi^c), \qquad \psi^c(\theta) = \psi(c\theta), \qquad \theta \in [-\tau, 0].$$

If for some $c > 0$, (2.2) has a monotone solution φ defined on $\mathbb{R}$ such that

$$\lim_{s \to -\infty} \varphi(s) = u_-, \qquad \lim_{s \to +\infty} \varphi(s) = u_+ \tag{2.3}$$

exist, then $u(t, x) = \varphi(x+ct)$ is called a *wavefront* with speed c.

In the remainder of this paper, we will use the usual notations for the standard ordering in $\mathbb{R}^n$. That is, for $u=(u_1, ..., u_n)^T$ and $v=(v_1, ..., v_n)^T$, we denote $u \leqslant v$ if $u_i \leqslant v_i$, $i=1, ..., n$, and $u<v$ if $u \leqslant v$ but $u \neq v$. In particular, we will denote $u \ll v$ if $u \leqslant v$ but $u_i \neq v_i$, $i=1, ..., n$. If $u \leqslant v$, we also denote $(u, v]=\{w \in \mathbb{R}^n : u<w \leqslant v\}$, $[u, v)=\{w \in \mathbb{R}^n : u \leqslant w<v\}$.

LEMMA 2.1 [11]. *If* (2.2) *and* (2.3) *have a monotone solution, then* $f^c(\hat{u}_+) = f^c(\hat{u}_-)=0$, *where* $u \in \mathbb{R}^n$, $\hat{u}$ *denotes the constant vector fuction on* $[-c\tau, 0]$ *taking the value u.*

Without loss of generality, we can assume $u_-=0$ and $u_+=K>0$. More precisely, we assume, throughout the remainder of this paper, the following holds

(H1) $f(\hat{0})=f(\hat{K})=0$.

Obviously, we should replace (2.3) with

$$\lim_{s \to -\infty} \varphi(s)=0, \qquad \lim_{s \to +\infty} \varphi(s)=K. \tag{2.4}$$

In this paper, we explore the existence of wave fronts of (2.1) where the reaction term f is monotone with respect to the delayed arguments. In other words, we assume the following quasimonotonicity condition:

(H2) There exists a matrix $\beta=\operatorname{diag}(\beta_1, ..., \beta_n)$ with $\beta_i \geqslant 0$ such that

$$f(\varphi)-f(\psi)+\beta(\varphi(0)-\psi(0)) \geqslant 0$$

for $\varphi, \psi \in C([-\tau, 0], \mathbb{R}^n)$ with $0 \leqslant \psi(s) \leqslant \varphi(s) \leqslant K$, $s \in [-\tau, 0]$.

Let $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ denotes the supremum norm in $C([-\tau, 0], \mathbb{R}^n)$. We also need the following continuty hypotheses:

(H3) There are two constants $\sigma>0$ and $L>0$ such that

$$|f(\varphi)-f(\psi)| \leqslant L\, \|\varphi-\psi\|^\sigma$$

for $\varphi, \phi \in C([-\tau, 0], \mathbb{R}^n)$ with $0 \leqslant \varphi(s)$, $\psi(s) \leqslant K$, $s \in [-\tau, 0]$.

Define the operator H: $C(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ by

$$H(\varphi)(t)=f^c(\varphi_t)+\beta\varphi(t), \qquad \varphi \in C(\mathbb{R}, \mathbb{R}^n). \tag{2.5}$$

Let

$$C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)=\{\varphi \in C(\mathbb{R}, \mathbb{R}^n) : 0 \leqslant \varphi(s) \leqslant K, s \in \mathbb{R}\}.$$

Then we have the following

LEMMA 2.2 [11]. *Assume that* (H1) *and* (H2) *hold. Then*

(i) $0 \leqslant H(\varphi)(t) \leqslant \beta K$, *for* $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$;

(ii) $H(\varphi)(t)$ *is nondecreasing in* $t \in \mathbb{R}$, *if* $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *is nondecreasing in* $t \in \mathbb{R}$;

(iii) $H(\psi)(t) \leqslant H(\varphi)(t)$ *for* $t \in \mathbb{R}$, *if* $\varphi, \psi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *are given so that* $\psi(t) \leqslant \varphi(t)$ *for* $t \in \mathbb{R}$.

Clearly, with the above notations, (2.2) is equivalent to the following system of ordinary differential equations

$$D\varphi''(t) - c\varphi'(t) - \beta\varphi(t) + H(\varphi)(t) = 0, \qquad t \in \mathbb{R}. \tag{2.6}$$

Without loss of generality, we assume that $\beta_i > 0$ for every $i = 1, ..., n$, and let

$$\lambda_{1i} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \qquad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i}.$$

Define the operator F: $C_{[0, K]}(\mathbb{R}, \mathbb{R}^n) \to C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ by

$$(F\varphi)_i(t) = \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} H_i(\varphi)(s)\, ds + \int_{t}^{\infty} e^{\lambda_{2i}(t-s)} H_i(\varphi)(s)\, ds \right] \tag{2.7}$$

for $i = 1, 2, ..., n$ and $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$.

It is easy to show that F: $C_{[0, K]}(\mathbb{R}, \mathbb{R}^n) \to C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ is a well defined map and and for any $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$,

$$D(F\varphi)'' - c(F\varphi)' - \beta(F\varphi) + H(\varphi) = 0.$$

Thus, a fixed point of F is a solution of (2.6). Furthermore, we have

LEMMA 2.3. *Assume that* (H1) *and* (H2) *hold. Then*

(i) $F\varphi(t)$ *is nondecreasing in* $\mathbb{R}$, *if* $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *is nondecreasing in* $t \in \mathbb{R}$;

(ii) $F\varphi(t) \leqslant F\varphi(t)$ *for* $t \in \mathbb{R}$, *if* $\varphi, \psi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *are given so that* $\psi(t) \leqslant \varphi(t)$ *for* $t \in \mathbb{R}$.

Proof. Part (ii) follows immediately from Lemma 2.2(iii).

