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Arhangel'skiĭ's solution to Alexandroff's problem: A survey

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Dedicated to A.V. Arhangel'skiĭ on the occasion of his sixty-fifth birthday

Abstract

In 1969, Arhangel'skiĭ proved that $|X| \leqslant 2^{\chi(X)L(X)}$ for every Hausdorff space X. This beautiful inequality solved a nearly fifty-year old question raised by Alexandroff and Urysohn. In this paper we survey a wide range of generalizations and variations of Arhangel'skiĭ's inequality. We also discuss open problems and an important legacy of the theorem: the emergence of the closure method as a fundamental unifying device in cardinal functions.

MSC: 54A25; 54A35

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1. The problem and the solution

In 1923, Alexandroff and Urysohn asked: Does every compact first-countable Hausdorff space have cardinality at most $2^{\aleph_0}$? Their question was obviously motivated by a theorem that they had proved earlier, in 1922, but did not publish until 1929; this 1922 result states that every compact perfectly normal Hausdorff space has cardinality at most $2^{\aleph_0}$. See Arhangel'skiĭ's paper *Mappings and Spaces* for a further discussion of the problem,

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including some of his early attempts at a solution; see [1] for comments by Arhangel'skiĭ on the special role played by Alexandroff in formulating the problem.

The solution to the problem was finally obtained almost 50 years later, in 1969. I refer, of course, to the following beautiful inequality in cardinal functions.

Theorem 1.1 *(Arhangel'skiĭ). For X Hausdorff,* $|X| \leqslant 2^{\chi(X)L(X)}$. *In particular, every first-countable Lindelöf Hausdorff space has cardinality at most* $2^{\aleph_0}$.

By 1969, the year in which Arhangel'skiĭ's inequality was published, cardinal functions was a growing, active area of research in set-theoretic topology. For example, in 1965 de Groot proved, among other results, that $|X| \leqslant 2^{hL(X)}$ whenever X is a Hausdorff space. And in 1967 Hajnal and Juhász published their two fundamental inequalities, namely $|X| \leqslant 2^{\chi(X)c(X)}$ for Hausdorff spaces and $|X| \leqslant 2^{\psi(X)s(X)}$ for T_1 spaces.

Given the two inequalities of Hajnal and Juhász, and the growing interest in cardinal functions, Arhangel'skiĭ's solution to Alexandroff's problem was a welcome addition to the field, and it immediately assumed its rightful position as the most important inequality in cardinal invariants. English translations of the proof were quickly available. Juhász published an English translation [19], Gillman distributed unpublished notes of a proof for the special case in which X is a first-countable compact Hausdorff space, and Roy published an alternate proof for this special case.

In 1970, Comfort gave a lecture on cardinal invariants at the International Conference on General Topology held at the University of Pittsburgh, and in his talk he outlined the key ideas of Arhangel'skiĭ's proof (for details see [10]). As Comfort himself stated, the Organizing Committee wanted Arhangel'skiĭ to give the talk, but he was unable to come.

What makes a theorem great? There are at least two criteria:

- solves a long-standing problem;
- introduces new techniques and generates new results and open problems.

Arhangel'skiĭ's Theorem obviously satisfies the first requirement. The remainder of this paper explains why it also satisfies the second.

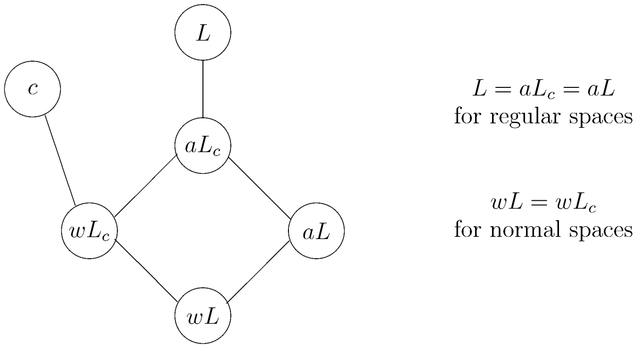
2. Definitions and examples

All spaces are at least T_1. Standard set-theoretic notation is used: κ and λ denote infinite cardinals and α, β, and γ denote ordinals. Notation for cardinal functions is also fairly standard: L, hL, wL, c, psw, χ, ψ, $\pi\chi$, and t denote Lindelöf degree, hereditary Lindelöf degree, weak Lindelöf degree, cellularity, point-separating weight, character, pseudo-character, π-character, and tightness; see [13,17,20]. We now define additional cardinal functions that are not quite so well known.

The *almost Lindelöf degree of* X, denoted $aL(X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{V}$ of X, there is a subcollection $\mathcal{V}_o$ of $\mathcal{V}$ such that $|\mathcal{V}_o| \leqslant \kappa$ and $\{V^-\colon V \in \mathcal{V}_o\}$ covers X. Note the following: $aL(X) = L(X)$ whenever X is regular; $wL(X) \leqslant aL(X)$; if X is an H-closed space, then $aL(X) = \omega$.

Recall that a subset A of a topological space X is an H-*set* if given any collection $\mathcal{V}$ of open sets in X that covers A, there is a finite subcollection of $\mathcal{V}$, say $\{V_1, \dots, V_k\}$, such that $\{V_1^-, \dots, V_k^-\}$ covers A. It is well known that a closed subset of an H-closed space need not be an H-set. This pathology carries over to the cardinal function aL, thereby giving rise to a stronger cardinal function aL_c (see [26]). The *almost Lindelöf degree of X with respect to closed sets*, denoted $aL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}_o$ of $\mathcal{V}$ such that $|\mathcal{V}_o| \leqslant \kappa$ and $\{V^-\colon V \in \mathcal{V}_o\}$ covers H. We have: $aL \leqslant aL_c \leqslant L$, and equality holds for regular spaces.

In a similar vein, the *weak Lindelöf degree of X with respect to closed sets*, denoted $wL_c(X)$, is the smallest infinite cardinal κ such that for every closed subset H of X and every collection $\mathcal{V}$ of open sets in X that covers H, there is a subcollection $\mathcal{V}_o$ of $\mathcal{V}$ such that $|\mathcal{V}_o| \leqslant \kappa$ and $H \subseteq (\bigcup \mathcal{V}_0)^-$. For normal spaces, $wL = wL_c$. The following diagram and the examples given below should clarify the relationship between these cardinal functions:



Example *($aL < aL_c < L$).* This example is discussed by Willard and Dissanayake in the paper [26]. Let $\kappa\omega$ denote the Katětov extension of ω with the discrete topology. Recall that $\kappa\omega = \omega \cup T$, where T is a set of cardinality 2^{2^ω} that indexes the collection of all free ultrafilters on ω. For $t \in T$ let $\mathcal{U}_t$ be the ultrafilter indexed by t; a local base for t is the collection $\{\{t\} \cup U\colon U \in \mathcal{U}_t\}$. The space $\kappa\omega$ has the following properties:

(a) countable tightness;
(b) countable pseudo-character;
(c) Urysohn (hence Hausdorff);
(d) H-closed, hence $aL(\kappa\omega) = \omega$;
(e) $aL_c(\kappa\omega) = 2^\omega$ (proof sketched below);
(f) $L(\kappa\omega) = 2^{2^\omega}$;
(g) separable, hence $wL_c(\kappa\omega) = \omega$.

