A graph-theoretic approach to a partial order of knots and links

Toshiki Endo [a], Tomoko Itoh [a], Kouki Taniyama [b],*

[a] *Research Institute for Mathematics and Information Science, Jiyu Gakuen College, Gakuencho 1-8-15, Higashikurume-shi, Tokyo 203-8521, Japan*
[b] *Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku-ku, Tokyo 169-8050, Japan*

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ABSTRACT

We say that a link L_1 is an s-major of a link L_2 if any diagram of L_1 can be transformed into a diagram of L_2 by changing some crossings and smoothing some crossings. This relation is a partial ordering on the set of all prime alternating links. We determine this partial order for all prime alternating knots and links with the crossing number less than or equal to six. The proofs are given by graph-theoretic methods.

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1. Introduction

Throughout this paper we work in the piecewise linear category. We use standard terminology and notation of knot theory, see for example [1] and [13], and graph theory, see for example [2] and [3]. In particular we denote the complete graph on n vertices by K_n, the n-cycle by C_n and a graph on two vertices and n multiple edges joining them by θ_n.

We assume that all links are unordered, unoriented and contained in the 3-sphere $\mathbb{S}^3$. By a *link diagram*, or simply a *diagram*, we mean a regular diagram of a link in the 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. Note that a diagram has only finitely many transversal double points each of which has over/under crossing information. We call such a double point a *crossing*. A diagram without over/under crossing information is called a *projection*.

We say that two links L_1 and L_2 are *equivalent*, denoted by $L_1 = L_2$, if there exists a possibly orientation reversing homeomorphism of $\mathbb{S}^3$ onto itself which maps L_1 to L_2. The equivalence class is called a link type. We do not distinguish between a link and its link type so long as no confusion occurs.

In [14] the third author defined a pre-ordering on the set of μ-component links as follows. Let L_1 and L_2 be μ-component links. Then we say that L_1 is a *major* of L_2, denoted by $L_1 \geqslant L_2$, if every projection of L_1 is also a projection of L_2. In other words every diagram of L_1 can be transformed into a diagram of L_2 by changing over/under information at some crossings of the diagram of L_1. Then we also say that L_2 is a *minor* of L_1. The third author studied this order for knots in [14] and for 2-component links in [15], and obtained two Hasse diagrams shown in Fig. 1, where each line segment means that the upper one is a major of the lower one. In the sequel, the number representing a link is due to the Rolfsen's knot table in [1].

We remark here that after [14] and [15] some related works are done. They are for example [5–7,9,11,17,18]. We also note that an application of the results in [14] and [15] to link signature are done in the forthcoming paper [12].

Note that this order is defined only to links with the same number of components, since any crossing change does not increase or decrease the number of link components. We now define an extended version of this order, which enables us to

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* Corresponding author.
E-mail addresses: end@prf.jiyu.ac.jp (T. Endo), tomokoitoh@prf.jiyu.ac.jp (T. Itoh), taniyama@waseda.jp (K. Taniyama).
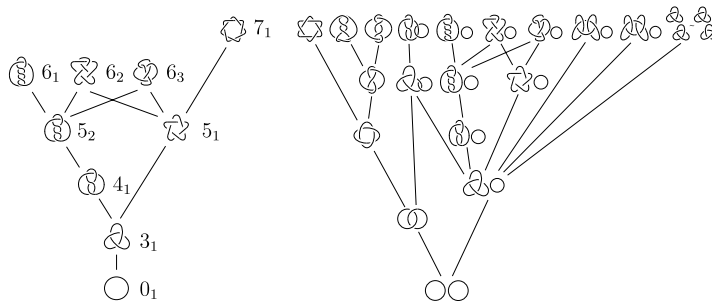
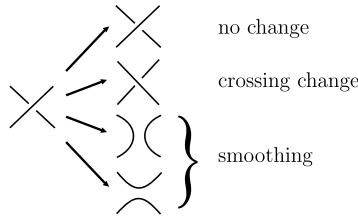**Fig. 1.** Pre-orders of knots and 2-component links.

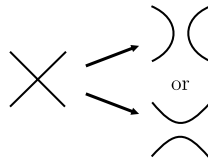**Fig. 2.** Four operations at a crossing point.