To prove (i), let $t \in \mathbb{R}$ and $s > 0$ be given. Then for any $i = 1, ..., n$, we obtain

$$\begin{aligned}
&(F\varphi)_i (t+s) - (F\varphi)_i (t) \\
&= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t+s} e^{\lambda_{1i}(t+s-\theta)} H_i(\varphi)(\theta)\, d\theta + \int_{t+s}^{\infty} e^{\lambda_{2i}(t+s-\theta)} H_i(\varphi)(\theta)\, d\theta \right] \\
&\quad - \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-\theta)} H_i(\varphi)(\theta)\, d\theta + \int_{t}^{\infty} e^{\lambda_{2i}(t-\theta)} H_i(\varphi)(\theta)\, d\theta \right] \\
&= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-\theta)} (H_i(\varphi)(s+\theta) - H_i(\varphi)(\theta))\, d\theta \right. \\
&\quad \left. + \int_{t}^{\infty} e^{\lambda_{2i}(t-\theta)} (H_i(\varphi)(s+\theta) - H_i(\varphi)(\theta))\, d\theta \right] \\
&\geqslant 0 \qquad \text{(by Lemma 2.2(ii))}.
\end{aligned}$$

This completes the proof.

Let $\rho > 0$ be such that $\rho < \min\{-\lambda_{1i}, \lambda_{2i}: i = 1, ..., n\}$, and let

$$B_\rho(\mathbb{R}, \mathbb{R}^n) = \{\varphi \in C(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} |\varphi(t)|\, e^{-\rho |t|} < \infty\},$$

$$|\varphi|_\rho = \sup_{t \in \mathbb{R}} |\varphi(t)|\, e^{-\rho |t|}.$$

Then it is easy to check that $(B_\rho(\mathbb{R}, \mathbb{R}^n), |\cdot|_\rho)$ is a Banach space.

LEMMA 2.4. *Assume that* (H1), (H2), *and* (H3) *hold. Then* $F: C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ *is continuous with respect to the norm* $|\cdot|_\rho$ *in* $B_\rho(\mathbb{R}, \mathbb{R}^n)$.

Proof. First of all, we claim that $H: C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to B_\rho(\mathbb{R}, \mathbb{R}^n)$ is continuous. In fact, for any fixed $\varepsilon > 0$, take $T > 0$ such that

$$2^\sigma L\, |K|^\sigma\, e^{-\rho T} < \varepsilon/2. \tag{2.8}$$

Let $\delta > 0$ be such that

$$\delta < \min\left\{ \left(\frac{\varepsilon}{2L}\right)^{1/\sigma} e^{-\rho(T+c\tau)}, \frac{\varepsilon}{2\, \|\beta\|} \right\}, \tag{2.9}$$

where $\|\beta\|$ denotes the matrix norm induced by the norm $|\cdot|$ in $\mathbb{R}^n$. Then, if $\varphi, \psi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ satisfy

$$|\varphi - \psi|_\rho = \sup_{t \in \mathbb{R}} |\varphi(t) - \psi(t)|\, e^{-\rho |t|} < \delta,$$

we have

$$|\varphi(t)-\psi(t)| \leqslant \delta e^{\rho(T+c\tau)} < \left(\frac{\varepsilon}{2L}\right)^{1/\sigma}, \qquad t \in [-T-c\tau, T].$$

Therefore, for $t \in [-T, T]$, we have

$$\begin{aligned}|H(\varphi)(t)-H(\psi)(t)|\, e^{-\rho|t|} &\leqslant |f^c(\varphi_t)-f^c(\psi_t)| + \|\beta\| \cdot |\varphi(t)-\psi(t)|\, e^{-\rho|t|}\\ &\leqslant L\, \|\varphi_t-\psi_t\|^\sigma + \|\beta\| \cdot |\varphi-\psi|_\rho\\ &< \varepsilon/2+\varepsilon/2=\varepsilon,\end{aligned}$$

and for $|t| \geqslant T$, we have

$$\begin{aligned}|H(\varphi)(t)-H(\psi)(t)|\, e^{-\rho|t|} &\leqslant |f^c(\varphi_t)-f^c(\psi_t)|\, e^{-\rho T}\\ &\quad + \|\beta\| \cdot |\varphi(t)-\psi(t)|\, e^{-\rho|t|}\\ &\leqslant 2^\sigma L\, |K|^\sigma\, e^{-\rho T} + \|\beta\| \cdot |\varphi-\psi|_\rho\\ &< \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Therefore, $|H(\varphi)-H(\psi)|_\rho \leqslant \varepsilon$. That is, $H\colon C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to B_\rho(\mathbb{R}, \mathbb{R}^n)$ is continuous.

Now, we show that $F\colon C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is continuous.

For $t \geqslant 0$, we find

$$\begin{aligned}&|(F\varphi)_i(t)-(F\psi)_i(t)|\\ &\quad\leqslant \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}\, |H_i(\varphi)(s)-H_i(\psi)(s)|\, ds\right.\\ &\quad\quad\left.+\int_t^\infty e^{\lambda_{2i}(t-s)}\, |H_i(\varphi)(s)-H_i(\psi)(s)|\, ds\right]\\ &\quad= \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)+\rho|s|}\, |H_i(\varphi)(s)-H_i(\psi)(s)|\, e^{-\rho|s|}\, ds\right.\\ &\quad\quad\left.+\int_t^\infty e^{\lambda_{2i}(t-s)+\rho|s|}\, |H_i(\varphi)(s)-H_i(\psi)(s)|\, e^{-\rho|s|}\, ds\right]\\ &\quad\leqslant \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\int_0^t e^{\lambda_{1i}(t-s)+\rho s}\, ds + \int_{-\infty}^0 e^{\lambda_{1i}(t-s)-\rho s}\, ds\right.\\ &\quad\quad\left.+\int_t^\infty e^{\lambda_{2i}(t-s)+\rho s}\, ds\right] |H(\varphi)-H(\psi)|_\rho\\ &\quad= \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\frac{\lambda_{2i}-\lambda_{1i}}{(\rho-\lambda_{1i})(\lambda_{2i}-\rho)}\, e^{\rho t} + \frac{2\rho}{\lambda_{1i}^2-\rho^2}\, e^{\lambda_{1i}t}\right] |H(\varphi)-H(\psi)|_\rho .\end{aligned}$$