- $aL_c(\kappa\omega) \geqslant 2^\omega$: This follows from the following lemma (see [9]): There is a collection $\{\mathcal{U}_\alpha\colon 0 \leqslant \alpha < 2^\omega\}$ of 2^ω free ultrafilters on ω such that for all $\alpha < 2^\omega$, there exists $U_\alpha \in \mathcal{U}_\alpha$ such that $U_\alpha^c \in \mathcal{U}_\beta$ for all $\beta \neq \alpha$.
- $aL_c(\kappa\omega) \leqslant 2^\omega$: This follows from the following observation. Let $A \subseteq T$, and for each $t \in A$ let $W_t = \{t\} \cup V_t$ be an open neighborhood of t. Define an equivalence relation

$\sim$ on A by $s \sim t \Leftrightarrow V_s = V_t$. The number of distinct equivalence classes is $\leqslant 2^\omega$, and if $s \sim t$, then $s \in W_t^-$.

Example *($wL_c < aL$).* Let S be the Sorgenfrey line. The space $S \times S$ is separable but not Lindelöf; $wL_c(S \times S) = \omega$ and $aL(S \times S) = 2^\omega$.

Example *($aL < wL_c$).* Let $\kappa > \omega$ and let X be the Katětov extension of κ with the discrete topology; X is an H-closed space and so $aL(X) = \omega$. On the other hand, $wL_c(X) = \kappa$. (The proof that $wL_c(X) \geqslant \kappa$ follows from the following observation: Given any infinite cardinal κ, there is a pairwise disjoint collection $\{A_\alpha\colon\ 0 \leqslant \alpha < \kappa\}$ of subsets of κ and a collection $\{\mathcal{U}_\alpha\colon\ 0 \leqslant \alpha < \kappa\}$ of κ free ultrafilters on κ such that $A_\alpha \in \mathcal{U}_\alpha$ for all $\alpha < \kappa$.)

The *closed pseudo-character* of a space X, denoted $\psi_c(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\{V(\alpha, x)\colon\ \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha<\kappa} V(\alpha, x)^- = \{x\}$. The *Hausdorff pseudo-character of* X, denoted $H\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\{V(\alpha, x)\colon\ \alpha < \kappa\}$ of open neighborhoods of x such that if $x \neq y$, then there exists $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$. These two cardinal functions are defined only for Hausdorff spaces. The *Urysohn pseudo-character of* X, denoted $U\psi(X)$, is similar to $H\psi(X)$ except that we require that $V(\alpha, x)^- \cap V(\beta, y)^- = \emptyset$. This cardinal function is defined only for Urysohn spaces and was introduced by Stavrova in [24]. The following hold:

(1) $\psi(X) \leqslant \psi_c(X) \leqslant H\psi(X) \leqslant U\psi(X) \leqslant \chi(X)$;
(2) $\psi_c(X) \leqslant \psi(X)L(X)$ for every Hausdorff space X (see [20, p. 15]).

Later we will need the following variation of (2).

Lemma 2.1. *Let X be a Urysohn space. Then $\psi_c(X) \leqslant \psi(X)aL_c(X)$.*

Proof. Let $\psi(X)aL_c(X) = \kappa$, let $x \in X$, and let $\{V(\alpha, x)\colon\ \alpha < \kappa\}$ be a collection of open neighborhoods of x such that $\bigcap_{\alpha<\kappa} V(\alpha, x) = \{x\}$. Fix $\alpha < \kappa$, let $H = X - V(\alpha, x)$, and for each $y \in H$ let U_y and V_y be open sets in X such that

(a) $x \in U_y$ and $y \in V_y$;
(b) $U_y^- \cap V_y^- = \emptyset$.

Now $\{V_y\colon\ y \in H\}$ is an open cover of the closed set H and $aL_c(X) \leqslant \kappa$; hence there is a subset A of H with $|A| \leqslant \kappa$ such that $H \subseteq \cup\{V_y^-\colon\ y \in A\}$. It easily follows that

$$\bigcap\bigl\{U_y^-\colon\ y \in A\bigr\} \subseteq V(\alpha, x).$$

In summary, for each $\alpha < \kappa$ there is a collection $\mathcal{U}_\alpha$ of open neighborhoods of x such that $|\mathcal{U}_\alpha| \leqslant \kappa$ and $\bigcap\{U^-\colon\ U \in \mathcal{U}_\alpha\} \subseteq V(\alpha, x)$. Since $\bigcap_{\alpha<\kappa} V(\alpha, x) = \{x\}$, we obtain $\psi_c(X) \leqslant \kappa$ as required. □

Let X be a topological space and let $H \subseteq X$. A point $x \in X$ is a *θ-limit point of* H if $V^- \cap H \neq \emptyset$ for every open neighborhood V of x. The *θ-closure* of H is the set $H^\theta = \{x\colon x \in H$ or x is a θ-limit point of $H\}$, and H is *θ-closed* if $H = H^\theta$. The following hold:

(1) $H^- \subseteq H^\theta$;
(2) every θ-closed set is closed;
(3) for regular spaces, $H^- = H^\theta$;
(4) for Urysohn spaces, $|H^\theta| \leqslant |H|^{\chi(X)}$ (due to Bella and Cammaroto);
(5) if H is a θ-closed set and $x \notin H$, then there is an open neighborhood V of x such that $V^- \cap H = \emptyset$ ("regularity with respect to θ-closed sets").

3. Generalizations and variations of $|X| \leqslant 2^{\chi(X)L(X)}$

In this section we will discuss a long list of theorems from the literature, each of which is either a generalization or a variation of Arhangel'skiĭ's original inequality. We begin with four generalizations; in each case the Hausdorff hypothesis is fixed and at least one of character or Lindelöf degree is weakened.

(1) $\lvert X\rvert \leqslant 2^{t(X)\psi(X)L(X)}$	Arhangel'skiĭ, Šapirovskiĭ, 1974
(2) $\lvert X\rvert \leqslant 2^{H\psi(X)L(X)}$	Hodel, 1991
(3) $\lvert X\rvert \leqslant 2^{t(X)\psi_c(X)\pi\chi(X)aL_c(X)}$	Willard–Dissanayake, 1984
(4) $\lvert X\rvert \leqslant 2^{t(X)\psi_c(X)aL_c(X)}$	Bella–Cammaroto, 1988

The inequality $|X| \leqslant 2^{t(X)\psi(X)L(X)}$ is perhaps our most elegant generalization of Arhangel'skiĭ's theorem. By replacing χ with ψ and t, we have isolated the precise properties of χ that are actually needed in the original proof. This result was first proved by Arhangel'skiĭ for regular spaces and later generalized to Hausdorff spaces by Šapirovskiĭ [22].

The inequality $|X| \leqslant 2^{H\psi(X)L(X)}$ generalizes χ in a different way; it replaces χ with $H\psi$, a local cardinal function that captures the Hausdorff property of X. At the same time $H\psi$ is a strengthening of ψ and so tightness can be omitted as a hypothesis.