Fig. 3. Smoothing a crossing of a projection.

compare links with different numbers of components. The allowable operation connecting them is a *smoothing* operation at a crossing point, shown in Fig. 2.

Definition 1.1. Let L_1 and L_2 be links. We say that L_1 is an *s-major* of L_2 if every diagram of L_1 can be transformed into a diagram of L_2 by applying one of the four operations illustrated in Fig. 2 at each crossing point of the diagram of L_1. We denote it by $L_1 \succcurlyeq L_2$. Then we also say that L_2 is an *s-minor* of L_1 and denote it by $L_2 \preccurlyeq L_1$. We call this order *smoothing order*. We note that L_1 is an s-major of L_2 if and only if every projection of L_1 can be transformed into a projection of L_2 by smoothing some crossing points of the projection of L_1 as illustrated in Fig. 3.

By definition we have the following proposition.

Proposition 1.2. *Let L_1 and L_2 be links of the same number of components. If $L_1 \geqslant L_2$ then $L_1 \succcurlyeq L_2$.*

We will show in Theorem 2.9 that the knot 6_1 is an s-major of the knot 5_1. However it is shown in [14] that 6_1 is not a major of 5_1. Therefore the converse of Proposition 1.2 does not hold.

Historical remark. This order has first defined by the third author and some results are announced without proof in [16]. In 2007 the second author re-defined this order without knowledge of [16]. We note that all results announced in [16] are contained in this paper.

Let $\mathcal{L}$ be the set of all links. The following two propositions immediately follow from the definition.

Proposition 1.3. *The pair $(\mathcal{L}, \succcurlyeq)$ is a pre-ordered set. Namely, for any L_1, L_2 and L_3 in $\mathcal{L}$ the following (1) and (2) hold.*

(1) $L_1 \succcurlyeq L_1$ (*reflexive law*).
(2) *If $L_1 \succcurlyeq L_2$ and $L_2 \succcurlyeq L_3$, then $L_1 \succcurlyeq L_3$* (*transitive law*).

Proposition 1.4. *Let L_1 and L_2 be links. Suppose that $L_1 \succcurlyeq L_2$. Then we have $c(L_1) \geqslant c(L_2)$ where $c(L)$ denotes the minimal crossing number of L.*
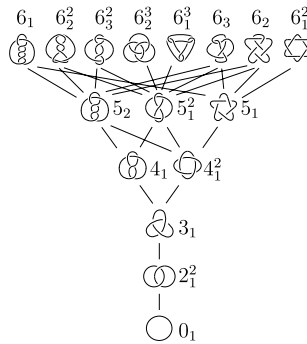
Fig. 4. Smoothing order of prime alternating knots and links with the crossing number $\leqslant 6$.

A link is said to be *prime* if it is non-splittable and every 2-sphere in $\mathbb{S}^3$ meeting the link transversely in two points bounds a trivial ball-arc pair. Thus we treat a trivial knot as a prime knot in this paper. It is known that every link is decomposed into finite number of prime links [4]. These prime links are called *prime factors* of the link. A link is called alternating if it has a diagram in which over-crossing and under-crossing appear alternately. We denote the set of all prime alternating links by $\mathcal{PAL}$. Then we have the following proposition.

Proposition 1.5. *The pair* $(\mathcal{PAL}, \succcurlyeq)$ *is a partially ordered set. Namely, in addition to the reflexive law and the transitive law the following holds.*

(3) *If* $L_1, L_2 \in \mathcal{PAL}$, $L_1 \succcurlyeq L_2$ *and* $L_2 \succcurlyeq L_1$, *then* $L_1 = L_2$ (*the antisymmetric law*).

Proof. Suppose that $L_1, L_2 \in \mathcal{PAL}$, $L_1 \succcurlyeq L_2$ and $L_2 \succcurlyeq L_1$. Then by Proposition 1.4 we have $c(L_1) = c(L_2)$. Let $\tilde{L}_1$ be a minimal crossing diagram of L_1. It is known that a minimal crossing diagram of a prime alternating link is always reduced alternating [8,10,19]. Since smoothing decreases the number of crossings we have that a diagram $\tilde{L}_2$ of L_2 is obtained by changing some crossings of $\tilde{L}_1$. Then $\tilde{L}_2$ is also a minimal diagram of L_2. Therefore $\tilde{L}_2$ is also alternating. Since these diagrams are connected we have that they are either identical or differ by all crossings. The latter case implies that L_2 is a mirror image of L_1. Then by definition we have $L_1 = L_2$. $\square$

Our results are summarized by the Hasse diagram shown in Fig. 4. Details are stated in the following section.