Hence, for $t \geqslant 0$, we have

$$
\begin{aligned}
&|(F\varphi)_i(t) - (F\psi)_i(t)|\, e^{-\rho\,|t|} \\
&\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\frac{\lambda_{2i} - \lambda_{1i}}{(\rho - \lambda_{1i})(\lambda_{2i} - \rho)} + \frac{2\rho}{\lambda_{1i}^2 - \rho^2}\, e^{(\lambda_{1i} - \rho)\, t} \right] |H(\varphi) - H(\psi)|_\rho \\
&\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\frac{\lambda_{2i} - \lambda_{1i}}{(\rho - \lambda_{1i})(\lambda_{2i} - \rho)} + \frac{2\rho}{\lambda_{1i}^2 - \rho^2} \right] |H(\varphi) - H(\psi)|_\rho. \quad (2.10)
\end{aligned}
$$

For $t < 0$, we find

$$
\begin{aligned}
&|(F\varphi)_i(t) - (F\psi)_i(t)| \\
&\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s) - \rho s}\, ds + \int_{t}^{0} e^{\lambda_{2i}(t-s) - \rho s}\, ds \right. \\
&\quad \left. + \int_{0}^{\infty} e^{\lambda_{2i}(t-s) + \rho s}\, ds \right] |H(\varphi) - H(\psi)|_\rho \\
&= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\frac{\lambda_{2i} - \lambda_{1i}}{-(\rho + \lambda_{1i})(\lambda_{2i} + \rho)}\, e^{-\rho t} \right. \\
&\quad \left. + \frac{2\rho}{(\lambda_{2i} - \rho)(\lambda_{2i} + \rho)}\, e^{\lambda_{2i} t} \right] |H(\varphi) - H(\psi)|_\rho.
\end{aligned}
$$

Hence, for $t < 0$, we have

$$
\begin{aligned}
&|(F\varphi)_i(t) - (F\psi)_i(t)|\, e^{-\rho\,|t|} \\
&\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\frac{\lambda_{2i} - \lambda_{1i}}{-(\rho + \lambda_{1i})(\lambda_{2i} + \rho)} + \frac{2\rho}{\lambda_{2i}^2 - \rho^2}\, e^{(\lambda_{2i} + \rho)\, t} \right] |H(\varphi) - H(\psi)|_\rho \\
&\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\frac{\lambda_{2i} - \lambda_{1i}}{-(\rho + \lambda_{1i})(\lambda_{2i} + \rho)} + \frac{2\rho}{\lambda_{2i}^2 - \rho^2} \right] |H(\varphi) - H(\psi)|_\rho. \quad (2.11)
\end{aligned}
$$

Thus, it follows from (2.10) and (2.11) that $F: C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \to C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is continuous with respect to the norm $|\cdot|_\rho$ in $B_\rho(\mathbb{R}, \mathbb{R}^n)$ and the proof is complete.

For convenience, we introduce the definition of an upper (or a lower) solution of (2.2).

Definition 2.1. A twice continuous differentiable function $\rho: \mathbb{R} \to \mathbb{R}^n$ is called an *upper solution* of (2.2), if ρ satisfies

$$
D\rho''(t) - c\rho'(t) + f^c(\rho_t) \leqslant 0, \qquad t \in \mathbb{R}. \tag{2.12}
$$

A *lower solution* of (2.2) is defined in a similar way by reversing the inequality in (2.12).

Now, we are in the position to state and prove the following existence theorem.

THEOREM 2.1. *Assume that* (H1), (H2), *and* (H3) *hold. Suppose that* (2.2) *has an upper solution* $\bar{\rho} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ *and a lower solution* $\underline{\rho} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ *satisfying*

(I) $\sup_{s \leqslant t} \underline{\rho}(s) \leqslant \bar{\rho}(t)$, *for* $t \in \mathbb{R}$;

(II) $f(\hat{u}) \neq 0$, *for* $u \in (0, \inf_{t \in \mathbb{R}} \bar{\rho}(t)] \cup [\sup_{t \in \mathbb{R}} \underline{\rho}(t), K)$.

Then (2.2) *and* (2.4) *have a monotone solution. That is,* (2.1) *has a traveling wavefront solution.*

Proof. Let

$$M = \sqrt{\frac{\max\{\beta_1, ..., \beta_n\}}{\min\{d_1, ..., d_n\}}},$$

and

$$\Gamma = \left\{ \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) : \quad \begin{array}{ll} \text{(i)} & \varphi \text{ is nondecreasing in } \mathbb{R}; \\ \text{(ii)} & \underline{\rho}(t) \leqslant \varphi(t) \leqslant \bar{\rho}(t), \ \forall t \in \mathbb{R}; \\ \text{(iii)} & |\varphi(u) - \varphi(v)| \leqslant M\, |u - v|, \ \forall u, v \in \mathbb{R}. \end{array} \right\}$$

Let $W(t) = (F\bar{\rho})(t) - \bar{\rho}(t)$, $t \in \mathbb{R}$. Since

$$D(F\bar{\rho})''(t) - c(F\bar{\rho})'(t) - \beta(F\bar{\rho})(t) + H(\bar{\rho})(t) = 0$$

and

$$D\bar{\rho}''(t) - c\bar{\rho}'(t) + f^c(\bar{\rho}_t) \leqslant 0,$$

it follows that

$$DW''(t) - cW'(t) - \beta W(t) \geqslant 0, \qquad t \in \mathbb{R}.$$

Denote $r(t) = DW''(t) - cW'(t) - \beta W(t)$. Then $r(t)$ is continuous, bounded, and non-negative on $\mathbb{R}$, and the fundamental theory of second order linear ordinary differential equations yields

$$\begin{aligned} W_i(t) &= e^{\lambda_{1i} t} c_{1i} + e^{\lambda_{2i} t} c_{2i} \\ &\quad - \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} r_i(s)\, ds + \int_t^{\infty} e^{\lambda_{2i}(t-s)} r_i(s)\, ds \right], \end{aligned} \tag{2.13}$$

where $c_{1i}, c_{2i} \in \mathbb{R}$ and $i=1, ..., n$. Since $W_i(t)$ is bounded on $\mathbb{R}$ and

$$\left|\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} r_i(s)\, ds + \int_{t}^{\infty} e^{\lambda_{2i}(t-s)} r_i(s)\, ds\right]\right|$$
$$\leqslant \frac{1}{\beta_i} \sup_{s\in\mathbb{R}} r_i(s) < \infty,$$

it follows from (2.13) that $c_{1i}=c_{2i}=0$, and hence

$$W_i(t) = -\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} r_i(s)\, ds + \int_{t}^{\infty} e^{\lambda_{2i}(t-s)} r_i(s)\, ds\right] \leqslant 0.$$

This proves that $(F\bar{\rho})(t) \leqslant \bar{\rho}(t)$, $t\in\mathbb{R}$.