In both (1) and (2) the hypothesis L is fixed and χ is generalized. In the two inequalities (3) and (4) the cardinal function L is weakened to aL_c. Willard and Dissanayake introduced this new cardinal function and then proved the inequality $|X| \leqslant 2^{t(X)\psi_c(X)\pi\chi(X)aL_c(X)}$. Somewhat later Bella and Cammaroto generalized their result by showing that $|X| \leqslant 2^{t(X)\psi_c(X)aL_c(X)}$. Note that (4) not only generalizes (3) but also (1) (recall the inequality $\psi_c(X) \leqslant \psi(X)L(X)$).

We now turn to variations of $|X| \leqslant 2^{\chi(X)L(X)}$; in each case the Hausdorff hypothesis is strengthened and at the same time the Lindelöf degree (and perhaps character) is weakened.

(5) For X normal, $\lvert X\rvert \leqslant 2^{\chi(X)wL(X)}$	Bell, Ginsburg, Woods, 1978
(6) For X regular, $\lvert X\rvert \leqslant 2^{\chi(X)wL_c(X)}$	Arhangel'skiĭ, 1979
(7) For X Urysohn, $\lvert X\rvert \leqslant 2^{\chi(X)wL_c(X)}$	Alas, 1993
(8) For X Urysohn, $\lvert X\rvert \leqslant 2^{\chi(X)aL(X)}$	Bella–Cammaroto, 1988

(9) For X Urysohn, $|X| \leqslant 2^{U\psi(X)aL(X)}$ Stavrova, 2000
(10) For X Urysohn, $|X| \leqslant 2^{t(X)\psi(X)aL_c(X)}$ (4) and Lemma 2.1

The inequality $|X| \leqslant 2^{\chi(X)wL(X)}$ for normal spaces is due to Bell et al. [6]; this was the first variation of Arhangel'skiĭ's inequality to use the cardinal function wL. At about the same time, Arhangel'skiĭ [3] proved the inequality $|X| \leqslant 2^{\chi(X)wL_c(X)}$ for regular spaces. Since $wL(X) = wL_c(X)$ for normal spaces, (6) is a generalization of (5).

In 1993 Alas proved that $|X| \leqslant 2^{\chi(X)wL_c(X)}$ holds for the class of Urysohn spaces, thereby generalizing both (5) and (6). Prior to this result, Bella and Cammaroto had obtained the inequality $|X| \leqslant 2^{\chi(X)aL(X)}$ for Urysohn spaces. Thus we have two variations of Arhangel'skiĭ's inequality for the class of Urysohn spaces, neither of which implies the other. Moreover, both proofs use an interesting new strategy: build up a θ-closed set that is all of X (instead of just a closed set; more on this later). We also note that the inequality $|H^\theta| \leqslant |H|^{\chi(X)}$ plays a key role in both proofs.

The Bella–Cammaroto inequality $|X| \leqslant 2^{\chi(X)aL(X)}$ does not hold for Hausdorff spaces. In 1998 Bella and Yaschenko obtained the following result: if κ is a non-measurable cardinal, then there is a first-countable almost Lindelöf Hausdorff space X such that $|X| > \kappa$.

In 2000 Stavrova [24] obtained a very nice generalization of the Bella–Cammaroto inequality by showing that $|X| \leqslant 2^{U\psi(X)aL(X)}$ for Urysohn spaces. This result is a "companion" of the inequality $|X| \leqslant 2^{H\psi(X)L(X)}$ for Hausdorff spaces. Finally, the inequality $|X| \leqslant 2^{t(X)\psi(X)aL_c(X)}$ for Urysohn spaces is a consequence of the two inequalities $|X| \leqslant 2^{t(X)\psi_c(X)aL_c(X)}$ (which holds for Hausdorff spaces) and $\psi_c(X) \leqslant \psi(X)aL_c(X)$ (which holds for Urysohn spaces; see Lemma 2.1). Compare this with the way in which (1) follows from (4) and $\psi_c(X) \leqslant \psi(X)L(X)$ for Hausdorff spaces.

Let us summarize the results discussed thus far. If we eliminate the inequalities that follow from more general ones, we have:

Hausdorff spaces	**Urysohn spaces**
(2) $|X| \leqslant 2^{H\psi(X)L(X)}$	(7) $|X| \leqslant 2^{\chi(X)wL_c(X)}$
(4) $|X| \leqslant 2^{t(X)\psi_c(X)aL_c(X)}$	(9) $|X| \leqslant 2^{U\psi(X)aL(X)}$

The proofs of these four inequalities have a common construction that is inspired by Arhangel'skiĭ's original proof. Theorem 3.1 below captures this common core; in most applications of the theorem, c will be the closure operator and d the identity function.

Theorem 3.1. *Let X be a set, let κ and λ be infinite cardinals with $\lambda \leqslant 2^\kappa$, let $c : P(X) \to P(X)$ and $d : P(X) \to P(X)$ be operators on X, and for each $x \in X$ let $\{V(\gamma, x)\colon \gamma < \lambda\}$ be a collection of subsets of X. Assume the following*:

- **(T)** *if $x \in c(H)$, then there exists $A \subseteq H$ with $|A| \leqslant \kappa$ such that $x \in c(A)$* (tightness *condition*);
- **(C)** *if $A \subseteq X$ with $|A| \leqslant \kappa$, then $|c(A)| \leqslant 2^\kappa$* (cardinality *condition*);

(**C-S**) *if $H \neq \emptyset$, $c(H) \subseteq H$, and $q \notin H$, then there exist $A \subseteq H$ with $|A| \leqslant \kappa$ and a function $f : A \to \lambda$ such that $H \subseteq d(\bigcup_{x \in A} V(f(x), x))$ and $q \notin d(\bigcup_{x \in A} V(f(x), x))$* (cover-separation *condition*).

Then $|X| \leqslant 2^\kappa$.

Proof. We will use the closure method (more on this in Section 6). Construct a sequence $\{H_\alpha\colon 0 \leqslant \alpha < \kappa^+\}$ of subsets of X such that for $0 \leqslant \alpha < \kappa^+$:

(a) $|H_\alpha| \leqslant 2^\kappa$ (H_0 is any non-empty subset of X of cardinality at most 2^κ);
(b) for all $A \subseteq \bigcup_{\beta<\alpha} H_\beta$ such that $|A| \leqslant \kappa$:
  (b1) $c(A) \subseteq H_\alpha$ (the cardinality condition (**C**) is used here);
  (b2) if $f : A \to \lambda$ and $d(\bigcup_{x \in A} V(f(x), x)) \neq X$,
  then $H_\alpha - d(\bigcup_{x \in A} V(f(x), x)) \neq \emptyset$ ($\lambda \leqslant 2^\kappa$ used here).

Let $H = \bigcup\{H_\alpha\colon \alpha < \kappa^+\}$.

- $|H| \leqslant 2^\kappa$;
- $c(H) \subseteq H$ (use the tightness condition (**T**) and (b1));
- $H = X$. Suppose that $q \notin H$. By the cover-separation condition (**C-S**), there exists $A \subseteq H$ with $|A| \leqslant \kappa$ and $f : A \to \lambda$ such that
  (i) $H \subseteq d(\bigcup_{x \in A} V(f(x), x))$;
  (ii) $q \notin d(\bigcup_{x \in A} V(f(x), x))$.