Our strategy for proofs is graph-theoretic and is different from the methods in [14,15]. It is well known that there is a correspondence between connected link diagrams on $\mathbb{S}^2$ and edge-signed plane graphs on $\mathbb{S}^2$. See for example [1]. If we ignore the over/under crossing information we have a correspondence between connected link projections on $\mathbb{S}^2$ and plane graphs on $\mathbb{S}^2$. We briefly review this correspondence.

Let L be a link, $\tilde{L}$ a connected diagram of L on $\mathbb{S}^2$ and $\hat{L}$ its underlying projection. Then $\hat{L}$ is a connected 4-regular plane graph on $\mathbb{S}^2$. Let $\mathcal{C}$ be a coloring of the regions $\mathbb{S}^2 - \hat{L}$ black and white such that adjacent regions have different colors. Note that there are two such colorings. Let $G(\hat{L}, \mathcal{C})$ be a plane graph on $\mathbb{S}^2$ contained in the closure of the union of black regions whose vertices lie in the black regions in one-to-one correspondence and whose edges are in one-to-one correspondence to the crossings of $\hat{L}$ so that each of them joins the vertices in two black regions meeting at a crossing.

Conversely, for a connected plane graph G on $\mathbb{S}^2$, we take disks on $\mathbb{S}^2$ such that each disk contains just one vertex and two disks containing two adjacent vertices meets at the middle point of the edge joining them. Then the boundary of the union of such disks is a 4-regular graph on $\mathbb{S}^2$ so that we may suppose it a link projection. We denote this link projection by $\hat{L}(G)$. Note that $\hat{L}(G(\hat{L}, \mathcal{C})) = \hat{L}$.

We say that a graph H is a *minor* of a graph G if H is obtained from G by a series of edge-contractions and taking subgraph. In addition we only consider edge-signed graphs on $\mathbb{S}^2$ and assume that edge-contractions and taking subgraph preserve the signs of the survived edges and edge-contractions are performed on $\mathbb{S}^2$ so that they respect the embedding of the graph into $\mathbb{S}^2$.

The following proposition can be shown by standard arguments in graph theory. We omit the proof.

Proposition 1.6. *Let G and H be connected graphs. Suppose that H is a minor of G. Then there is a sequence of connected graphs $G = G_0, G_1, \ldots, G_n = H$ such that G_{i+1} is obtained from G_i by deleting an edge or by contracting an edge for each $i \in \{0, 1, \ldots, n-1\}$.*

Then we immediately have the following proposition that is a key to prove the theorems of the next section.

Proposition 1.7. *Let $\hat{L}$ be a connected link projection. Let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. Suppose that a connected graph H is a minor of $G(\hat{L}, \mathcal{C})$. Then we have that the link projection $\hat{L}(H)$ can be obtained from $\hat{L}$ by smoothing some crossings of $\hat{L}$.*
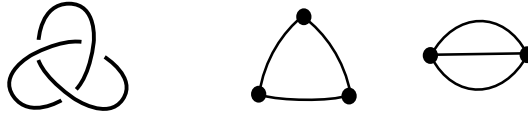
Fig. 5. The Hopf link and its graph.

Fig. 6. The trefoil knot and its graphs.

2. Determining the smoothing order of prime alternating knots and links up to six crossings

2.1. Links that are s-majors of the trivial knot

Proposition 2.1. *An s-major of a non-split link is non-splittable.*

Proof. A split link has a disconnected diagram from which no connected diagram arises. Therefore a split link cannot be an s-major of a non-splittable link. □

Theorem 2.2. *A link is an s-major of a trivial knot if and only if it is non-splittable.*

Proof. By Proposition 2.1 a split link is not an s-major of a trivial knot. Let L be a non-splittable link and $\hat{L}$ a projection of L. Then $\hat{L}$ is connected. Let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. Then K_1 is a minor of $G(\hat{L}, \mathcal{C})$. Note that $\hat{L}(K_1)$ is a projection of a trivial knot. Then by Proposition 1.7 L is an s-major of a trivial knot. □

2.2. Links that are s-majors of the Hopf link

A diagram of the Hopf link and a plane graph corresponding to its underlying projection are illustrated in Fig. 5.