In a similar way, we may show that $(F\underline{\rho})(t) \geqslant \underline{\rho}(t)$, $t\in\mathbb{R}$.

Let $\bar{\varphi}(t) = \sup_{s\leqslant t} \underline{\rho}(s)$, then $\bar{\varphi}(t)$ is nondecreasing in $\mathbb{R}$ and it follows from Conditon (I) that

$$\underline{\rho}(t) \leqslant \bar{\varphi}(t) \leqslant \bar{\rho}(t), \qquad t\in\mathbb{R}$$

which together with Lemma 2.3 yields

$$\underline{\rho}(t) \leqslant (F\underline{\rho})(t) \leqslant (F\bar{\varphi})(t) \leqslant (F\bar{\rho})(t) \leqslant \bar{\rho}(t), \qquad t\in\mathbb{R}.$$

It is also easy to check that

$$|(F\bar{\varphi})(u) - (F\bar{\varphi})(v)| \leqslant M\,|u-v|, \qquad u, v\in\mathbb{R}.$$

Consequently, $F\bar{\varphi}\in\Gamma$, implying that Γ is nonempty. It is also easy to show that Γ is convex and compact in $B_\rho(\mathbb{R}, \mathbb{R}^n)$.

Moreover, by Lemma 2.3 and a similar argument as above, we may prove that

$$F(\Gamma) \subset \Gamma.$$

Therefore, the well-known Schauder's fixed point theorem implies that F has a fixed point $\varphi\in\Gamma$, which is a solution of (2.2).

Also, we have

$$0 \leqslant \varphi_- =: \lim_{t\to-\infty} \varphi(t) \leqslant \inf_{t\in\mathbb{R}} \bar{\rho}(t) \tag{2.14}$$

and

$$\sup_{t\in\mathbb{R}} \underline{\rho}(t) \leqslant \varphi_+ =: \lim_{t\to\infty} \varphi(t) \leqslant K. \tag{2.15}$$

Hence, by Lemma 2.1, we have

$$f(\hat{\varphi}_-) = f^c(\hat{\varphi}_-) = 0, \qquad f(\hat{\varphi}_+) = f^c(\hat{\varphi}_+) = 0.$$

Therefore, it follows from (2.14), (2.15), and Condition (II) that

$$\varphi_- = \lim_{t \to -\infty} \varphi(t) = 0, \qquad \varphi_+ = \lim_{t \to \infty} \varphi(t) = K.$$

Thus $\varphi(t)$ is a monotone solution of (2.2) and (2.4), and this completes the proof.

DEFINITION 2.2. A continuous function $\varphi: \mathbb{R} \to \mathbb{R}^n$ is called a *supersolution* of (2.2), if there exist constants T_i $(i = 1, ..., m)$ such that $\varphi(t)$ is twice continuously differentiable in $\mathbb{R} \setminus \{T_i: i = 1, ..., m\}$ and satisfies

$$D\varphi''(t) - c\varphi'(t) + f^c(\varphi_t) \leqslant 0, \qquad \text{a.e. on } \mathbb{R}. \tag{2.16}$$

A *subsolution* of (2.2) is defined in a similar way by reversing the inequality in (2.16).

LEMMA 2.5. *If* $\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *is a supersolution of* (2.2) *and* $\varphi'(t+) \leqslant \varphi'(t-)$, $\forall t \in \mathbb{R}$, *then* $F\varphi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}^n)$ *is an upper solution of* (2.2).

Proof. Without loss of generality, we assume that $\varphi(t)$ is continuously differentiable in $\mathbb{R} \setminus \{T_i: i = 1, ..., m\}$ with $T_m < T_{m-1} < \cdots < T_1$. Denote $T_0 = +\infty$ and $T_{m+1} = -\infty$. For any $1 \leqslant i \leqslant n$ and $t \in (T_{k+1}, T_k)$, $0 \leqslant k \leqslant m$, it follows from (2.16) that

$$\begin{aligned}
(F\varphi)_i(t) &\leqslant \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} (-d_i \varphi_i''(s) + c\varphi_i'(s) + \beta_i \varphi_i(s))\, ds \right. \\
&\quad \left. + \int_t^\infty e^{\lambda_{2i}(t-s)} (-d_i \varphi_i''(s) + c\varphi_i'(s) + \beta_i \varphi_i(s))\, ds \right] \\
&= \varphi_i(t) + \frac{1}{\lambda_{2i} - \lambda_{1i}} \left[\sum_{j=k+1}^{m} e^{\lambda_{1i}(t - T_j)} (\varphi_i'(T_j +) - \varphi_i'(T_j -)) \right. \\
&\quad \left. + \sum_{j=1}^{k} e^{\lambda_{2i}(t - T_j)} (\varphi_i'(T_j +) - \varphi_i'(T_j -)) \right] \\
&\leqslant \varphi_i(t)
\end{aligned}$$

which together with Lemma 2.1 yields

$$\begin{aligned} D(F\varphi)''(t) - c(F\varphi)'(t) + f^c((F\varphi)_t) &= D(F\varphi)''(t) - c(F\varphi)'(t) - \beta F\varphi(t) + H(F\varphi)(t) \\ &\leqslant D(F\varphi)''(t) - c(F\varphi)'(t) - \beta F\varphi(t) + H(\varphi)(t) \\ &= 0. \end{aligned}$$

Noting that $F\varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) \cap C^2(\mathbb{R}, \mathbb{R}^n)$, we conclude that $F\varphi$ is an upper solution of (2.2). The proof is complete.

LEMMA 2.6. *If $\varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is a subsolution of (2.2) and $\varphi'(t+) \geqslant \varphi'(t-)$, $\forall t \in \mathbb{R}$, then $F\varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ is a lower solution of (2.2).*

Proof. The proof is similar to that of Lemma 2.5 and is omitted.

