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On the perfect matching of disjoint compact sets by noncrossing line segments in $\mathbb{R}^n$

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Abstract

Let V be a family of even disjoint line segments in $\mathbb{R}^n$ of f-equal width for a direction $f \in (\mathbb{R}^n)^*(n \geqslant 2)$, or even disjoint curve segments in $\mathbb{R}^n$ of f_n-equal width, where f_n is the normal direction for bases ($n \geqslant 2$), or even disjoint twisted triangular prisms in $\mathbb{R}^3$ of f_3-equal width. We prove that V has a perfect matching by open disjoint line segments in the complementary domain of the union of all the elements of V.

Keywords: Perfect matching; Compact sets; General dimension

0. Introduction

It is well known that any finite set of even points in the n-dimensional Euclidean space $\mathbb{R}^n$ admits a perfect matching by disjoint line segments. We seek a suitable generalization of this result from points to disjoint compact sets. For instance, consider disjoint line segments instead of points in $\mathbb{R}^n$. The case in the plane is treated in [4]. Some related topics in the plane are discussed in [2,3,5,6]. In this paper, we answer questions for several cases in higher dimensions. Our approach is geometric and includes a new construction procedure for obtaining such a perfect matching.

We introduce some fundamental notions: CL-figure, CL-perfect matching, f-equal width and so on in Section 1. In Section 2, we present the Local Matching Principle. Lemma I gives us the algorithm which constructs the global perfect matching by patching the local matchings. In concrete cases, we have only to show the validity of this principle. Lemmas II and III are technical tools in the general dimensional case of Section 5. In Section 3, we prove the existence of a CL-perfect matching of a set of even disjoint translates of a triangle in the plane. We prove the existence of a

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CL-perfect matching of even disjoint line segments in $\mathbb{R}^n$ with the same f-width for a $f \in (\mathbb{R}^n)^*(n \geqslant 2)$ in Section 4. Let D be a compact set in $\mathbb{R}^{n-1}$ where the embedding is specified ($n \geqslant 2$). A compact set E in $\mathbb{R}^n$ is said to be a D-cylinder, if $f_n^{-1}(\{c\}) \cap E$ is a set of translates of $D \times \{0\}$ in $\mathbb{R}^n$, for every $c \in \mathbb{R}$ such that $f_n^{-1}(\{c\}) \cap E \neq \phi$ where $f_n : (x_1, x_2, \ldots, x_{n-1}, x_n) \to x_n$. In Section 5, we state the cylinder theorem which contains a principle of descent of dimension. We derive the following results from this theorem:

(1) Under very weak restrictions, there exists a CL-perfect matching for an even disjoint family of T-cylinders with the same f_3-width in $\mathbb{R}^3$, where T is a triangle in $\mathbb{R}^2$.

(2) Under very weak restrictions, there exists a CL-perfect matching for an even disjoint family of P-cylinders with the same f_n-width in $\mathbb{R}^n$, where P is a point in $\mathbb{R}^{n-1}(n \geqslant 2)$.

1. Fundamental notions

We consider a family of disjoint objects. For such a finite family V, we call V an *even disjoint family* or an *odd disjoint family* if Card(V) is even or odd, respectively, where Card(V) denotes the cardinality of V.

Definition 1.1. Let V be a disjoint family of compact sets in $\mathbb{R}^n$, denoted by $\{C_a \mid a \in A\}$, and let L be a set of line segments in $\mathbb{R}^n$. The pair $F = (V, L)$ is said to be a *CL-figure* in $\mathbb{R}^n$, if V and L satisfy the following conditions:

(i) Each endpoint of any line segment of L is on the boundary of a compact set C_a ($a \in A$).

(ii) Any line segment of L has no common points with other line segments of L except possibly at common endpoints.

(iii) Any line segment of L has no common point with any C_a ($a \in A$) except for its two endpoints.

If we regard the elements of V and the elements of L as vertices and edges, respectively, maintaining the incidence relation between V and L, then we can obtain a graph called the *skeleton* of the CL-figure $F = (V, L)$. Definitions of graph theory terms can be found in [1]. Henceforth, we will use the graph-theoretic terms properly belonging to the skeleton of F on the CL-figure F itself.

Definition 1.2. For a disjoint family V of compact sets, we define a *CL-matching* of V by a CL-figure $M = (V, L)$, if no two elements of L are adjacent to each other. The set of elements of V which are the ends of some $l \in L$ comprise the *saturated ends set* $V(M)$. If $V(M) = V, M$ is called a *CL-perfect matching* of V.

For an odd disjoint family V of compact sets and for an element E of V, we define a *CL-perfect matching* (of V) *with residue* E by a CL-matching (V, L) of V such that $(V \setminus \{E\}, L)$ is a CL-perfect matching.

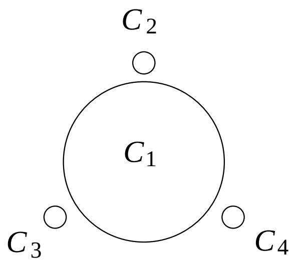

Fig. 1.

For instance, if $V=\{C_1,C_2,C_3,C_4\}$ is a family of closed discs in $\mathbb{R}^2$ as shown in Fig. 1, then there does not exist any CL-perfect matching of V.

We define the technical terms for the general dimensional case.

Definition 1.3. Let X^* denote the dual space of a linear space X. For a compact set C of $\mathbb{R}^n$ and for a non-trivial linear function $f\in(\mathbb{R}^n)^*$, let $m(C,f)=\min\{f(x)\mid x\in C\}$ and $M(C,f)=\max\{f(x)\mid x\in C\}$. $m(C,f)$ is called the *f-supporting value* of C. Define $d(C,f)$ by $d(C,f)=M(C,f)-m(C,f)$; $d(C,f)$ is called the *f-width* of C. If $d(C_1,f)=d(C_2,f)$ for two compact sets C_1,C_2 of $\mathbb{R}^n$ and $f\in(\mathbb{R}^n)^*$, then C_1 and C_2 are said to have *f-equal width*. Let $V=\{C_a\mid a\in A\}$ be a family of compact sets in $\mathbb{R}^n$. V is said to have *f-equal width* for $f\in(\mathbb{R}^n)^*$ if C_a and C_b have f-equal width for every $a,b\in A$. Confer with Fig. 3.

2. The local matching principle and three lemmas

In this section, we present the following principle.