We say that a crossing c of a link projection $\hat{L}$ is *nugatory* if the number of the connected components of $\hat{L} - c$ is greater than that of $\hat{L}$. We say that $\hat{L}$ is *reduced* if it has no nugatory crossings. We note that a nugatory crossing appears as a loop or a cut edge in the corresponding plane graph. It is easy to see that we may only consider reduced projections. That is, we have the following proposition.

Proposition 2.3. *Let L_1 and L_2 be links. Suppose that for any reduced projection $\hat{L}_1$ of L_1 there is a projection $\hat{L}_2$ of L_2 that is obtained from $\hat{L}_1$ by smoothing some crossings. Then L_1 is an s-major of L_2.*

From now on all link projections are supposed to be reduced.

Theorem 2.4. *A link is an s-major of a Hopf link if and only if it is non-splittable and it is not a trivial knot.*

Proof. The 'only if' part follows from Proposition 2.1 and the fact that a trivial knot is not an s-major of a Hopf link. Let L be a non-splittable link that is not a trivial knot and $\hat{L}$ a reduced projection of L. Let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. Then $G(\hat{L}, \mathcal{C})$ is a connected graph without loops nor cut edges. Since L is not a trivial knot we have that $G(\hat{L}, \mathcal{C})$ has at least two edges. Then we have that a 2-cycle is a minor of $G(\hat{L}, \mathcal{C})$. Then by Proposition 1.7 we have the result. □

2.3. Links that are s-majors of the trefoil knot

A diagram of the trefoil knot and plane graphs C_3 and θ_3 corresponding to its underlying projection are illustrated in Fig. 6.

Theorem 2.5. *A link is an s-major of the trefoil knot if and only if it is non-splittable and it has a prime factor that is not a Hopf link.*

Proof. It is easily seen that a connected sum of some Hopf links is not an s-major of the trefoil knot. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor which is not a Hopf link. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. Then $G(\hat{L}, \mathcal{C})$ is a connected graph without loops nor cut edges. Note that $G(\hat{L}, \mathcal{C})$ may have some cut vertices. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds to not necessarily one but
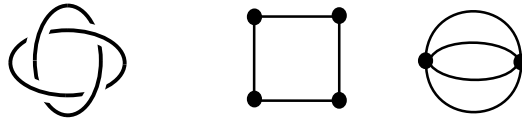
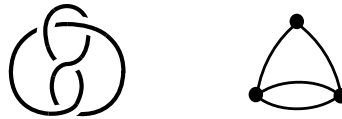**Fig. 7.** The $(2, 4)$-torus link and its graphs.

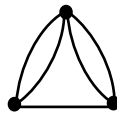**Fig. 8.** The figure eight knot and its graph.

Fig. 9.

some prime factors of L such that at least one of them is not a Hopf link. If H has a cycle of length three or more then we have that the 3-cycle C_3 is a minor of H, hence of $G(\hat{L}, \mathcal{C})$. Suppose that the length of any cycle of H is less than or equal to two. Then H must be the graph on two vertices and three or more edges joining them. Then θ_3 is a minor of H and $G(\hat{L}, \mathcal{C})$. By Proposition 1.7 we have the result. □

2.4. Links that are s-majors of the $(2, 4)$-torus link

A diagram of the $(2, 4)$-torus link and plane graphs C_4 and θ_4 corresponding to its underlying projection are illustrated in Fig. 7.

We also illustrate here in Fig. 8 a diagram of the figure eight knot 4_1 and a plane graph corresponding to its underlying projection.

Theorem 2.6. *A link is an s-major of the $(2, 4)$-torus link if and only if it is non-splittable and it has a prime factor that is none of a Hopf link, a trefoil knot and a figure eight knot.*

Proof. It is easily seen that a connected sum of some Hopf links, trefoil knots and figure eight knots is not an s-major of the $(2, 4)$-torus link. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor which is none of a Hopf link, a trefoil knot and a figure eight knot. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds to not necessarily one but some prime factors of L such that at least one of them is none of a Hopf link, a trefoil knot and a figure eight knot. Then we have that H has four or more edges. If H has a cycle of length four or more, then we have that a 4-cycle C_4 is a minor of H and hence of $G(\hat{L}, \mathcal{C})$. Thus, in this case, L is an s-major of the $(2, 4)$-torus link. Suppose that the length of any cycle of H is less than or equal to three. If H has two vertices connected by four or more internally disjoint paths, then we have that θ_4 is a minor of H and hence of $G(\hat{L}, \mathcal{C})$ and we have that L is an s-major of the $(2, 4)$-torus link.