THEOREM 2.2. *Assume that (H1), (H2), and (H3) hold. Suppose that (2.2) has a supersolution $\bar{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ and a subsolution $\underline{\varphi} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ satisfying*

(I) $\sup_{s \leqslant t} \underline{\varphi}(s) \leqslant \bar{\varphi}(t)$, *for* $t \in \mathbb{R}$;

(II) $f(\hat{u}) \neq 0$, *for* $u \in (0, \inf_{t \in \mathbb{R}} \bar{\varphi}(t)] \cup [\sup_{t \in \mathbb{R}} \underline{\varphi}(t), K)$;

(III) $\bar{\varphi}'(t+) \leqslant \bar{\varphi}'(t-)$, *for* $t \in \mathbb{R}$;

(IV) $\underline{\varphi}'(t+) \geqslant \underline{\varphi}'(t-)$, *for* $t \in \mathbb{R}$.

Then (2.2) and (2.4) have a monotone solution. That is, (2.1) has a traveling wavefront solution.

Proof. Let $\bar{\rho}(t) = (F\bar{\varphi})(t)$ and $\underline{\rho}(t) = (F\underline{\varphi})(t)$. Then by (III), (IV), Lemma 2.5, and Lemma 2.6, $\bar{\rho} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ and $\underline{\rho} \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n)$ are an upper solution and a lower solution of (2.2), respectively.

Denote $\hat{\varphi}(t) = \sup_{s \leqslant t} \underline{\varphi}(s)$. Then $\hat{\varphi}(t)$ is nondecreasing in $t \in \mathbb{R}$. It follows from Lemma 2.3 and Condition (I) that $F\hat{\varphi}(t)$ is nondecreasing in $t \in \mathbb{R}$ and

$$\sup_{s \leqslant t} \underline{\rho}(s) \leqslant \sup_{s \leqslant t} F\hat{\varphi}(s) = F\hat{\varphi}(t) \leqslant \bar{\rho}(t), \qquad t \in \mathbb{R}. \tag{2.17}$$

On the other hand, by Conditions (III), (IV) and a similar argument as used in the proof of Lemma 2.5, we can show that

$$\bar{\rho}(t) \leqslant \bar{\varphi}(t), \qquad \underline{\rho}(t) \geqslant \underline{\varphi}(t), \qquad t \in \mathbb{R}. \tag{2.18}$$

It follows from Condition (II) and (2.18) that

$$f(\hat{u}) \neq 0, \qquad u \in (0, \inf_{t \in \mathbb{R}} \bar{\rho}(t)] \cup [\sup_{t \in \mathbb{R}} \underline{\rho}(t), K). \tag{2.19}$$

Thus, by virtue of Theorem 2.1, (2.2) and (2.4) have a monotone solution and this completes the proof.

Theorem 2.2 reduces the existence of traveling wavefronts of (2.1) to the existence of an admissible pair of subsolution and supersolution of (2.2) satisfying some additional conditions. The following theorem shows that there is a natural way to construct a supersolution of (2.2).

THEOREM 2.3. *Assume that* (H1) *and* (H2) *hold. For* $\lambda_i \geqslant 0$, $T_i \in \mathbb{R}$, $i = 1, ..., n$, *let*

$$\varphi_i(t) = \min\{K_i e^{\lambda_i(t - T_i)}, K_i\}, \qquad i = 1, ..., n.$$

If for every i, we have

$$f_i^c(\varphi_t) \leqslant 0, \qquad t \in [T_i, T_i + c\tau],$$

$$d_i \lambda_i^2 - c\lambda_i + \frac{1}{K_i} \sup_{t \leqslant T_i} e^{-\lambda_i(t - T_i)} f_i^c(\varphi_i) \leqslant 0,$$

then $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))^T$ *is a supersolution of* (2.2) *satisfying* $\varphi'(t+) \leqslant \varphi'(t-)$, $\forall t \in \mathbb{R}$.

Proof. The proof is easy and therefore is omitted.

3. APPLICATIONS

In this section, we shall give some applications of our main results obtained in previous section.

At first, we consider the delayed predator-prey model with diffusion

$$\begin{cases} \dfrac{\partial}{\partial t} u(x, t) = d_1 \dfrac{\partial^2}{\partial x^2} u(x, t) + ru(x, t) \left[\left(1 - \dfrac{u(x, t)}{P} \right) - av(x, t) \right]; \\ \dfrac{\partial}{\partial t} v(x, t) = d_2 \dfrac{\partial^2}{\partial x^2} v(x, t) + v(x, t)[-v + bu(x, t - \tau)], \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}$, $\tau \geqslant 0$, r, a, b, v and P, the prey carrying capacity, are positive constants. $u(x, t)$, $v(x, t)$ denote the population density of prey and predator, respectively. d_1 and d_2 are the diffusion coefficients. For a detailed description of this model equation, we refer to Murray [17]. We mention that in the case where $\tau = 0$, by using some techniques different from ours, the existence of traveling wavefront solutions of (3.1) have been studied by Murray [17] (for $d_1 = 0$, $d_2 > 0$) and by Dunbar [18, 19] (for $d_1 > 0$, $d_2 > 0$).

We are only interested in the existence of waves of pursuit and evasion. In other words, we shall consider here only the case where $d_1 > 0$, $d_2 > 0$.

In the sequel, we always assume (3.1) has a positive steady state, i.e., we assume that

$$P > \frac{v}{b}. \tag{3.2}$$

In what follows, we will seek a traveling wavefront solution of (3.1) with the boundary conditions

$$\begin{cases} u(-\infty, t) = 0, & v(-\infty, t) = 0; \\ u(+\infty, t) = \dfrac{v}{b}, & v(+\infty, t) = \dfrac{1}{a}\left(1 - \dfrac{v}{Pb}\right). \end{cases} \tag{3.3}$$

Clearly, the wave equation corresponding to (2.2) in this case is

$$\begin{cases} c\phi_1'(t) = d_1\phi_1''(t) + r\phi_1(t)[(1 - \phi_1(t)/P) - a\phi_2(t)]; \\ c\phi_2'(t) = d_2\phi_2''(t) + \phi_2(t)[-v + b\phi_1(t - c\tau)], \end{cases} \tag{3.4}$$

and the corresponding asymptotic boundary condition is

$$\begin{cases} \lim_{t\to-\infty} \phi_1(t) = 0, & \lim_{t\to-\infty} \phi_2(t) = 0; \\ \lim_{t\to+\infty} \phi_1(t) = \dfrac{v}{b}, & \lim_{t\to+\infty} \phi_2(t) = \dfrac{1}{a}\left(1 - \dfrac{v}{Pb}\right). \end{cases} \tag{3.5}$$

Define $f(\phi) = (f_1(\phi), f_2(\phi))^T$ by

$$f_1(\phi) = r\phi_1(0)[(1 - \phi_1(0)/P) - a\phi_2(0)];$$

$$f_2(\phi) = \phi_2(0)[-v + b\phi_1(-\tau)].$$

It is easy to check that in this case (H1), (H2), and (H3) are satisfied.