2.1. Local matching principle

Let V be an even disjoint family of compact sets in $\mathbb{R}^n$. If V satisfies the following condition for a linear function $f\in(\mathbb{R}^n)^*$ and a real number $c\in\mathbb{R}$, we say that V satisfies the *local matching principle* (LMP) at the f-value c.

Condition. Consider any CL-matching $M=(V,L)$ of V such that $V(M)=\{C\in V\mid C\subsetneq f^{-1}((-\infty,c))\}$. Define V_c by $V_c=\{C\in V\mid C\cap f^{-1}(\{c\})\neq\phi\}$. Then either of the following conditions LMP-I or LMP-II holds:

(LMP-I) Card(V_c) *is odd*: There exists $\tilde{C}$ whose f-supporting value is minimum among the elements of $V\setminus(V(M)\cup V_c)$. Then there exists a set L_c of line segments in $\mathbb{R}^n$ such that the pair $(V_c\cup\{\tilde{C}\},L_c)$ is a CL-perfect matching of $V_c\cup\{\tilde{C}\}$ and that the pair $(V,L\cup L_c)$ is a CL-matching of V.

(LMP-II) Card(V_c) *is even*: There exists a set L_c of line segments in $\mathbb{R}^n$ such that the pair (V_c,L_c) is a CL-perfect matching of V_c and that the pair $(V,L\cup L_c)$ is a CL-matching of V.

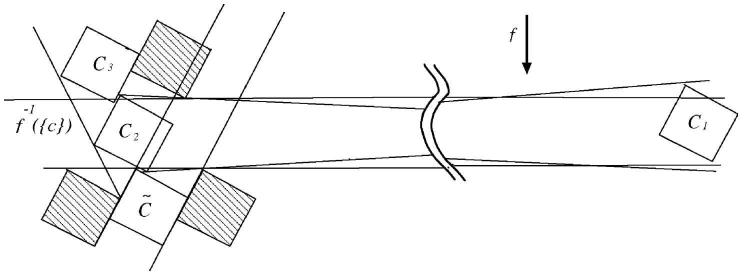

Fig. 2.

Remark 2.1. Let V be an even disjoint family of translates of a square in $\mathbb{R}^2$ and let $V_c = \{C_1, C_2, C_3\}$ as shown in Fig. 2. Then V does not satisfy the LMP at the f-value c.

We state the algorithm which constructs the global perfect matching by patching the local matchings.

Lemma I (Algorithm for the construction of a CL-perfect matching). *Let V be an even disjoint family of compact sets in* $\mathbb{R}^n$ *which has f-equal width for a linear function f. If V satisfies the following conditions, then there exists a CL-perfect matching of V.*

(i) *The f-supporting value of each element of V is distinct from the others.*

(ii) *V satisfies the LMP at the f-value c for every* $c \in \mathbb{R}$ *except for a finite set of real numbers.*

Proof. We shall construct the ascending chain $M_1, M_2, \ldots$ of CL-matchings of V. Since $L_1 \subsetneq L_2 \subsetneq \cdots$ holds for $M_j = (V, L_j)$, then $V(M_1) \subsetneq V(M_2) \subsetneq \cdots$ also holds for the saturated ends sets. Thus $V = V(M_J)$ holds for some natural number $J \in \mathbb{N}$ by the finiteness of V. Therefore, a CL-perfect matching of V results. Consider $V = \{C_1, C_2, \ldots, C_{2N}\}$ such that $c_1 < c_2 < \cdots < c_{2N}$ holds for $c_i = m(C_i, f)$ $(1 \leqslant i \leqslant 2N)$ by condition (i). Since V has f-equal width, we simply denote the common value $d(C_i, f)$ for every i by d.

(A) *The construction of the CL-matching* M_1 *of* V: Since only C_1 and C_2 among the elements of V intersects the closed convex region $f^{-1}([c_1, c_2])$, we can join C_1 to C_2 by some line segment l_1 in this region. Then $(V, \{l_1\})$ is a CL-matching of V. Let $L_1 = \{l_1\}, M_1 = (V, L_1)$. Then choose some positive number ε_1 such that $c_2 < c_2 + \varepsilon_1 < \min\{c_2 + d, c_3\}$ holds and the value $c_2 + \varepsilon_1 + md$ is never equal to c_3 for every $m \in \mathbb{N}$.

(B) *The construction of* M_{j+1} *by the CL-matching* M_j of V: For the constructed CL-matching $M_j = (V, L_j)$ of V and a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j$, let $V(M_j) = \{C_1, C_2, \ldots, C_{2k}\}$ and $L_j = \{l_1, l_2, \ldots, l_k\}$. In particular, suppose that the value $c_{2k} + \varepsilon_j + md$ is never equal to c_{2k+1} for every $m \in \mathbb{N}$.

We can now construct a CL-matching M_{j+1} of V. There exists a unique natural number $m(j)$ by the choice of ε_j such that $c_{2k+1} < c_{2k} + \varepsilon_j + m(j)d < c_{2k+1} + d$. Let

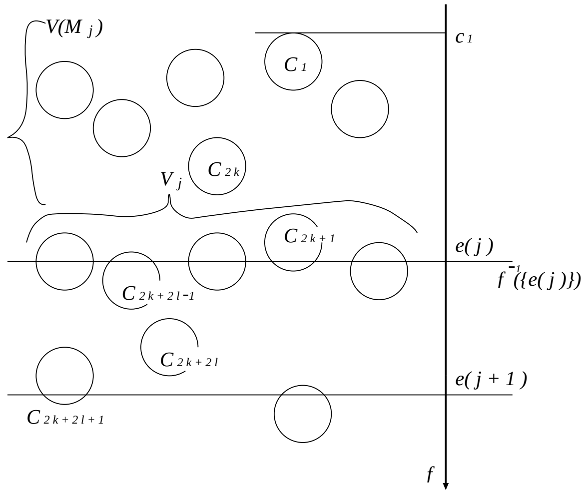

Fig. 3.

$e(j) = c_{2k} + \varepsilon_j + m(j)d$. Then, all the elements of $V(M_j)$ are included in the open half-space $f^{-1}((-\infty, e(j)))$, and $C_{2k+1} \in V_j$ holds for $V_j = \{C \in V \mid C \cap f^{-1}(\{e(j)\}) \neq \phi\}$. Therefore, either of the following conditions (I) or (II) holds since V satisfies the LMP at the f-value $e(j)$.