Suppose that every pair of vertices of H has at most three internally disjoint paths between them. Then we have that either H contains the graph illustrated in Fig. 9 from which θ_4 is obtained by an edge-contraction, or H is a proper minor of the graph in illustrated in Fig. 9. Then it is easy to check that the graph only corresponds to a trivial knot, a Hopf link, a trefoil knot or a figure eight knot. □

2.5. Links that are s-majors of the figure eight knot

Theorem 2.7. *A link is an s-major of the figure eight knot if and only if it is non-splittable and it has a prime factor that is none of $(2, n)$-torus knots and links.*

Proof. It is easily seen that a connected sum of some $(2, n)$-torus knots and links is not an s-major of the figure eight knot. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor which is none of $(2, n)$-torus knots and links. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds to not necessarily one but some prime factors of L such that at least one of them is none of

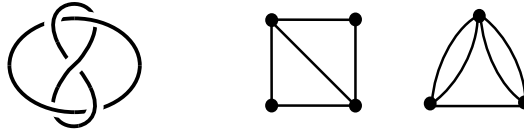**Fig. 10.** Graphs corresponding to the $(2, n)$-torus link.

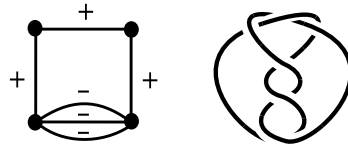**Fig. 11.** The Whitehead link and its graphs.

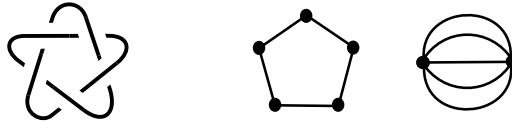**Fig. 12.** A graph corresponding to a Whitehead link.

Fig. 13. The $(2, 5)$-torus knot and its graphs.

$(2, n)$-torus knots and links (see Fig. 10). Then we have that H is neither a cycle nor a θ_n-curve. Then it is easy to see that the graph illustrated in Fig. 8 is a minor of H. □

2.6. Links that are s-majors of the Whitehead link

A diagram of the Whitehead link 5_1^2 and plane graphs corresponding to its underlying projection are illustrated in Fig. 11. One of them is a 4-cycle with one diagonal, and the other is the dual which is a 3-cycle with two multiple edges.

Theorem 2.8. *A link is an s-major of the Whitehead link if and only if it is non-splittable and it has a prime factor that is none of $(2, n)$-torus knots and links and twist knots.*

Proof. It is easily seen that a connected sum of some $(2, n)$-torus knots and links and twist knots is not an s-major of the Whitehead link. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor which is none of $(2, n)$-torus knots and links and twist knots. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds not necessarily one but some prime factors of L such that at least one of them is none of $(2, n)$-torus knots and links and twist knots. If H has a cycle of length greater than or equal to four and the cycle has a diagonal path, or if H has a cycle of length three or more such that at least two edges of the cycle are multiple edges, then we obtain a graph illustrated in Fig. 11.

If H has a cycle of length greater than or equal to four with multiple three edges, then we get the graph in Fig. 12, which also corresponds to a Whitehead link.

Thus, we may assume that if H has a cycle of length greater than or equal to four then the cycle has no diagonals, no multiple three edges, and no distinct multiple edges. If H has a 3-cycle then it has no distinct multiple edges. In these cases we only have $(2, n)$-torus knots and links and twist knots. □

2.7. Links that are s-majors of the $(2, 5)$-torus knot

A diagram of the $(2, 5)$-torus knot and plane graphs C_5 and θ_5 corresponding to its underlying projection is illustrated in Fig. 13. A graph shown in Fig. 14 also corresponds to a $(2, 5)$-torus knot. It will appear in the following proof.