Now choose $\alpha < \kappa^+$ such that $A \subseteq \bigcup_{\beta<\alpha} H_\beta$. By (ii) and (b2), $H_\alpha - d(\bigcup_{x \in A} V(f(x), x)) \neq \emptyset$. This contradicts (i). □

We emphasize that the statement of Theorem 3.1 is not far removed from General Theorem 2 in Arhangel'skiĭ's original paper! In particular, for c the closure operator, condition (**T**) and a variation of (**C**) both appear. To emphasize this point, let us use Theorem 3.1 to prove a result that Arhangel'skiĭ derives from General Theorem 2 in [1].

Theorem 3.2 *(Arhangel'skiĭ). Let X be a sequential Lindelöf Hausdorff space with $\psi(X) \leqslant 2^\omega$. Then $|X| \leqslant 2^\omega$.*

Proof. We need to check the following.

(**T**) If $x \in H^-$, then there is a countable set $A \subseteq H$ such that $x \in A^-$; this follows from the fact that every sequential space has countable tightness.
(**C**) If $A \subseteq X$ and A is countable, then $|A^-| \leqslant 2^\omega$ (this holds in any sequential Hausdorff space).
(**C-S**) This follows from the Lindelöf and the pseudo-character hypotheses. □

We will now use Theorem 3.1 to prove (7) and (2) and leave (4) and (9) to the reader. Each of (7) and (2) has a "non-standard" choice for c. Let us also mention another common thread in the proofs of all four of these inequalities: in each case the verification of the cardinality condition (**C**) is inspired by the standard proof of the inequality $|X| \leqslant d(X)^{\chi(X)}$.

Proof that $|X| \leqslant 2^{\chi(X)wL_c(X)}$ for Urysohn spaces. Let $\chi(X)wL_c(X) = \kappa$, for each $x \in X$ let $\{V(\gamma, x)\colon \gamma < \kappa\}$ be a local base for x, let $c(H) = H^\theta$, and let $d(H) = H^-$. We need to check the following.

- (**T**) If $x \in H^\theta$, then there exists $A \subseteq H$ with $|A| \leqslant \kappa$ such that $x \in A^\theta$ (the hypothesis $\chi(X) \leqslant \kappa$ proves this).
- (**C**) If $A \subseteq X$ and $|A| \leqslant \kappa$, then $|A^\theta| \leqslant 2^\kappa$ (this follows from the inequality $|H^\theta| \leqslant |H|^{\chi(X)}$ in [7]; the Urysohn hypothesis is used here).
- (**C-S**) Let H be a θ-closed set and let $q \notin H$. There exists $\gamma < \kappa$ such that $V(\gamma, q)^- \cap H = \emptyset$. For each $x \in H$ there exists $\gamma_x < \kappa$ such that $V(\gamma_x, x) \cap V(\gamma, q) = \emptyset$. Now H is closed, $\{V(\gamma_x, x)\colon x \in H\}$ covers H, and $wL_c(X) \leqslant \kappa$; it follows that there exists $A \subseteq H$ with $|A| \leqslant \kappa$ such that $H \subseteq (\bigcup_{x\in A} V(\gamma_x, x))^-$. Now $(\bigcup_{x\in A} V(\gamma_x, x)) \cap V(\gamma, q) = \emptyset$ and so $q \notin (\bigcup_{x\in A} V(\gamma_x, x))^-$. The required function $f : A \to \kappa$ is $f(x) = \gamma_x$. $\square$

Proof that $|X| \leqslant 2^{H\psi(X)L(X)}$ for Hausdorff spaces. Let $H\psi(X)L(X) = \kappa$, and for each $x \in X$ let $\mathcal{U}_x$ be a collection of open neighborhoods of x with $|\mathcal{U}_x| \leqslant \kappa$, closed under finite intersections, and such that if $x \neq y$, then there exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U \cap V = \emptyset$. Let $\mathcal{U}_x = \{V(\gamma, x)\colon \gamma < \kappa\}$. Define c and d by $c(H) = \{x\colon V(\gamma, x) \cap H \neq \emptyset$ for all $\gamma < \kappa\}$ and $d(H) = H$. We need to check the following.

- (**T**) If $x \in c(H)$, then there exists $A \subseteq H$ with $|A| \leqslant \kappa$ such that $x \in c(A)$. (Here we see why tightness can be omitted—c is not the usual closure operator!)
- (**C**) If $A \subseteq X$ and $|A| \leqslant \kappa$, then $|c(A)| \leqslant 2^\kappa$ (follows from $|c(A)| \leqslant |A|^\kappa$; see the proof of Theorem 3.3 below for details).
- (**C-S**) Again see the proof of Theorem 3.3. $\square$

In recent years there has been considerable interest in relative versions of cardinal function inequalities; see, for example, [4,16,24]. Theorem 3.3 below gives a unified approach to several such results that are related to Arhangel'skiĭ's inequality. For example, we will use Theorem 3.3 to obtain a relative version of (9). Note that the cardinality condition (**C**) of Theorem 3.1 has been incorporated into the proof of Theorem 3.3.

Theorem 3.3. *Let X be a set, let $Y \subseteq X$, and for each $x \in X$ let $\{V(\gamma, x)\colon \gamma < \kappa\}$ be a collection of subsets of X such that $x \in V(\gamma, x)$ for all $\gamma < \kappa$. Assume the following*:

- (**I**) *given $\alpha, \beta < \kappa$, there exists $\gamma < \kappa$ such that $V(\gamma, x) \subseteq V(\alpha, x) \cap V(\beta, x)$* (intersection *condition*);
- (**H**) *if $x \neq y$, then there exists $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$* (Hausdorff *condition*);

(**C**) *if $f: X \to \kappa$, then there exists $A \subseteq X$ with $|A| \leqslant \kappa$ such that $Y \subseteq \bigcup_{x \in A} V(f(x), x)$* (cover *condition*).

Then $|Y| \leqslant 2^\kappa$.

Proof. For $A \subseteq X$, define $c(A) = \{x\colon\ V(\gamma, x) \cap A \neq \emptyset$ for all $\gamma < \kappa\}$. We first verify the inequality

$$\bigl|c(A)\bigr| \leqslant |A|^\kappa.$$

Let $x \in c(A)$; there exists $A_x \subseteq A$ such that $|A_x| \leqslant \kappa$ and $x \in c(A_x)$. Define $\Phi: c(A) \to P_\kappa(P_\kappa(A))$ by

$$\Phi(x) = \bigl\{V(\gamma, x) \cap A_x\colon\ \gamma < \kappa\bigr\}.$$

By condition (**I**), $x \in c(V(\gamma, x) \cap A_x)$ for all $\gamma < \kappa$. From this it follows that Φ is one-to-one (use condition (**H**) here).