(I) Card(V_j) *is odd* (Fig. 3): Let $V_j = \{C_{2k+1}, C_{2k+2}, \ldots, C_{2k+2l-1}\}$. Then by LMP-I, there exist $(V_j \cup \{C_{2k+2l}\}, L'_j)$ of a CL-perfect matching of $V_j \cup \{C_{2k+2l}\}$ and $(V, L_j \cup L'_j)$ of a CL-matching of V. Since we can choose some positive number ε_{j+1} such that $\varepsilon_{j+1} < d$ and the value $c_{2k+2l} + \varepsilon_{j+1} + md$ is never equal to $c_{2k+2l+1}$ for every $m \in \mathbb{N}$, we can construct $M_{j+1} = (V, L_{j+1})$ for $L_{j+1} = L_j \cup L'_j$.

(II) Card(V_j) *is even*: Let $V_j = \{C_{2k+1}, C_{2k+2}, \ldots, C_{2k+2l}\}$. Then by LMP-II, there exist (V_j, L'_j) of a CL-perfect matching of V_j and $(V, L_j \cup L'_j)$ of a CL-matching of V. Since we can choose some positive number ε_{j+1} such that $\varepsilon_{j+1} < d$ and the value $c_{2k+2l} + \varepsilon_{j+1} + md$ is never equal to $c_{2k+2l+1}$ for every $m \in \mathbb{N}$, we can construct $M_{j+1} = (V, L_{j+1})$ for $L_{j+1} = L_j \cup L'_j$. □

This lemma reduces the existence of a perfect matching in the concrete cases to the local problem.

Definition 2.1. Let $V = \{C_1, C_2, \ldots, C_N\}$ be a disjoint family of compact sets in $\mathbb{R}^n$ such that $m(C_i, f) < m(C_{i+1}, f)$ $(1 \leqslant i \leqslant N-1)$ for a linear function f. Then C_k is said to be f*-even* in V or f*-odd* in V, respectively, if k is even or odd.

Lemma II. *Let V be an odd disjoint family of compact sets in* $\mathbb{R}^n$ *which has f-equal width for a linear function f and let E be f-odd in V. If V satisfies the following conditions, then V has a CL-perfect matching with residue E.*

(i) *The f-supporting value of each element of V is distinct from the others.*

(ii) *V satisfies the LMP at the f-value c and* $(-f)$*-value c for every* $c \in \mathbb{R}$ *except for a finite set of real numbers.*

Proof. Let $V_-(E) = \{C \in V \mid m(C,f) < m(E,f)\}$ and $V_+(E) = \{C \in V \mid M(C,f) > M(E,f)\}$. Since V has f-equal width, $V_-(E) \cap V_+(E) = \phi$ and $V_-(E) \cup V_+(E) = V \setminus \{E\}$. Note that both Card$(V_-(E))$ and Card$(V_+(E))$ are even. By an argument similar to the one used in Lemma I for $V_-(E)$ and $(-f)$, or $V_+(E)$ and f, we can construct a set L_- or L_+ of line segments in $f^{-1}((-\infty, m(E,f)))$ or $f^{-1}((M(E,f), +\infty))$ such that (V_-, L_-) or (V_+, L_+) is a CL-perfect matching, respectively. Let $L = L_- \cup L_+$. Then $(V \setminus \{E\}, L)$ is a CL-perfect matching of $V \setminus \{E\}$ and (V, L) is a CL-matching of V. □

Definition 2.2. Let T be a set of points in $\mathbb{R}^n$. T is said to be *weakly* 2*-general* if for every point $P \in T$, there exists another point $Q \in T$ such that the line PQ does not contain any other point of T.

In the following, we denote a dense set $\{f \in (\mathbb{R}^n)^* \mid ||f|| = 1\}$ by $S((\mathbb{R}^n)^*)$, where $|| \cdot ||$ is the operator norm.

Lemma 2.1. *Let T be a weakly* 2*-general set of odd points in* $\mathbb{R}^n$ $(n \geqslant 2)$. *Then for every dense subset* $F \subset S((\mathbb{R}^n)^*)$ *and every point* $P \in T$, *there exists two linear functions* $f_1, f_2 \in F$ *which satisfy the following conditions*:

(i) *The* f_i*-supporting value of each point of T is distinct from the others for* $i = 1, 2$.

(ii) *P is* f_1*-odd and* f_2*-even in T.*

Proof. Take a point $P \in T$ and a dense subset $F \subset S((\mathbb{R}^n)^*)$. Since there exists another point $Q \in T$ such that the line PQ does not contain any other point of T, we can take some linear function $f \in S((\mathbb{R}^n)^*)$ such that $f^{-1}(f(P)) \cap T = \{P, Q\}$. Let $T_- = \{R \in T \mid f(R) < f(P)\}$ and $T_+ = \{R \in T \mid f(R) > f(P)\}$. By moving the direction f slightly in $S((\mathbb{R}^n)^*)$, we can obtain two linear functions $g_1, g_2 \in F$ which satisfy the following conditions (Fig. 4):

(a) The g_i-supporting value of each point of T is distinct from the others for $i = 1, 2$.

(b) $T_- = \{R \in T \mid g_2(R) < \min\{g_2(P), g_2(Q)\}\}$ and $T_+ = \{R \in T \mid g_1(R) > \max\{g_1(P), g_1(Q)\}\}$.

(c) $g_2(P) < g_2(Q)$ and $g_1(P) > g_1(Q)$.

Then, if P is g_1-odd or g_1-even, then P is g_2-even or g_2-odd, respectively, by virtue of Q on $f^{-1}(f(P))$. By letting $f_1 = g_1, f_2 = g_2$ or that $f_1 = g_2, f_2 = g_1$, we complete the proof. □

Definition 2.3. Let C be a compact set in $\mathbb{R}^n$. Let V be a family of translates of C in $\mathbb{R}^n$, denoted by $\{C + \overrightarrow{OP_i} \mid 1 \leqslant i \leqslant N\}$. We call the point set $\{P_i \mid 1 \leqslant i \leqslant N\}$ the *translation set* of V, denoted by *Trans*(V). V is said to be *weakly* 2*-general* if Trans(V) is weakly 2-general.