LEMMA 3.1. *Let $c > 2\sqrt{d_1 r}$ and*

$$\lambda_0 =: \frac{c - \sqrt{c^2 - 4d_1 r}}{2d_1}.$$

Define

$$\bar{\varphi}_1(t) = \min\left\{\frac{v}{b}e^{\lambda_0 t}, \frac{v}{b}\right\}, \qquad \bar{\varphi}_2(t) = \left(\frac{v}{b}, \frac{1}{a}\left(1 - \frac{v}{Pb}\right)\right)$$

and

$$\underline{\varphi}_1(t) = \max\left\{\frac{v}{b}\,[1 - Me^{\varepsilon t}]\, e^{\lambda_0 t}, 0\right\}, \qquad \underline{\varphi}_2(t) = 0,$$

where $M > 0$, $\varepsilon > 0$. *Then* $\bar{\varphi}(t) = (\bar{\varphi}_1(t), \bar{\varphi}_2(t))^T$ *is a supersolution of* (3.4), *and if* $M > 0$ *is sufficiently large and* $\varepsilon > 0$ *is sufficiently small, then* $\underline{\varphi}(t) = (\underline{\varphi}_1(t), \underline{\varphi}_2(t))^T$ *is a subsolution of* (3.4). *Furthermore, we have*

(i) $\sup_{s \leqslant t} \underline{\varphi}(s) \leqslant \bar{\varphi}(t)$, *for* $t \in \mathbb{R}$;

(ii) $\bar{\varphi}'(t+) \leqslant \bar{\varphi}'(t-)$, $t \in \mathbb{R}$;

(iii) $\underline{\varphi}'(t+) \geqslant \underline{\varphi}'(t-)$, $t \in \mathbb{R}$.

Proof. Denote

$$K = \left(\frac{v}{b}, \frac{1}{a}\left(1 - \frac{v}{Pb}\right)\right),$$

and $\lambda_1 = \lambda_0 > 0$, $\lambda_2 = 0$. Since

$$f_1^c(\bar{\varphi}_t) \leqslant 0, \qquad f_2^c(\bar{\varphi}_t) \leqslant 0,$$

and

$$\begin{aligned} d_1\lambda_1^2 - c\lambda_1 + \frac{1}{K_1}\sup_{t \leqslant 0} e^{-\lambda_1 t} f_1^c(\bar{\varphi}_t) &= d_1\lambda_0^2 - c\lambda_0 + \frac{rv}{Pb} \\ &< d_1\lambda_0^2 - c\lambda_0 + r = 0, \\ d_2\lambda_2^2 - c\lambda_2 + \frac{1}{K_2}\sup_{t \leqslant 0} e^{-\lambda_2 t} f_2^c(\bar{\varphi}_t) &= \frac{1}{K_2}\sup_{t \leqslant 0} f_2^c(\bar{\varphi}_t) \leqslant 0, \end{aligned}$$

by Theorem 2.3, we know that $\bar{\varphi}(t)$ is a supersolution of (3.4) satisfying (i).

Next, we will show that if $M > 0$ is sufficiently large and $\varepsilon > 0$ is sufficiently small, then $\underline{\varphi}(t)$ is a subsolution of (3.4).

Assume that $M > 1$. Then for some $t^* < 0$,

$$\begin{aligned} \underline{\varphi}_1(t) &= \frac{v}{b}\,[1 - Me^{\varepsilon t}]\, e^{\lambda_0 t} \geqslant 0, \qquad & t < t^*, \\ \underline{\varphi}_1(t) &= 0, & t \geqslant t^*. \end{aligned}$$

Notice that the equation

$$d_1\lambda^2 - c\lambda + r = 0$$

has exactly two positive real zeros $0<\lambda_0<\lambda^*$, so we can choose $\varepsilon>0$ sufficiently small such that

$$0<\varepsilon<\lambda_0, \qquad d_1(\lambda_0+\varepsilon)^2-c(\lambda_0+\varepsilon)+r<0. \tag{3.6}$$

Let $M>1$ be sufficiently large so that

$$-M(d_1(\lambda_0+\varepsilon)^2-c(\lambda_0+\varepsilon)+r)\geqslant\frac{rv}{Pb}. \tag{3.7}$$

For $t>t^*$, we have

$$d_1\underline{\varphi}_1''(t)-c\underline{\varphi}_1'(t)+f_1^c(\underline{\varphi}_t)=0,$$

and for $t<t^*$, from (3.6) and (3.7), we also have

$$\begin{aligned}
&d_1\underline{\varphi}_1''(t)-c\underline{\varphi}_1'(t)+f_1^c(\underline{\varphi}_t)\\
&\quad=\frac{v}{b}[d_1\lambda_0^2-c\lambda_0]\,e^{\lambda_0 t}-\frac{v}{b}M[d_1(\lambda_0+\varepsilon)^2-c(\lambda+\varepsilon)]\,e^{(\lambda_0+\varepsilon)t}\\
&\qquad+r\underline{\varphi}_1(t)\left[\left(1-\frac{\underline{\varphi}_1(t)}{P}\right)\right]\\
&\quad\geqslant\frac{v}{b}[d_1\lambda_0^2-c\lambda_0+r-r\frac{\underline{\varphi}_1(t)}{P}]\,e^{\lambda_0 t}\\
&\qquad-\frac{v}{b}M[d_1(\lambda_0+\varepsilon)^2-c(\lambda_0+\varepsilon)+r]\,e^{(\lambda_0+\varepsilon)t}\\
&\quad\geqslant\frac{v}{b}\left[-\frac{rv}{Pb}e^{(\lambda_0-\varepsilon)t}-M(d_1(\lambda_0+\varepsilon)^2-c(\lambda_0+\varepsilon)+r)\right]e^{(\lambda_0+\varepsilon)t}\\
&\quad\geqslant\frac{v}{b}\left[-\frac{rv}{Pb}-M(d_1(\lambda_0+\varepsilon)^2-c(\lambda_0+\varepsilon)+r)\right]e^{(\lambda_0+\varepsilon)t}\\
&\quad\geqslant 0.
\end{aligned}$$

Therefore, noting that for any $t\in\mathbb{R}$,

$$d_2\underline{\varphi}_2''(t)-c\underline{\varphi}_2'(t)+f_2^c(\underline{\varphi}_t)=0,$$

$\underline{\varphi}(t)$ is a subsolution of (3.4) satisfying (iii).