Now construct a sequence $\{H_\alpha\colon\ \alpha < \kappa^+\}$ of subsets of X such that for $0 \leqslant \alpha < \kappa^+$,

(a) $|H_\alpha| \leqslant 2^\kappa$ (H_0 is any non-empty subset of X of cardinality at most 2^κ);
(b) for all $A \subseteq \cup_{\beta<\alpha} H_\beta$ such that $|A| \leqslant \kappa$:
   (b1) $c(A) \subseteq H_\alpha$ (the inequality $|c(A)| \leqslant |A|^\kappa$ is used here);
   (b2) if $f: A \to \kappa$, $W = \bigcup_{x \in A} V(f(x), x)$, and $Y - W \neq \emptyset$, then $(H_\alpha \cap Y) - W \neq \emptyset$.

Let $H = \bigcup\{H_\alpha\colon\ \alpha < \kappa^+\}$. Clearly $|H| \leqslant 2^\kappa$, and $c(H) = H$ by (b1). It remains to prove that $Y \subseteq H$. Suppose that $q \in Y - H$.

- For each $x \in H$ choose $\gamma_x < \kappa$ such that $q \notin V(\gamma_x, x)$.
- For each $x \notin H$ choose $\gamma_x < \kappa$ such that $V(\gamma_x, x) \cap H = \emptyset$ (recall that $c(H) = H$).

Define $f: X \to \kappa$ by $f(x) = \gamma_x$. By the cover condition (**C**), there exists $B \subseteq X$ with $|B| \leqslant \kappa$ such that $Y \subseteq \bigcup_{x \in B} V(f(x), x)$. Let $A = B \cap H$, so $A \subseteq H$ and $|A| \leqslant \kappa$. Moreover, $\{V(f(x), x)\colon\ x \in A\}$ covers $H \cap Y$. Let $W = \bigcup\{V(f(x), x)\colon\ x \in A\}$, and note that $(H \cap Y) \subseteq W$ and $q \notin W$. Choose $\alpha < \kappa^+$ such that $A \subseteq \bigcup_{\beta<\alpha} H_\beta$. By (b2), there exists $z \in (H_\alpha \cap Y) - W$, a contradiction of $(H \cap Y) \subseteq W$. □

From Theorem 3.3 we can derive the two inequalities $|X| \leqslant 2^{H\psi(X)L(X)}$ and $|X| \leqslant 2^{U\psi(X)aL(X)}$. In fact, let us extend the latter to a relative inequality. For this we need a relative version of aL. Let X be a topological space and let $Y \subseteq X$. The cardinal function $aL(Y, X)$ is the smallest infinite cardinal κ such that if $\mathcal{V}$ is any open cover of X, then there is a subcollection $\mathcal{V}_0$ of $\mathcal{V}$ such that $|\mathcal{V}_0| \leqslant \kappa$ and $\{V^-\colon\ V \in \mathcal{V}_0\}$ covers Y. For $Y = X$ this reduces to $aL(X)$.

Corollary 3.4. *Let X be a Urysohn space and let $Y \subseteq X$. Then $|Y| \leqslant 2^{U\psi(X)aL(Y,X)}$.*

Proof. Let $U\psi(X)aL(Y, X) = \kappa$, and for each $x \in X$ let $\mathcal{U}_x$ be a collection of open neighborhoods of x with $|\mathcal{U}_x| \leqslant \kappa$, closed under finite intersections, and such that if $x \neq y$, then there exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U^- \cap V^- = \emptyset$. Let $\mathcal{U}_x = \{U(\gamma, x)\colon \gamma < \kappa\}$, and for $x \in X$ and $\gamma < \kappa$ let $V(\gamma, x) = U(\gamma, x)^-$. The three conditions (**I**), (**H**), and (**C**) of Theorem 3.3 are easy to check. □

Corollary 3.4 also generalizes Theorem 12 in [8]: Let X be a Urysohn space and let Y be a relatively H-closed subset of X (if $\mathcal{V}$ is any open cover of X, then there is a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ such that $\{V^-\colon V \in \mathcal{V}_0\}$ covers Y). Then $|Y| \leqslant 2^{\chi(X)}$.

Theorems 3.1 and 3.3 are tailored to prove cardinal function inequalities that are related to Arhangel'skiĭ's inequality. Arhangel'skiĭ has a much more general approach (an "algorithm") for proving relative versions of cardinal inequalities. In the paper *A generic theorem in the theory of cardinal invariants of topological spaces* he states:

> *We formulate a general technical theorem, after which the proofs of many original results on cardinal inequalities acquire almost algorithmic character—they turn into a rather easy* (*though still not quite routine*) *verification of certain natural* (*mostly, technical*) *conditions.*

An example of an application of his algorithm is the following partial solution to Question 2 in Section 5; for details, see [4]. We state the countable version only. Stavrova [23] has also given a unified approach to a wide range of inequalities in cardinal invariants.

Theorem 3.5 *(Arhangel'skiĭ). Let X be a first-countable Hausdorff space such that the following holds for every closed subset H of X: if $\mathcal{V} = \bigcup \mathcal{V}_n$ is a collection of open sets in X that covers H, then each $\mathcal{V}_n$ has a countable subcollection $\mathcal{W}_n$ such that $H \subseteq \bigcup_{n\in\omega}(\bigcup \mathcal{W}_n)^-$. Then $|X| \leqslant 2^\omega$.*

Finally we arrive at what is undoubtedly the most complicated (to prove) variation of Arhangel'skiĭ's inequality. Recall that a space X is *linearly Lindelöf* if every increasing open cover of X has a countable subcover. This is equivalent to: every uncountable subset of X of regular cardinality has a complete accumulation point. In recent years Arhangel'skiĭ has emphasized the following general problem: What theorems about Lindelöf spaces extend to linearly Lindelöf spaces? Arhangel'skiĭ and Buzyakova prove:

Theorem 3.6 *(Arhangel'skiĭ–Buzyakova). Let X be a completely regular space that is sequential, linearly Lindelöf, and has $\psi(X) \leqslant 2^\omega$. Then $|X| \leqslant 2^\omega$.*

Arhangel'skiĭ and Buzyakova first prove:

Theorem 3.7. *Let X be a T_1 space such that*

(1) *X is ω_1-Lindelöf* (*every open cover of X of cardinality $\leqslant \omega_1$ has a countable subcover*);
(2) *X has countable tightness*;
(3) *if $A \subseteq X$ with $|A| \leqslant 2^\omega$, then $|A^-| \leqslant 2^\omega$*;

(4) *for each closed subset H of X of cardinality $\leqslant 2^{\omega}$, there is a collection $\mathcal{V}_H$ of open sets in X with $|\mathcal{V}_H| \leqslant 2^{\omega}$ and $\bigcap \mathcal{V}_H = H$.*

Then $|X| \leqslant 2^{\omega}$.

First note that every linearly Lindelöf space is ω_1-Lindelöf. The proof of Theorem 3.7 is given below and is an interesting variation of the closure arguments given thus far. Moreover, the statement itself is reminiscent of General Theorem 2 in [1]. We emphasize that the derivation of Theorem 3.6 from Theorem 3.7 requires considerable effort. The difficult to verify property is (4). Note that (4) is easy to prove if (a) X is Lindelöf or (b) CH holds. But for the case in which X is just linearly Lindelöf, the verification of (4) is extremely delicate; see [5] for details.