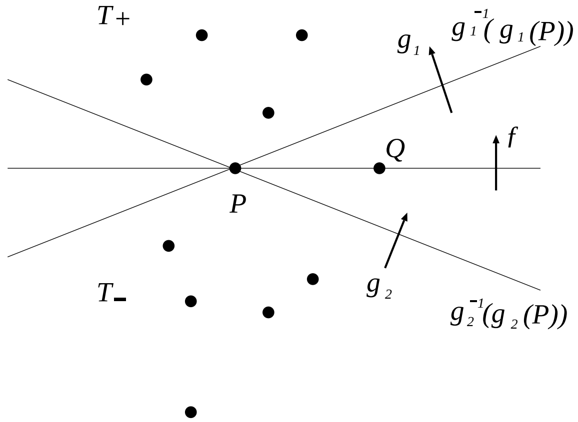

Fig. 4.

Lemma 2.2 (The parity lemma). *Let V be a weakly 2-general odd family of translates of a compact set C in $\mathbb{R}^n$ ($n \geqslant 2$). Then for any dense subset $F \subset S((\mathbb{R}^n)^*)$ and for every $E \in V$, there exist two linear functions $f_1, f_2 \in F$ which satisfy the following conditions*:

(i) *The f_i-supporting value of each element of V is distinct from the others ($i = 1, 2$).*

(ii) *E is f_1-odd and f_2-even in V.*

Proof. By applying Lemma 2.1 to Trans(V), we obtain this result easily. □

Lemma III (The residue element lemma). *Let V be a weakly 2-general odd family of translates of a compact set C in $\mathbb{R}^n$ ($n \geqslant 2$) and let F be any dense subset of $S((\mathbb{R}^n)^*)$. If V and F satisfy the following conditions, then V has a CL-perfect matching with residue E for every element $E \in V$.*

(i) *If $f \in F$, then $(-f) \in F$.*

(ii) *The f-supporting value of each element of V is distinct from the others for every $f \in F$.*

(iii) *For every $f \in F$, V satisfies the LMP at the f-value c for every $c \in \mathbb{R}$ with finite exceptions.*

Proof. For every $E \in V$, we can take a linear function $f \in F$ such that E is f-odd in V by virtue of Lemma 2.2. Hence, by Lemma II, we obtain the desired result. □

3. The existence of a CL-perfect matching of triangles

In this section, we propose the following theorem as a concrete example in $\mathbb{R}^2$ which applies our principle.

Theorem 3.1. *Let V be a disjoint family of translates of a triangle T in $\mathbb{R}^2$.*

(I) *If* Card(V) *is even, then there exists a CL-perfect matching of V.*

(II) *If V is weakly* 2*-general and* Card(V) *is odd, then there exists a CL-perfect matching with residue E for every element $E \in V$.*

Proof. In this proof, we use the next terminology: Let A and B be pairwise disjoint convex sets. If a line s intersects both A and B, then the closure of the subset of s consisting of the points between A and B is called the *line segment AB on s*.

Since Card(V) is finite, we may assume that the f-supporting value of any element of V is distinct from the others for some linear function f in $\mathbb{R}^2$. Note that for sufficiently many $c \in \mathbb{R}$, the lines $f^{-1}(\{c\})$ are not tangent to any element of V, that is, the boundary of each triangle intersects some line $f^{-1}(\{c\})$ precisely twice. We show that for such a linear function $f \in (\mathbb{R}^2)^*$ and $c \in \mathbb{R}$, V satisfies the LMP at the f-value c. Suppose that the line $f^{-1}(\{c\})$ is parallel to the x-axis, where it is normal to the direction f in $\mathbb{R}^2$. We take a CL-matching $M=(V,L)$ of V such that $V(M)=\{C \in V \mid C \subset f^{-1}((-\infty, c))\}$, and let $V_c = \{C \in V \mid C \cap f^{-1}(\{c\}) \neq \phi\}$.

For $T = \triangle ABC$ we can assume that A or C are on a non-negative portion of the x- or y-axis, respectively, and B is in the first quadrant. Denote the line passing through A and B by l, denote the parallel line through C to l by l' and let D be the point of intersection of the x-axis and l'. For every triangle $T_i = \triangle A_iB_iC_i \in V$, let A_i, B_i or C_i be correspondent to A, B or C on T, respectively. Moreover, let D_i, l_i or l'_i for T_i be correspondent to D, l or l' for T, respectively. For a line t, denote the x-coordinate of the point of intersection of x-axis and t by $x(t)$. Again, let $V_c = \{T_1, T_2, \ldots, T_K\}$ indexed according to their order in the line $f^{-1}(\{c\})$ such that $x(l_i) < x(l_{i+1})$ $(1 \leqslant i \leqslant K-1)$. We call T_{2i-1} or T_{2i} an *odd* or *even triangle*, respectively.

(I) (1) If Card(V_c) is even, then the validity of LMP-II is trivial. In fact, for $V_c = \{T_1, T_2, \ldots, T_{2k}\}$ and $L_c = \{T_1T_2, T_3T_4, \ldots, T_{2k-1}T_{2k}$: line segments on the line $f^{-1}(\{c\})\}$, (V_c, L_c) is a CL-perfect matching of V_c and $(V, L \cup L_c)$ is a CL-matching of V.

(2) In case Card(V_c) is odd, let $V_c = \{T_1, T_2, \ldots, T_{2k-1}\}$ and let T_{2k} be the element which has the minimum f-value in $V \setminus (V(M) \cup V_c)$. Now, if we can join T_{2k} to an odd triangle T_{2p-1} of V_c by a line segment m which does not intersect any other element of V, then for $L_c = \{T_1T_2, T_3T_4, \ldots, T_{2p-3}T_{2p-2}, T_{2p}T_{2p+1}, \ldots, T_{2k-2}T_{2k-1}$: line segments on $f^{-1}(\{c\})\} \cup \{m\}$, $(V_c \cup \{T_{2k}\}, L_c)$ is a CL-perfect matching of $V_c \cup \{T_{2k}\}$ and $(V, L \cup L_c)$ is a CL-matching of V. Hence, LMP-I is valid then.

For any T_i, let R_i^- or R_i^+ be the closed region surrounded by l_i, l'_i, and C_iA_i or C_iB_i, respectively, as shown in Fig. 5. Then we claim by the convexity of a triangle that R_i^- never intersects any T_j for $m(T_j, f) > m(T_i, f)$, and R_i^+ also never intersects any T_j for $m(T_j, f) < m(T_i, f)$.