Before we describe our next result, we shall confirm some links and their corresponding graphs which are appeared in the statement.
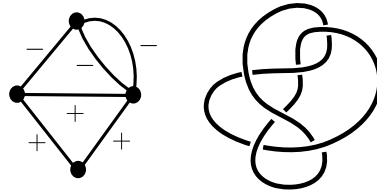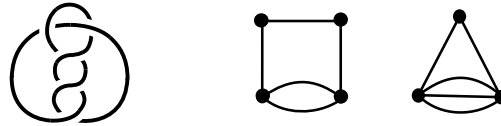
Fig. 14. A graph corresponding to a $(2, 5)$-torus knot.

Fig. 15. 5_2 and its graphs.

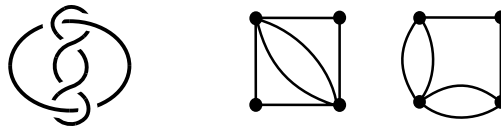

Fig. 16. 6_2^2 and its graph.

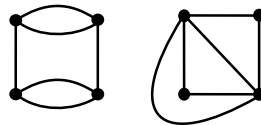**Fig. 17.** 6_3^2 and its graphs.
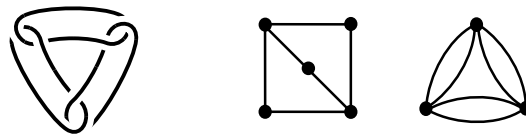

Fig. 18. Graphs corresponding to 6_3^2.

Fig. 19. 6_1^3 and its graphs.

A graph corresponding to 5_2 is a graph consists of a 4-cycle with multiple edges. Its dual is a graph consists of a 3-cycle with multiple three edges. See Fig. 15.

A graph corresponding to 6_2^2 is a graph consists of a 4-cycle with multiple three edges. See Fig. 16.

A graph corresponding to 6_3^2 is a graph consists of a 4-cycle with one diagonal multiple edges. Its dual consists of a 4-cycle with two adjacent multiple edges. See Fig. 17. Graphs illustrated in Fig. 18 also represent 6_3^2, which appears in the following proof.

A graph corresponding to 6_1^3 is the complete bipartite graph $K_{2,3}$. Its dual is a 3-cycle with three multiple edges. See Fig. 19.

A graph corresponding to the Borromean rings 6_2^3 is the complete graph K_4. See Fig. 20.

A graph corresponding to 7_1^3 consists of a 4-cycle with three multiple edges. Its dual is the graph obtained from the complete bipartite graph $K_{2,4}$ by an edge-contraction. See Fig. 21.

A graph corresponding to 8_1^4 consists of a 4-cycle with four multiple edges. Its dual is the complete bipartite graph $K_{2,4}$. See Fig. 22.
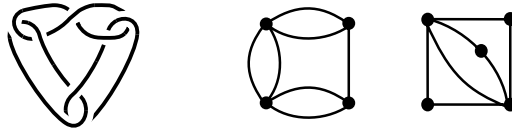
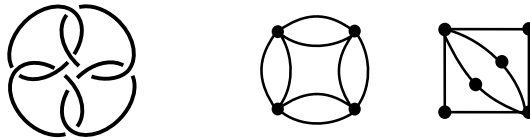**Fig. 20.** 6_2^3 and its graph.

Fig. 21. 7_1^3 and its graphs.

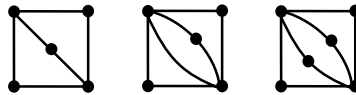**Fig. 22.** 8_1^4 and its graphs.

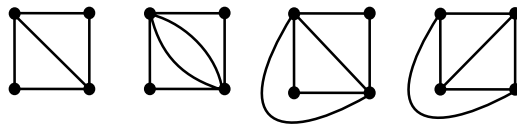**Fig. 23.** 4-cycle with a diagonal path of length 2.

Fig. 24. 4-cycle with a diagonal.