Proof of Theorem 3.7. Construct a sequence $\{H_\alpha\colon \alpha < \omega_1\}$ of *closed* subsets of X (use hypothesis (3) here) such that for $0 \leqslant \alpha < \omega_1$:

(a) $|H_\alpha| \leqslant 2^{\omega}$;
(b) if A is a countable subset of $\bigcup\{H_\beta\colon \beta < \alpha\}$, then $A^- \subseteq H_\alpha$;
(c) if W is a countable union of elements of $\{V\colon V \in \mathcal{V}_{H_\beta}$ and $\beta < \alpha\}$ and $W \neq X$, then $H_\alpha - W \neq \emptyset$.

Let $H = \bigcup\{H_\alpha\colon \alpha < \omega_1\}$; clearly $|H| \leqslant 2^{\omega}$. Now verify: H is closed [by (b) and $t(X) \leqslant \omega$]; $H = X$ [use (c) and the fact that H is ω_1-Lindelöf]. □

It is not known if the Arhangel'skiĭ–Buzyakova inequality extends to regular or Hausdorff spaces. However, Buzyakova has recently proved:

Theorem 3.8 *(Buzyakova). Every first-countable ω_1-Lindelöf Hausdorff space has cardinality at most $2^{2^{\omega}}$.*

4. Gryzlov's theorems and generalizations

In 1980 Gryzlov proved two variations of Arhangel'skiĭ's equality, each of which answers the original question of Alexandroff and Urysohn. His first result is as follows.

Theorem 4.1 *(Gryzlov). Let X be a compact T_1 space. Then $|X| \leqslant 2^{\psi(X)}$.*

Recall that a compact T_1-space need not be Hausdorff. The proof of Theorem 4.1 is a standard closure argument: construct a set $H = \bigcup\{H_\alpha\colon \alpha < \kappa^+\}$, where each H_α has size at most 2^{κ}, and then prove that $H = X$. However, the sequence $\{H_\alpha\colon \alpha < \kappa\}$ is constructed to insure that H is compact (as opposed to closed as in previous proofs). The key is Lemma 4.2 below, easily the most difficult and ingenious step in Gryzlov's proof (given that closure arguments are by now well understood).

Lemma 4.2. *Let X be a compact T_1 space with $\psi(X) \leqslant \kappa$ and let H be a subset of X such that every infinite subset of H of cardinality $\leqslant \kappa$ has a complete accumulation point in H. Then H is compact.*

Once Lemma 4.2 is available, the proof of Theorem 4.1 proceeds as follows.

Proof of Theorem 4.1. Let X be a compact T_1-space, let $\psi(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a collection of open neighborhoods of x such that $|\mathcal{V}_x| \leqslant \kappa$ and $\bigcap\{V\colon V \in \mathcal{V}_x\} = \{x\}$. Construct a sequence $\{H_\alpha\colon \alpha < \kappa^+\}$ of subsets of X such that for $0 \leqslant \alpha < \kappa^+$:

(a) $|H_\alpha| \leqslant 2^\kappa$;
(b) if A is an infinite subset of $\bigcup_{\beta<\alpha} H_\beta$ of cardinality $\leqslant \kappa$, then some point of H_α is a complete accumulation point of A;
(c) if W is a finite union of elements of $\{V\colon V \in \mathcal{V}_x$ and $x \in \bigcup_{\beta<\alpha} H_\beta\}$ and $W \neq X$, then $H_\alpha - W \neq \emptyset$.

Let $H = \bigcup\{H_\alpha\colon \alpha < \kappa^+\}$; clearly $|H| \leqslant 2^\kappa$. Now check: H is compact (use (b) and Lemma 4.2); $H = X$ (use (c) and the compactness of H). □

In 1983 Stephenson generalized Gryzlov's Theorem as follows.

Theorem 4.3 *(Stephenson). Let X be a 2^κ-total T_1 space with $\psi(X) \leqslant \kappa$. Then $|X| \leqslant 2^\kappa$ and X is compact.*

Stephenson's proof is interesting from the following point of view: he proves the compactness of X by first obtaining an upper bound on the cardinality of X. A space X is *κ-total* if for every subset H of X with $|H| \leqslant \kappa$, every filter base on H has an adherent point in X. This class of spaces was introduced by Vaughan in connection with problems on compactness-like properties of product spaces.

Now let us turn to Gryzlov's second theorem.

Theorem 4.4 *(Gryzlov). Let X be an H-closed space with $\psi_c(X) = \omega$. Then $|X| \leqslant 2^\omega$.*

Again the proof is a closure argument, but the technical details are even more delicate. The idea is to construct the sequence $\{H_\alpha\colon \alpha < \omega_1\}$ so that the union is an H-set. (Recall that a closed subset of an H-closed space need not be an H-set.) To do this, Gryzlov uses θ-accumulation points rather than complete accumulation points. However, this method of proof does not seem to extend to higher cardinality, and in 1982 Porter and Dow used a quite different attack to prove the general case.

Theorem 4.5 *(Dow–Porter). Let X be an H-closed space. Then $|X| \leqslant 2^{\psi_c(X)}$.*

A suitable modification of Gryzlov's original construction does extend to higher cardinality. The key is to replace θ-accumulation points with θ-cluster points. We now outline this approach. First of all, we will work with nets of the form $\xi = \{x_F\colon F \in \kappa^{<\omega}\}$;

here $\kappa^{<\omega}$ is the directed set that consists of all finite subsets of κ. Let $A \subseteq X$ and let $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ be a net in A. A point $x \in A$ is a *θ-cluster point of ξ relative to A* if given any open set R of X with $x \in R$ and any $\alpha < \kappa$, there exists $F \in \kappa^{<\omega}$ such that $\alpha \in F$ and $x_F \in (R \cap A)^-$. If $A = X$, we say that x is a *θ-cluster point of ξ*. It is easy to prove that for an H-closed space X, every net $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ in X has a θ-cluster point. We will use the following well-known characterization of H-sets.

Lemma 4.6. *Let $A \subseteq X$. The following are equivalent*:

(1) *A is an H-set.*
(2) *If $\mathcal{V}$ is a collection of open sets in X, closed under finite intersections, and such that $A \cap V \neq \emptyset$ for all $V \in \mathcal{V}$, then there exists $x \in A$ such that $x \in V^-$ for all $V \in \mathcal{V}$.*

The following replaces Lemma 3 in [15].

Lemma 4.7. *Let X be an H-closed space with $\psi_c(X) \leqslant \kappa$ and let A be a subset of X with the following property* (θ**CP**):

For every net $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ in A, there exists $x \in A$ such that x is a θ-cluster point of ξ relative to A.

Then A is an H-set.

Proof. First note that (θ**CP**) implies that A has the following cover property (**C**):

If $\{R_\alpha\colon\ \alpha < \kappa\}$ is a collection of open sets in X that covers A, then there exists $F \in \kappa^{<\omega}$ such that $A \subseteq \bigcup_{\alpha\in F}(R_\alpha \cap A)^-$.

Indeed, the definition of a θ-cluster point is motivated by the proof that (θ**CP**) implies (**C**).