(2.1) Suppose that R_{2k}^- does not intersect any even triangle.

(i) If R_{2k}^- intersects an odd triangle, we can take a line segment m parallel to l in R_{2k}^-. Then, $(V_c \cup \{T_{2k}\}, L_c)$ is a CL-perfect matching of $V_c \cup \{T_{2k}\}$ and $(V, L \cup L_c)$ is a CL-matching of V.

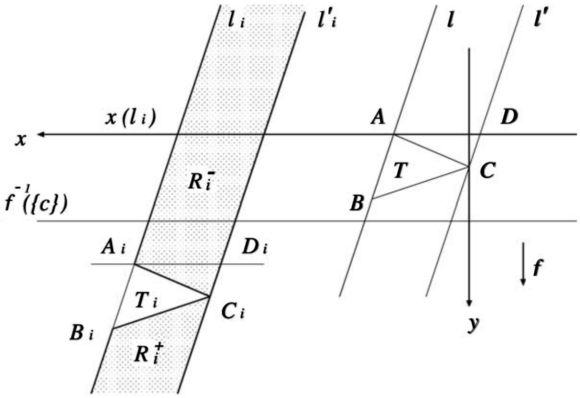

Fig. 5.

(ii) If R_{2k}^{-} does not intersect any odd triangle, that is, it does not intersect any element of V_c, then the line segment $f^{-1}(\{c\}) \cap R_{2k}^{-}$ is inevitably neighbor of a unique odd triangle, denoted by Δ. Then we can find a line segment m, joining T_{2k} to Δ, on a line passing through A_{2k} or D_{2k}.

(2.2) Suppose that R_{2k}^{-} intersects an even triangle T_{2p} $(1 \leqslant p \leqslant k-1)$.

(i) If $x(l_{2k}) \geqslant x(l_{2p})$, we can join T_{2k} to T_{2p+1} by the line segment passing through A_{2k} which does not intersect any other element of V.

(ii) $x(l_{2k}) < x(l_{2p})$: If $x(l'_{2p}) \leqslant x(l_{2p-1}) < x(l_{2p})$ and $m(T_{2p-1}, f) > m(T_{2p}, f)$, we can join T_{2k} to T_{2p-1} or T_{2p+1} by the line segment on l_{2p-1}. If $x(l'_{2p}) \leqslant x(l_{2p-1}) < x(l_{2p})$ and $m(T_{2p-1}, f) < m(T_{2p}, f)$, we claim that l'_{2p} never intersects any element of $V_c \setminus \{T_{2p-1}, T_{2p}\}$ since V is a disjoint family of translates of a triangle. Thus, we can join T_{2k} to T_{2p-1} by the line segment on a line obtained by moving l'_{2p} slightly in a parallel motion with l. If $x(l'_{2k}) \leqslant x(l_{2p-1}) < x(l'_{2p})$, we can take the line segment on l_{2p-1}. Last, if $x(l_{2p-1}) < x(l'_{2k})$, we can take the line segment through D_{2k}.

(II) For every element E, we can obtain a linear function f such that E is f-odd by Lemma 2.2. Therefore, by Lemma III, the theorem follows. □

4. The existence of a CL-perfect matching of line segments

The following theorem is a concrete example in the general dimension which applies our principle.

Theorem 4.1. *Let f be a non-trivial linear function in $\mathbb{R}^n$ and let V be an even disjoint family of line segments in $\mathbb{R}^n$ $(n \geqslant 2)$. If V has f-equal width such that the f-supporting value of each element of V is distinct from the others, then there exists a CL-perfect matching of V.*

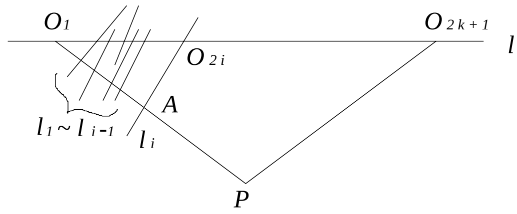

Fig. 6.

We first show a few lemmas.

Lemma 4.1. *Let $O_1, O_2, \ldots, O_{2n-1}$ be odd points on a line l located in the order of their indices in $\mathbb{R}^2$ and let $L = \{l_1, l_2, \ldots, l_{n-1}\}$ be a disjoint family of line segments in $\mathbb{R}^2$ such that l_i intersects l at the point O_{2i} for every i $(1 \leqslant i \leqslant n-1)$. Then, for a point P in $\mathbb{R}^2$ which is neither on l nor on any element of L, there exists some line segment PO_{2i-1} $(1 \leqslant i \leqslant n)$ which does not intersect any element of L.*

Proof. By induction on n. It is trivial when $n = 1$. When $n = 2$, l_1 does not intersect PO_1 or PO_3. Suppose that the proposition holds for $n \leqslant k$ and consider the case $n = k + 1$. If PO_1 does not intersect any element of L, then PO_1 satisfies the proposition. Thus, suppose that PO_1 intersects some $l_i \in L$ and let A be the point of intersection of PO_1 and l_i (Fig. 6). Since AO_{2i} does not intersect any l_j $(1 \leqslant j \leqslant i-1)$, each l_j intersects AO_1, i.e., PO_1, or the endpoint of l_j is in the interior region of $\triangle O_1AO_{2i}$. Therefore every PO_{2m+1} $(i \leqslant m \leqslant k)$ does not intersect any l_j $(1 \leqslant j \leqslant i-1)$. We can apply the induction hypothesis for the odd points on l, $\{O_{2i+1}, O_{2i+2}, \ldots, O_{2k+1}\}$ and the disjoint family of line segments $\{l_{i+1}, l_{i+2}, \ldots, l_k\}$. □

Lemma 4.2. *Given a line l and an odd disjoint family of line segments $\{l_1, l_2, \ldots, l_{2n-1}\}$ in $\mathbb{R}^2$ such that l_i intersects l at the point O_i for every i $(1 \leqslant i \leqslant 2n-1)$ where $O_1, O_2, \ldots, O_{2n-1}$ are located on l in the order of their indices. Let P be a point in $\mathbb{R}^2$ which is neither on l nor on l_i $(1 \leqslant i \leqslant 2n-1)$. Then, there exists some line segment m satisfying the following conditions*:

(i) *One of the endpoints of m is P and the other is on some odd line segment l_{2k-1}.*

(ii) *The set of points in m except for the endpoints are on the same side of P with respect to l.*

(iii) *m does not intersect the other l_i's except l_{2k-1} $(1 \leqslant i \leqslant 2n-1, i \neq 2k-1)$.*

Proof. We call l_{2k-1} or l_{2k} an *odd* or *even line segment*, respectively. If we apply Lemma 4.1 to the odd points $O_1, O_2, \ldots, O_{2n-1}$ and the disjoint family of even line segments $L = \{l_2, l_4, \ldots, l_{2n-2}\}$, then there exists some line segment PO_{2r-1} $(1 \leqslant r \leqslant n)$ which does not intersect any even line segment, i.e., PO_{2r-1} only intersects some odd line segments. In fact, PO_{2r-1} intersects at least one odd line segment l_{2r-1}. Let l_{2k-1}

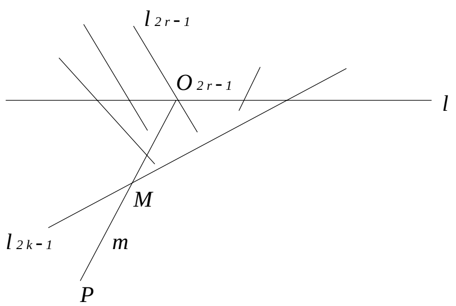

Fig. 7.

be an odd line segment such that l_{2k-1} has the point of intersection of PO_{2r-1} which is the nearest to P (Fig. 7). Therefore, we can take PM for m, where M is the point of intersection of PO_{2r-1} and l_{2k-1}. □

Lemma 4.3. *Let V be a finite set of non-collinear points in $\mathbb{R}^n$ $(n \geqslant 2, \mathrm{Card}(V) \geqslant 3)$. We can construct the polygon P such that the vertex set of P is V and the edge set of P is a set of line segments.*

Proof. When $n = 2$, let $\{p, q, r\}$ be three points on the boundary of the convex hull of V such that p is adjacent to both q and r. Let l be the half-line passing through q with the initial point p. Rotate l for the supporting point p until l meets r. During this operation, the line l meets all the elements of V. When l meets some points at the kth time, we denote the l by l_k $(0 \leqslant k \leqslant s)$. Moreover, denote the points on l_k except p by $v(k,1), v(k,2), \ldots, v(k,m_k)$ such that $v(k,i)$ is nearer to p than $v(k,j)$ for $i < j$. Then, we can construct P according to the following sequence:

$p \to q = v(0,1) \to v(0,2) \to \cdots \to v(0,m_0) \to v(1,1) \to \cdots \to v(1,m_1) \to \cdots \to v(k,1) \to \cdots \to v(k,m_k) \to \cdots \to v(s-1,m_{s-1}) \to v(s,m_s) \to v(s,m_s-1) \to \cdots \to v(s,1) = r \to p$.

For $n \geqslant 3$, we take a plane $H \subset \mathbb{R}^n$ such that $H \cap V = \phi$. If we consider the projection of a suitable direction from V to H, our problem is reduced to the case $n = 2$ on H. □

We can now prove the theorem.

Proof of Theorem 4.1. Denote the common f-width of any element of V by d. Then the values of $c + nd$ are not equal to the supporting value of each element of V for sufficiently many $c \in \mathbb{R}$ and every integer $n \in Z$. We show that V satisfies the LMP at the f-value c for such $c \in \mathbb{R}$. Take a CL-matching $M = (V, L)$ such that $V(M) = \{l \in V \mid l \subset f^{-1}((-\infty, c))\}$ and let $V_c = \{l \in V \mid l \cap f^{-1}(\{c\}) \neq \phi\}$.

(1) If $\mathrm{Card}(V_c)$ is even, the validity of LMP-II is trivial. In fact, let O_i be the point of intersection of the hyperplane $f^{-1}(\{c\})$ and l_i for $V_c = \{l_1, l_2, \ldots, l_{2k}\}$. We obtain

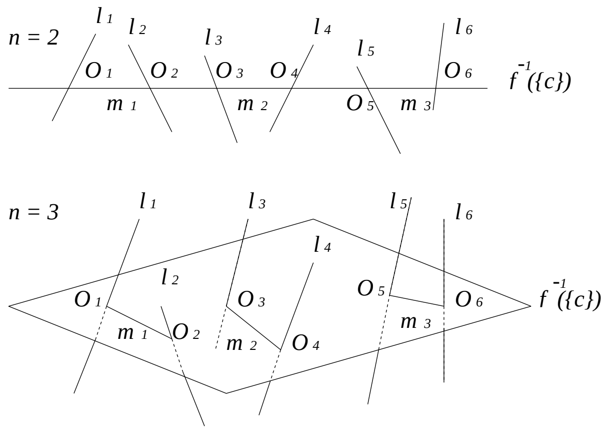

Fig. 8.

a CL-perfect matching of the set of points $\{O_1, O_2, \ldots, O_{2k}\}$ on $f^{-1}(\{c\})$ by using a set of k line segments $L_c = \{m_1, m_2, \ldots, m_k\}$ on $f^{-1}(\{c\})$ as shown in Fig. 8. Hence (V_c, L_c) is a CL-perfect matching of V_c and $(V, L \cup L_c)$ is a CL-matching of V.

(2) If Card(V_c) is odd, we show the validity of LMP-I. For $V_c = \{l_1, l_2, \ldots, l_{2k-1}\}$, let l_{2k} be the element having the smallest f-supporting value of $V \setminus \{V(M) \cup V_c\}$ and let P be one of the endpoints of l_{2k} which has the smaller f-value.

(2.1) All the elements of V_c are on a plane π .

Let l be the line of intersection of the hyperplane $f^{-1}(\{c\})$ and the plane π, and let O_i be the point of intersection of the element l_i of V_c and $f^{-1}(\{c\})$, i.e., of l_i and l $(1 \leqslant i \leqslant 2k-1)$. Suppose that $O_1, O_2, \ldots, O_{2k-1}$ are located on l in the order indicated by their indices.

(i) P is not on π: If we take some line segment $m = PO_{2r-1}$ $(1 \leqslant r \leqslant k)$, then $(V_c \cup \{l_{2k}\}, L_c)$ is a CL-perfect matching and $(V, L \cup L_c)$ is a CL-matching for $L_c = \{O_1O_2, O_3O_4, \ldots, O_{2r-3}O_{2r-2}, m, O_{2r}O_{2r+1}, \ldots, O_{2k-2}O_{2k-1}\}$.