Theorem 2.9. *A link is an s-major of the $(2, 5)$-torus knot if and only if it is non-splittable and it has a prime factor whose crossing number is greater than four that is none of 5_2, 5_1^2, 6_2^2, 6_3^2, 6_1^3, 6_2^3, 7_1^3, and 8_1^4.*

Proof. It is easily seen that a connected sum of some of knots and links of crossing number less than five, 5_2, 5_1^2, 6_2^2, 6_3^2, 6_1^3, 6_2^3, 7_1^3, and 8_1^4 is not an s-major of the $(2, 5)$-torus knot. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor with crossing number greater than four which is none of 5_2, 5_1^2, 6_2^2, 6_3^2, 6_1^3, 6_2^3, 7_1^3, and 8_1^4. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds to not necessarily one but some prime factors of L such that at least one of them has crossing number greater than four and it is none of 5_2, 5_1^2, 6_2^2, 6_3^2, 6_1^3, 6_2^3, 7_1^3, and 8_1^4.

If H has a cycle of length greater than or equal to five, or if H has two vertices connected by five or more internally disjoint paths, then we obtain graphs corresponding to a $(2, 5)$-torus knot.

If H has a 4-cycle with multiple edges and a diagonal, then we obtain the graph corresponding to a $(2, 5)$-torus knot.

Thus we may assume that all cycles of H have length at most four, that every pair of vertices has at most four internally disjoint paths between them, and that if H has a 4-cycle then it does not have multiple edges and diagonals simultaneously.

If H has a 4-cycle with a diagonal path of length two, such graphs are only three types shown in Fig. 23.

If H has a 4-cycle with a diagonal, such graphs are only four types shown in Fig. 24.

If H has a 4-cycle with multiple edges, such graphs are only six types shown in Fig. 25.

If H has a 3-cycle with multiple edges, such graphs are only three types shown in Fig. 26. This completes the proof. □

2.8. Links that are s-majors of 5_2

Theorem 2.10. *A link is an s-major of 5_2 if and only if it is non-splittable and it has a prime factor with crossing number greater than four that is none of $(2, n)$-torus knots and links, 5_1^2, 6_1^3, and 6_2^3.*
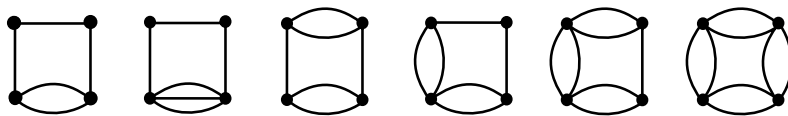
Fig. 25. 4-cycle with multiple edges.
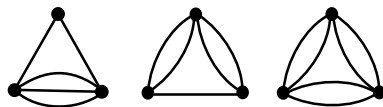

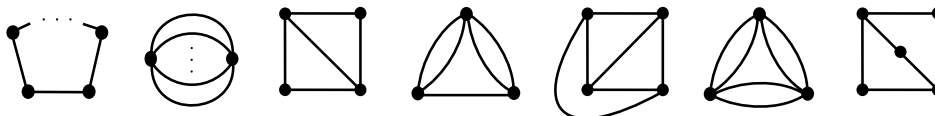**Fig. 26.** 3-cycle with multiple edges.

Fig. 27. Graphs that satisfies the conditions in the proof of Theorem 2.10.

Proof. It is easily seen that a connected sum of some of knots and links of crossing number less than five, $(2, n)$-torus knots and links, 5_1^2, 6_1^3, and 6_2^3 is not an s-major of 5_2. Therefore the 'only if' part follows. Let L be a non-splittable link that has a prime factor with crossing number greater than four which is none of $(2, n)$-torus knots and links, 5_1^2, 6_1^3, and 6_2^3. Let $\hat{L}$ be a reduced projection of L and let $\mathcal{C}$ be a coloring of $\mathbb{S}^2 - \hat{L}$. By the assumption there is a block H of $G(\hat{L}, \mathcal{C})$ that corresponds to not necessarily one but some prime factors of L such that at least one of them has crossing number greater than four and it is none of $(2, n)$-torus knots and links, 5_1^2, 6_1^3, and 6_2^3.

If H has two vertices which are connected by three internally disjoint paths one of which is of length at least three, or if H has two vertices which are connected by a path of length two and by another at least three internally disjoint paths, then we can get a graph of 5_2.

Thus, we may assume that every pair of vertices has one path of length at least three and one another path, that every pair of vertices has one path of length two and at least two other paths, or that every pair of vertices has paths of length one. In this case, the complete list of such graphs is given in Fig. 27 by the similar way as the previous section, and we get graphs of the $(2, n)$-torus link, 5_1^2, 6_1^3, and 6_2^3. □

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