We will prove that A is an H-set by verifying condition (2) of Lemma 4.6. Let $\mathcal{V}$ be a collection of open sets in X, closed under finite intersections, and such that $V \cap A \neq \emptyset$ for all $V \in \mathcal{V}$. By Zorn's Lemma, we may assume that $\mathcal{V}$ is maximal with respect to these two properties. (Thus, if R is an open set and $R \notin \mathcal{V}$, then there exists $V \in \mathcal{V}$ such that $R \cap A \cap V = \emptyset$.) Since X is H-closed, there exists $x \in X$ such that $x \in V^-$ for all $V \in \mathcal{V}$. The proof is complete if we can show that $x \in A$.

Since $\psi_c(X) \leqslant \kappa$, there is a collection $\{W_\alpha\colon\ \alpha < \kappa\}$ of open neighborhoods of x such that $\bigcap_{\alpha<\kappa} W_\alpha^- = \{x\}$. For each $\alpha < \kappa$, there exists $V_\alpha \in \mathcal{V}$ such that $(V_\alpha \cap A)^- \subseteq W_\alpha^-$. To see this, let $R = X - W_\alpha^-$. Clearly $x \notin R^-$ and so $R \notin \mathcal{V}$. Hence there exists $V_\alpha \in \mathcal{V}$ such that $V_\alpha \cap A \cap R = \emptyset$, and from this we obtain $(V_\alpha \cap A)^- \subseteq W_\alpha^-$.

Property (**C**) can now be used to show that $A \cap (\bigcap_{\alpha<\kappa}(V_\alpha \cap A)^-) \neq \emptyset$. (Otherwise, $\{X - (V_\alpha \cap A)^-\colon\ \alpha < \kappa\}$ is an open collection in X that covers A; use the fact that $A \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_k} \neq \emptyset$ for any $\alpha_1, \ldots, \alpha_k \in \kappa$.) Let $p \in A \cap (\bigcap_{\alpha<\kappa}(V_\alpha \cap A)^-)$. We then have: $p \in A$ and $p \in \bigcap_{\alpha<\kappa} W_\alpha^-$; so $p = x$ and $x \in A$ as required. □

The following replaces Lemma 4 in [15].

Lemma 4.8. *Let X be an H-closed space with $\chi(X) \leqslant \kappa$ and let $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ be a net in X. Then there is a subset A_ξ of X such that*

(1) $\xi \subseteq A_\xi$ *and* $|A_\xi| \leqslant \kappa$;
(2) *some point of A_ξ is a θ-cluster point of ξ relative to A_ξ.*

Proof. Let x be a θ-cluster point of ξ, and for each $y \in X$ let $\{V(\gamma, y)\colon\ \gamma < \kappa\}$ be a local base for y.

- We have: for each $\gamma < \kappa$ and each $\alpha < \kappa$, there exists $F \in \kappa^{<\omega}$ such that $\alpha \in F$ and $x_F \in V(\gamma, x)^-$.
- We want: for each $\gamma < \kappa$ and each $\alpha < \kappa$, there exists $F \in \kappa^{<\omega}$ such that $\alpha \in F$ and $x_F \in (V(\gamma, x) \cap A_\xi)^-$.

For each $\gamma < \kappa$ and each $F \in \kappa^{<\omega}$ such that $x_F \in V(\gamma, x)^-$, let $x(\gamma, F, \beta) \in V(\gamma, x) \cap V(\beta, x_F)$, $0 \leqslant \beta < \kappa$. The required set is

$$A_\xi = \{x\} \cup \xi \cup \bigl\{x(\gamma, F, \beta)\colon\ \gamma < \kappa,\ F \in \kappa^{<\omega},\ x_F \in V(\gamma, x)^-,\ \beta < \kappa\bigr\}. \qquad \square$$

Theorem 4.9 *(Gryzlov for $\kappa = \omega$; Dow–Porter). Let X be an H-closed space. Then $|X| \leqslant 2^{\chi(X)}$.*

Proof. Let $\chi(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a local base for x such that $|\mathcal{V}_x| \leqslant \kappa$. For each net $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ in X let A_ξ be a subset of X that satisfies (1) and (2) of Lemma 4.8. Construct a sequence $\{A_\alpha\colon\ \alpha < \kappa^+\}$ of subsets of X such that for $0 \leqslant \alpha < \kappa^+$:

(a) $|A_\alpha| \leqslant 2^\kappa$;
(b) if $\xi = \{x_F\colon\ F \in \kappa^{<\omega}\}$ is a net in $\bigcup_{\beta<\alpha} A_\beta$, then $A_\xi \subseteq A_\alpha$;
(c) if W is a finite union of elements of $\{V\colon\ V \in \mathcal{V}_x$ and $x \in \bigcup_{\beta<\alpha} H_\beta\}$ and $W^- \neq X$, then $A_\alpha - W^- \neq \emptyset$.

Let $A = \bigcup\{A_\alpha\colon\ \alpha < \kappa^+\}$; clearly $|A| \leqslant 2^\kappa$. By (b), the set A satisfies (θ**CP**); Lemma 4.7 now applies and A is an H-set. Finally, $A = X$ by (c) and the fact that A is an H-set. $\square$

Let $\langle X, \tau\rangle$ be an H-closed space and let $RO(X)$ be the collection of all regular open subsets of X. Then $RO(X)$ is a base for a courser topology τ_S on X. Moreover, $\langle X, \tau_S\rangle$ is an H-closed space and $\chi(\langle X, \tau_S\rangle) \leqslant \psi_c(\langle X, \tau\rangle)$. From these observations we have:

Corollary 4.10. *Let X be an H-closed space. Then $|X| \leqslant 2^{\psi_c(X)}$.*

5. Problems

This section is devoted to problems that are related to generalizations or variations of Arhangel'skiĭ's inequality. The first two come from Section 3.

Question 1. Let X be a regular space. Is $|X| \leqslant 2^{\chi(X)wL(X)}$?

This question is due to Bell, Ginsburg, and Woods and also Arhangel'skiĭ [3]. For Hausdorff spaces, the answer is NO; see Example 2.3 in [6]. On the other hand, for normal spaces, the answer is YES (see inequality (5) in Section 3). What happens for Urysohn spaces? For regular spaces?

Question 2. Let X be a Hausdorff space. Is $|X| \leqslant 2^{\chi(X)wL_c(X)}$?

This question is due to Arhangel'skiĭ (1979). The answer is YES for Urysohn spaces (due to Alas). A positive solution to Question 2 would give a unified approach to these two fundamental inequalities for Hausdorff spaces: $|X| \leqslant 2^{\chi(X)L(X)}$ and $|X| \leqslant 2^{\chi(X)c(X)}$.

Now let us turn to questions of the following general form. Suppose that X is a Lindelöf space with countable pseudo-character. What can we say about the cardinality of X? Note that X is T_1; for now we do not assume the Hausdorff hypothesis. This question is very natural and was raised by Arhangel'skiĭ in [2]. Tall's paper [25] is the best and most comprehensive source of information on questions of this type. The situation is rather negative and can be summarized by the following two results.