(ii) P is on π (Fig. 9): This case is equivalent to the proposition for $n = 2$. Now if we apply Lemma 4.2 on π, then we can join P to some odd line segment l_{2r-1} $(1 \leqslant r \leqslant k)$ by some line segment m on the same side of P with respect to l so that m does not intersect any other l_i $(i \neq 2r-1)$. Here, if m intersects some l_j for $m(l_j, f) > m(l_{2k}, f)$, then we can move l_j slightly. Hence LMP-I is valid.

(2.2) All the elements of V_c are not on any plane (Fig. 10).

Let c' be a real number such that $c \leqslant c' \leqslant m(l_{2k}, f)$ and let O'_i be the point of intersection of $f^{-1}(\{c'\})$ and l_i $(1 \leqslant i \leqslant 2k-1)$ such that $\{O'_1, O'_2, \ldots, O'_{2k-1}\}$ are not collinear. By Lemma 4.3, we can construct a polygon by joining $O'_1, O'_2, \ldots, O'_{2k-1}$ by some line segments on $f^{-1}(\{c'\})$. Then we can arrange the indices by which $O'_1, O'_2, \ldots, O'_{2k-1}$ are located on the boundary in the order indicated. If we take a line segment $m = PO'_{2k-1}$, then $(V_c \cup \{l_{2k}\}, L_c)$ is a CL-perfect matching and $(V, L \cup L_c)$ is

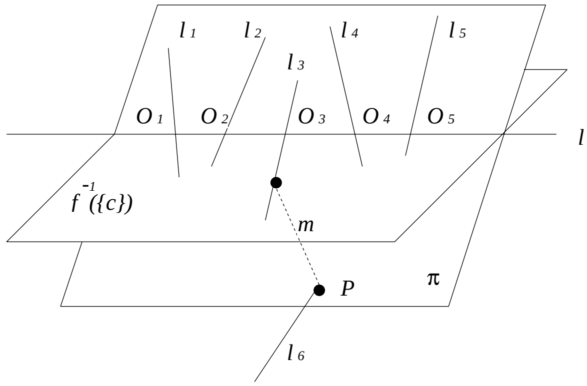

Fig. 9.

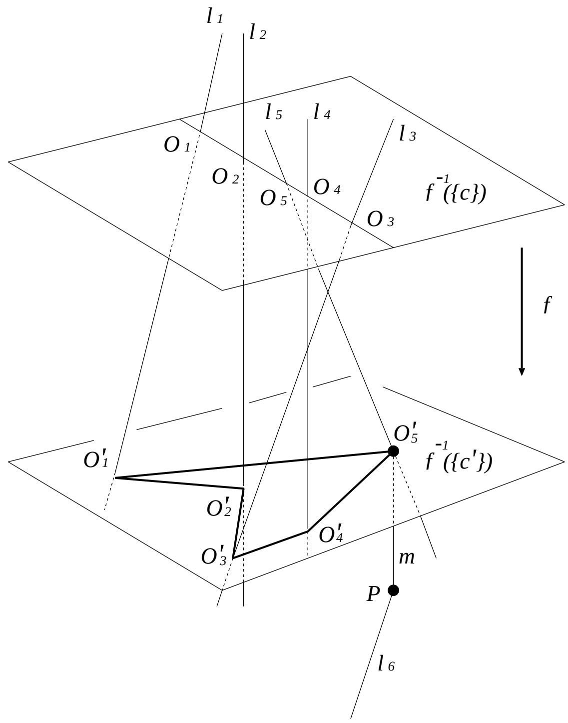

Fig. 10.

a CL-matching for $L_c = \{O_1'O_2', O_3'O_4', \ldots, O_{2k-3}'O_{2k-2}', m\}$. Here, if PO_{2k-1}' intersects some other l_i, then we may consider $m = PQ$, where Q is the point of intersection PO_{2k-1}' and l_i. $\square$

5. The cylinder theorem

For the basis vector $e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)$ in $\mathbb{R}^n$ $(1 \leqslant i \leqslant n)$, let $\{f_i \mid 1 \leqslant i \leqslant n\}$ be the dual basis of $\{e_i \mid 1 \leqslant i \leqslant n\}$ in $(\mathbb{R}^n)^*$. Let D be a compact set in $\mathbb{R}^{n-1}$. A compact set E in $\mathbb{R}^n$ is said to be a *D-cylinder*, if $f_n^{-1}(\{c\}) \cap E$ is a family of translates of $D \times \{0\}$ in $\mathbb{R}^n$, for every $c \in \mathbb{R}$ such that $f_n^{-1}(\{c\}) \cap E \neq \phi$.

Theorem 5.1 (The cylinder theorem). *Let D be a compact set in $\mathbb{R}^{n-1}$ $(n \geqslant 2)$ which satisfies the following condition: For a disjoint weakly 2-general family V of translates of D, there exists a dense subset $F \subset S((\mathbb{R}^{n-1})^*)$ such that V satisfies the LMP at the f-value c for sufficiently many $c \in \mathbb{R}$ and $f \in F$. Then, there exists a CL-perfect matching of any even disjoint family W of D-cylinders in $\mathbb{R}^n$ if W satisfies the following conditions:*

(i) *W has f_n-equal width.*
(ii) *The f_n-supporting value of each element of W is distinct from the others.*
(iii) *For sufficiently many $c \in \mathbb{R}$, if the set $\{E \cap f_n^{-1}(\{c\}) \mid E \in W$ and $E \cap f_n^{-1}(\{c\}) \neq \phi\}$ is not empty, then it is weakly 2-general.*

Proof. Let W be a disjoint family of D-cylinders in $\mathbb{R}^n$ which satisfies the above conditions. According to Lemma I, it is sufficient to show that for the linear function f_n and for sufficiently many $c \in \mathbb{R}$, W satisfies the LMP. We take a CL-matching $M = (W, L)$ of W such that $V(M) = \{C \in W \mid C \subset f_n^{-1}((-\infty, c))\}$. Let $W_c = \{C \in W \mid C \cap f_n^{-1}(\{c\}) \neq \phi\}$ and $W'_c = \{C \cap f_n^{-1}(\{c\}) \mid C \in W_c\}$. Note that W'_c is a disjoint weakly 2-general family of translates of $D \times \{c\} \cong D$ in $f_n^{-1}(\{c\}) \cong \mathbb{R}^{n-1}$.