Finally, (i) follows from the fact that $\bar{\varphi}(t)$ is nondecreasing in $t\in\mathbb{R}$, and $\underline{\varphi}(t)\leqslant\bar{\varphi}(t)$, $\forall t\in\mathbb{R}$. The proof is complete.

It is easy to see that (3.1) has exactly three steady states,

$$0=(0,0)^T, \qquad K=\left(\frac{v}{b}, \frac{1}{a}\left(1-\frac{v}{Pb}\right)\right)^T, \qquad (P,0)^T.$$

Thus, by (3.2), there are exactly two zeros of $g(x, y) =: f(\hat{x}, \hat{y})$, $(x, y) \in \mathbb{R}^2$, in the interval $[0, K] \subset \mathbb{R}^2$ and

$$f(\hat{x}, \hat{y}) \neq 0, \qquad \forall (x, y)^T \in (0, \inf_{t\in\mathbb{R}} \bar{\varphi}(t)] \cup [\sup_{t\in\mathbb{R}} \underline{\varphi}(t), K).$$

Therefore, all conditions of Theorem 2.2 are satisfied and we have the following

THEOREM 3.1. *For every* $c > 2\sqrt{d_1 r}$, (3.1) *and* (3.3) *have a traveling wavefront solution* $u(x, t) = \varphi_1(x+ct)$, $v(x, t) = \varphi_2(x+ct)$ *with the asymptotic behavior*

$$\varphi_1(s) = K_0 \exp\left(\frac{(c-\sqrt{c^2-4d_1 r})\, s}{2d_1}\right)[1+o(1)] \qquad \text{as} \quad s \to -\infty,$$

where K_0 *is some positive constant.*

Remark 3.1. Theorem 3.1 claims that the existence of traveling wavefronts for the predator-prey model (3.1) is independent of the time delay.

Now, we consider the system of reaction-diffusion equations with the well-known Belousov–Zhabotinskii reaction

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \dfrac{\partial^2}{\partial x^2} u(x,t) + u(x,t)[1-u(x,t)-rv(x,t)]; \\ \dfrac{\partial}{\partial t} v(x,t) = \dfrac{\partial^2}{\partial x^2} v(x,t) - bu(x,t)\, v(x,t), \end{cases} \tag{3.8}$$

where $r > 0$ and $b > 0$ are constants, u and v correspond respectively to the bromic acid and bromide ion concentrations. This system can also be regarded as a model for many other more complex biochemical and biological processes. We also refer the readers to Murray [17] for a detailed description of this model equation.

For (3.8), the following boundary conditions have been proposed (see [3, 4, 6, 9, 11]).

$$\begin{cases} u(-\infty, t) = 0, & v(-\infty, t) = 1; \\ u(+\infty, t\} = 1, & v(+\infty, t) = 0. \end{cases} \tag{3.9}$$

By incorporating a discrete delay $\tau \geqslant 0$ into system (3.8), X. Zou [11] considered the system

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + u(x, t)[1 - u(x, t) - rv(x, t-\tau)]; \\ \frac{\partial}{\partial t} v(x, t) = \frac{\partial^2}{\partial x^2} v(x, t) - bu(x, t)\, v(x, t), \end{cases} \tag{3.10}$$

Corresponding to (3.10) and (3.9), we get the wave equation

$$\begin{cases} c\phi_1'(t) = \phi_1''(t) + \phi_1(t)[1 - \phi_1(t) - r\phi_2(t - c\tau)]; \\ c\phi_2'(t) = \phi_2''(t) - b\phi_1(t)\, \phi_2(t), \end{cases} \tag{3.11}$$

and the boundary conditions

$$\begin{cases} \lim_{t \to -\infty} \phi_1(t) = 0, \qquad \lim_{t \to +\infty} \phi_1(t) = 1; \\ \lim_{t \to -\infty} \phi_2(t) = 1, \qquad \lim_{t \to +\infty} \phi_2(t) = 0. \end{cases} \tag{3.12}$$

By making change of variables $\phi_1^* = \phi_1$, $\phi_2^* = 1 - \phi_2$ and omitting the asterisks for notational simplicity, (3.11) and (3.12) become, respectively,

$$\begin{cases} c\phi_1'(t) = \phi_1''(t) + \phi_1(t)[1 - r - \phi_1(t) + r\phi_2(t - c\tau)]; \\ c\phi_2'(t) = \phi_2''(t) + b\phi_1(t)[1 - \phi_2(t)], \end{cases} \tag{3.13}$$

and

$$\begin{cases} \lim_{t \to -\infty} (\phi_1(t), \phi_2(t)) = (0, 0); \\ \lim_{t \to +\infty} (\phi_1(t), \phi_2(t)) = (1, 1). \end{cases} \tag{3.14}$$

Define $f(\phi) = (f_1(\phi), f_2(\phi))^T$ by

$$f_1(\phi) = \phi_1(0)[1 - r - \phi_1(0) + r\phi_2(-\tau)],$$

$$f_2(\phi) = b\phi_1(0)[1 - \phi_2(0)].$$

It is easy to verify that f satisfies (H1), (H2), and (H3). It is also easy to see that $(0, \theta)$, $\theta \in [0, 1]$ and $(1, 1)$ are zeros of $g(x, y) =: f(\hat{x}, \hat{y})$, $x, y \in \mathbb{R}^2$.