Theorem 5.1 *(Arhangel'skiĭ). Let X be a Lindelöf space with countable pseudo-character. Then $|X| <$ first measurable cardinal (if it exists).*

Theorem 5.2. (Juhász [20]) *Let $\lambda_0 = \omega$, $\lambda_{n+1} = 2^{2^{\lambda_n}}$, and $\kappa = \sup\{\lambda_n\colon n \in \omega\}$. Then there is a Lindelöf space with countable pseudo-character and of cardinality κ. (This construction works for any choice of λ_0 below the first measurable cardinal.)*

The cardinal κ of Theorem 5.2 has countable cofinality. In [25] Tall proves: it is consistent that for each regular cardinal $\kappa <$ first measurable cardinal, there is a Lindelöf space with countable pseudo-character and of cardinality κ.

Two partial (positive) result are:

Theorem 5.3 *(Charlesworth). Let X be a Lindelöf space with countable pseudo-character and a separating open cover $\mathcal{S}$ such that* $\operatorname{ord}(x, \mathcal{S}) \leqslant 2^\omega$ *for all $x \in X$. Then $|X| \leqslant 2^\omega$.*

Theorem 5.4 *(Buzyakova). Let X be a countably paracompact Lindelöf space with countable pseudo-character and countable tightness. Then $|X| \leqslant 2^\omega$.*

Now suppose that X is a Lindelöf space with countable pseudo-character that is also Hausdorff. Again, what can we say about the cardinality of X? The following result by Shelah (proved in 1978 but not published until 1996) shows that we cannot prove $|X| \leqslant 2^\omega$ without extra set-theoretic axioms.

Theorem 5.5 *(Shelah). The following is consistent with* ZFC $+ 2^\omega = \aleph_1$: *there is a zero-dimensional Lindelöf space with countable pseudo-character of cardinality $\aleph_2$.*

Can we prove that $|X| \leqslant \aleph_2, \aleph_3$, and so on? The answer is NO.

Theorem 5.6 *(Gorelic). Let κ be a cardinal. The following is consistent with* ZFC $+ 2^\omega = \aleph_1$: *$\kappa < 2^{\aleph_1}$ and there is a zero-dimensional Lindelöf space with countable pseudo-character and of cardinality $2^{\aleph_1}$.*

There is a partial (positive) result.

Lemma 5.7 *(Toroni). Let X be a pseudo-radial space. Then $t(X) \leqslant \psi(X)$.*

Theorem 5.8 *(Toroni). Let X be a pseudo-radial Hausdorff space. Then $|X| \leqslant 2^{\psi(X)L(X)}$.*

Proof. Follows from Lemma 5.7 and the inequality $|X| \leqslant 2^{\psi(X)t(X)L(X)}$. □

The consistency result of Gorelic suggests the following question.

Question 3. Let X be a Lindelöf Hausdorff space with countable pseudo-character. Can we prove that $|X| \leqslant 2^{\aleph_1}$?

There is a theorem in [25], proved independently by Alas and Tall, that gives a partial solution to Question 3 and perhaps shows that the question is not unreasonable. Recall that every $\aleph_1$-compact meta-Lindelöf space is Lindelöf.

Theorem 5.9 *(Alas–Tall). Let X be an $\aleph_1$-compact space with countable pseudo-character such that every subspace of X of cardinality at most $2^{\aleph_1}$ is meta-Lindelöf. Then $|X| \leqslant 2^{\aleph_1}$.*

A somewhat more general result, with proof, follows.

Theorem 5.10. *Let X be an $\aleph_1$-compact space, and assume that X satisfies*

(1) $\psi(X) \leqslant 2^{\aleph_1}$;
(2) *if $Y \subseteq X$ and $|Y| \leqslant 2^{\aleph_1}$, then Y is meta-Lindelöf.*

Then $|X| \leqslant 2^{\aleph_1}$.

Proof. For each $x \in X$ let $\mathcal{V}_x$ be a collection of open neighborhoods of x such that $|\mathcal{V}_x| \leqslant 2^{\aleph_1}$ and $\bigcap\{V\colon V \in \mathcal{V}_x\} = \{x\}$. Construct a sequence $\{H_\alpha\colon \alpha < \aleph_2\}$ of subsets of X such that for $0 \leqslant \alpha < \aleph_2$:

(a) $|H_\alpha| \leqslant 2^{\aleph_1}$;
(b) if $A \subseteq \bigcup_{\beta<\alpha} H_\beta$ with $|A| = \aleph_1$, then some point of H_α is a limit point of A;
(c) if W is a countable union of elements of $\{V\colon V \in \mathcal{V}_x$ and $x \in \bigcup_{\beta<\alpha} H_\beta\}$ and $W \neq X$, then $H_\alpha - W \neq \emptyset$.

Let $H = \bigcup\{H_\alpha\colon \alpha < \aleph_2\}$; clearly $|H| \leqslant 2^{\aleph_1}$, and by (b), H is $\aleph_1$-compact. By (2), H is also meta-Lindelöf. Now use (c) and the fact that H is Lindelöf to prove that $H = X$. □

Finally, there is an obvious question about which very little is known.

Question 4. Let X be a Lindelöf first-countable T_1-space. Can we prove that $|X| \leqslant 2^{\aleph_0}$?

6. The closure method and elementary submodels

Throughout this paper we have used the closure method to prove a wide range of inequalities, all inspired by Arhangel'skiĭ's original theorem. The closure method emerged from Arhangel'skiĭ's original proof, with simplifications by Šapirovskiĭ [22] and Pol [21]. As an early example of a proof by the closure method, let us mention Rudin's 1964 proof that every countably compact space with a point-countable base has a countable base (see [11]).

Pol [21] used the closure method to prove the Arhangel'skiĭ inequality $|X| \leqslant 2^{\chi(X)L(X)}$ and also the Hajnal–Juhász inequality $|X| \leqslant 2^{\chi(X)c(X)}$; somewhat later I used this method to prove the inequality $|X| \leqslant 2^{\psi(X)s(X)}$. The original proofs of the two inequalities by Hajnal and Juhász used the Erdös–Rado Partition Theorem, which itself can be proved using the closure method (see [18]). By now it is generally recognized that the closure method is a unifying device for most of the deeper inequalities in cardinal invariants. In summary, the development of the closure method is one of the most important legacies of the Arhangel'skiĭ inequality.

In 1980 Hajnal and Juhász proved the following remarkable reflection theorem: if every subspace of X of cardinality $\leqslant \aleph_1$ has a countable base, then X itself has a countable base. Their proof is a highly original application of the closure method. In 1988, Dow answered a question raised by Juhász by proving the following equally remarkable reflection theorem: if X is countably compact, and every subspace of X of cardinality $\leqslant \aleph_1$ is metrizable, then X itself is metrizable. Dow's method of proof introduced a new technique into set-theoretic topology: elementary submodels. Roughly speaking, this is a deeper and more sophisticated version of the closure method. See [12] for a detailed discussion of Dow's proof.

A recent paper by Fedeli and Watson is highly recommended to anyone who wants to understand and use this important new tool. They show that a wide variety of results in set-theoretic topology can be proved using elementary submodels. An especially nice feature of their paper is the use of the Lowenheim–Skölem Theorem to give a clear explanation of why the method works. Another nice feature is that on several occasions the authors give two proofs: a "formal" proof in which the required formulas are actually constructed, and an "in practice" proof in which the language of elementary submodels is used. Finally, we note that in each of the papers [12,14] there is a proof of the Arhangel'skiĭ inequality using elementary submodels.

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