(1) If Card(W_c) is even, then W'_c has a CL-perfect matching in $f_n^{-1}(\{c\})$ by virtue of the assumptions and Lemma I. Hence, LMP-II for W is valid at the f_n-value c.

(2) In case Card(W_c) is odd, we denote the element of $W \setminus (V(M) \cup W_c)$ which has the minimum f_n-value by $\tilde{C}$. Then, we can take some element $E \in W_c$ and a line segment $m \subset f_n^{-1}([c, m(\tilde{C}, f)])$ such that $(W, \{m\})$ is a CL-matching and $(\{E, \tilde{C}\}, \{m\})$ is a CL-perfect matching. On the other hand, by the assumptions and Lemma III, W'_c has a CL-perfect matching with residue $E \cap f_n^{-1}(\{c\})$, i.e., there exists a set of line segments L'_c in $f_n^{-1}(\{c\})$ such that (W'_c, L'_c) is a CL-matching of W'_c in $f_n^{-1}(\{c\})$ and such that $(W'_c \setminus \{E \cap f_n^{-1}(\{c\})\}, L'_c)$ is a CL-perfect matching of $W'_c \setminus \{E \cap f_n^{-1}(\{c\})\}$ in $f_n^{-1}(\{c\})$. Thus, $(W_c \cup \{\tilde{C}\}, L'_c \cup \{m\})$ is a CL-perfect matching of $W_c \cup \{\tilde{C}\}$ and $(W, L \cup L'_c \cup \{m\})$ is a CL-matching of W in $\mathbb{R}^n$. Hence LMP-I is valid. □

Last, we obtain the following concrete results for the cylinder theorem.

Theorem 5.2 (The twisted triangular prism theorem). *Let T be a triangle in $\mathbb{R}^2$. Then there exists a CL-perfect matching of any even disjoint family V of T-cylinders in $\mathbb{R}^3$ if V satisfies the following conditions* (Fig. 11):

(i) *V has f_3-equal width.*
(ii) *The f_3-supporting value of each element of V is distinct from the others.*

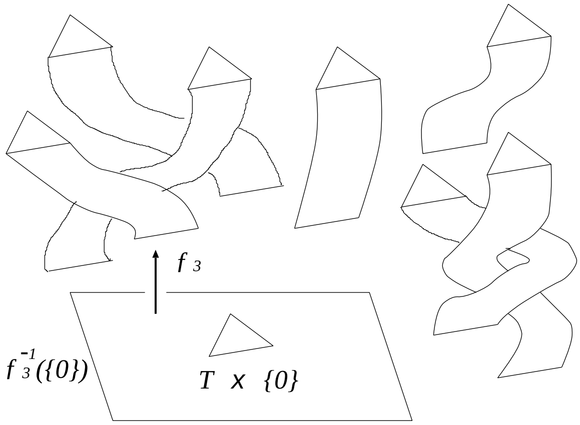

Fig. 11.

(iii) *For sufficiently many* $c \in \mathbb{R}$*, if the set* $\{E \cap f_3^{-1}(\{c\}) \mid E \in V$ and $E \cap f_3^{-1}(\{c\}) \neq \phi\}$ *is not empty, then it is weakly* 2*-general.*

Proof. By virtue of the proof of Theorem 3.1, the conditions of Theorem 5.1 are satisfied. □

Theorem 5.3 (The curve segment theorem)**.** *Let* P *be a point in* $\mathbb{R}^{n-1}$ $(n \geqslant 2)$*. Then, there exists a CL-perfect matching of any even disjoint family* V *of* P*-cylinders in* $\mathbb{R}^n$ *if* V *satisfies the following conditions*:

(i) V *has* f_n*-equal width.*

(ii) *The* f_n*-supporting value of each element of* V *is distinct from the others.*

(iii) *If the set of points* $\{E \cap f_n^{-1}(\{c\}) \mid E \in V$ *and* $E \cap f_n^{-1}(\{c\}) \neq \phi\}$ *is not empty for sufficiently many* $c \in \mathbb{R}$*, then it is weakly* 2*-general.*

Proof. The conditions of Theorem 5.1 are satisfied for every 2-general family of points in $\mathbb{R}^{n-1}$ $(n \geqslant 2)$. □

Remark 5.1. The condition (iii) is essential to Theorem 5.3. In fact, let P be a point in $\mathbb{R}^1$. If $V = \{l_{-1}, l_0, l_1, l_2, l_3, l_4, l_5, l_6\}$ is an even disjoint family of P-cylinders which has f_2-equal width as shown in Fig. 12, then V does not satisfy the LMP at the f-value c.

6. Conjectures

We propose the most general conjecture.

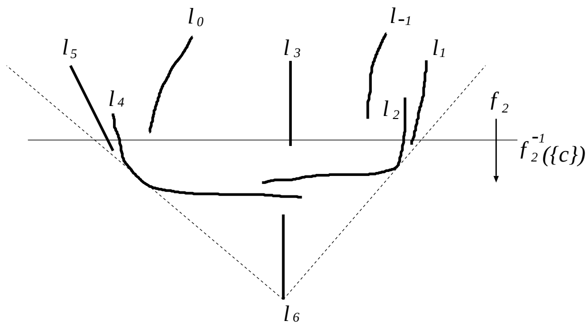

Fig. 12.

Definition 6.1. Let V be a family of compact convex sets in $\mathbb{R}^n$. The family V is said to have *equal width*, if V has f-equal width for every $f \in (\mathbb{R}^n)^*$.

Conjecture 6.1. *Let V be an even disjoint family of compact convex sets in $\mathbb{R}^n$. If V has equal width, then there exists a CL-perfect matching of V. In particular, we are interested in the family of congruent balls in $\mathbb{R}^n$.*

The following version is by virtue of Remark 5.1.

Conjecture 6.2. *Let P be a point in $\mathbb{R}^1$ and let V be an even disjoint family of P-cylinders in $\mathbb{R}^2$. If V satisfies the following conditions, then there exists a CL-perfect matching of V.*

(i) *V has f_2-equal width.*
(ii) *The f_2-supporting value of each element of V is distinct from the others.*

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