LEMMA 3.2. *Let* $0 < r < 1$ *and* $c > 2\sqrt{1-r}$. *Assume*

$$\lambda = \lambda_0 =: \frac{c - \sqrt{c^2 - 4(1-r)}}{2}$$

satisfies the inequality

$$\lambda^2 - c\lambda + be^{-\lambda c\tau} \leqslant 0.$$

Define

$$\bar{\varphi}_1(t) = \min\{e^{\lambda_0 t}, 1\}, \qquad \bar{\varphi}_2(t) = \min\{e^{\lambda_0(t+c\tau)}, 1\}$$

and

$$\underline{\varphi}_1(t) = \max\{(1-r)[1 - Me^{\varepsilon t}]\, e^{\lambda_0 t}, 0\}, \qquad \underline{\varphi}_2(t) = 0,$$

where $M > 0$, $\varepsilon > 0$. *Then* $\bar{\varphi}(t) = (\bar{\varphi}_1(t), \bar{\varphi}_2(t))^T$ *is a supersolution of* (3.13), *and if* $M > 0$ *is sufficiently large and* $\varepsilon > 0$ *is sufficiently small, then* $\underline{\varphi}(t) = (\underline{\varphi}_1(t), \underline{\varphi}_2(t))^T$ *is a subsolution of* (3.4). *Furthermore, we have*

(i) $\sup_{s \leqslant t} \underline{\varphi}(s) \leqslant \bar{\varphi}(t)$, *for* $t \in \mathbb{R}$;

(ii) $\bar{\varphi}'(t+) \leqslant \bar{\varphi}'(t-)$, $t \in \mathbb{R}$;

(iii) $\underline{\varphi}'(t+) \geqslant \underline{\varphi}'(t-)$, $t \in \mathbb{R}$.

Proof. The proof is similar to that of Lemma 3.1 and is omitted.

By virtue of Lemma 3.2 and a similar argument as before, we may get the following

THEOREM 3.2. *Let* $0 < r < 1$ *and* $c > 2\sqrt{1-r}$ *be such that*

$$b \exp\left(-\frac{c\tau(c - \sqrt{c^2 - 4(1-r)})}{2}\right) \leqslant 1 - r. \tag{3.15}$$

Then (3.13) *and* (3.14) *have a monotone solution, that is,* (3.10) *and* (3.9) *have a traveling wavefront solution* $u(x, t) = \varphi_1(x + ct)$, $v(x, t) = \varphi_2(x + ct)$ *with the asymptotic behavior*

$$\varphi_1(s) = K_0 \exp\left(\frac{(c - \sqrt{c^2 - 4(1-r)})\, s}{2}\right)[1 + o(1)] \qquad \text{as} \quad s \to -\infty,$$

where K_0 *is some positive constant.*

Remark 3.2. In the case where $\tau = 0$, (3.15) becomes $b \leqslant 1 - r$, and Theorem 3.2 reduces to the main theorem (Theorem 4.1) of Ye and Wang [9] and improves the results in Kanel [3], Kapel [4], and Troy [6]. In the case where $\tau > 0$, (3.15) yields $b \leqslant 1 - r$; therefore, Theorem 3.2 improves Theorem 2.5.2.2 in Zou [11].

Remark 3.3. If $r \in (0, 1)$, $b > 1 - r$ and $c \in (2\sqrt{1-r}, 2\sqrt{b})$, then (3.10) and (3.9) may have no traveling wavefront solutions with speed c for small

$\tau \geqslant 0$. But, it follows from (3.15) that there exists a $\tau^* = \tau^*(r, c) \geqslant 0$ such that for every $\tau \geqslant \tau^*$, (3.10) and (3.9) have a wavefront solution with speed c. So time delay not only reduces the minimal wave velocity (see [5, 8]), but also gives rise to traveling wavefront solutions.

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REFERENCES

1. A. N. Kolomgorov, I. G. Petrovskii, and N. S. Piskunov, Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem, *Byul. Mosk. Gos. Univ. Ser. A Mat. Mekh.* **1** (1937), 1–26.
2. R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937), 355–369.
3. Ya. I. Kanel, Existence of a traveling-wave solution of the Belousov–Zhabotinskii system, *Differentsial'nye Uravneniya* **26** (1990), 652–660.
4. A. Ya. Kapel, Existence of traveling-wave type solutions for the Belousov–Zhabotinskii system equations, *Sibirsk. Mat. Zh.* **32** (1991), 47–59.
5. K. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, *Trans. Amer. Math. Soc.* **302** (1987), 587–615.
6. W. C. Troy, The existence of traveling wavefront solutions of a model of the Belousov–Zhabotinskii reaction, *J. Differential Equations* **36** (1980), 89–98.
7. A. I. Volpert, V. A. Volpert, and V. A. Volpert, "Traveling Wave Solutions of Parabolic Systems," Translations of Mathematical Monographs, Vol. 140, Amer. Math. Soc., Providence, 1994.
8. J. Wu, "Theory and Applications of Partial Functional Differential Equations," Springer-Verlag, New York, 1996.
9. Q. Ye and M. Wang, Traveling wavefront solutions of Noyes-field system for Belousov–Zhabotinskii reaction, *Nonlinear Anal.* **11** (1987), 1289–1302.
10. X. Zou and J. Wu, Existence of traveling wavefronts in delayed reaction-diffusion system via monotone iteration method, *Proc. Amer. Math. Soc.* **125** (1997), 2589–2598.
11. X. Zou, "Traveling Wave Solutions of Delayed Reaction-Diffusion Systems and Lattice Differential Equations," Ph.D. thesis, York University, North York, Ontario, 1997.
12. R. Gardner, Existence of traveling wave solutions of predator-prey systems via the connection index, *SIAM J. Appl. Math.* **44** (1984), 56–79.
13. C. Conley and R. Gardner, An application of the generalized Morse index to traveling wave solutions of a competitive reaction-diffusion model, *Indiana Univ. Math. J.* **44** (1984), 319–343.
14. R. Gardner, Existence and stability of traveling wave solutions of competition models: A degree theoretic approach, *J. Differential Equations* **44** (1982), 343–364.
15. M. M. Tang and P. Fife, Propagating fronts for competing species equations with diffusion, *Arch. Rational Mech. Anal.* **73** (1980), 69–77.

16. J. H. van Vuuren, The existence of traveling plane waves in a general class of competition-diffusion systems, *IMA J. Appl. Math.* **55** (1995), 135–148.
17. J. Murray, "Mathematical Biology," 2nd corrected ed., Springer-Verlag, New York, 1993.
18. S. R. Dunbar, Traveling wave solutions of diffusive Lotka–Volterra equations, *J. Math. Biol.* **17** (1983), 11–32.
19. S. R. Dunbar, Traveling wave solutions of diffusive Lotka-Volterra equations: A heteroclinic connection in $\mathbb{R}^4$, *Trans. Amer. Math. Soc.* **268** (1984), 557–594.