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A penalty function method based on bilevel programming for solving inverse optimal value problems☆

Yibing Lv[a,*], Zhong Chen[a], Zhongping Wan[b]

[a] School of Information and Mathematics, Yangtze University, Jingzhou 434023, PR China

[b] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China

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ABSTRACT

In this work, we reformulate the inverse optimal value problem equivalently as a corresponding nonlinear bilevel programming (BLP) problem. For the nonlinear BLP problem, the duality gap of the lower level problem is appended to the upper level objective with a penalty, and then a penalized problem is obtained. On the basis of the concept of partial calmness, we prove that the penalty function is exact. Then, an algorithm is proposed and an inverse optimal value problem is resolved to illustrate the algorithm.

1. Introduction

An inverse optimization problem consists of inferring the value of the model parameters, such as the objective function and constraint coefficient. A standard inverse optimization problem is as follows: given an optimization problem with a linear objective $P : \min_x\{c^{\mathrm{T}}x | x \in X\}$ and a desired optimal solution $x^* \in X$, find a cost vector c^* such that $x^* \in X$ is an optimal solution of P, and at the same time c^* is required to satisfy some additional conditions—such that, given a preferred cost c', the deviation $\|c^* - c'\|_p$ is to be minimum under some ℓ_p norm. Geophysical scientists were the first ones to study inverse problems. The book by Tarantola [1] gives a comprehensive discussion of the theory of inverse problems in the geophysical sciences. From then on, many researchers have devoted their attention to this challenging topic. For detailed expositions, the reader may consult [2–4].

Motivated by an application in telecommunication bandwidth pricing as proposed by Paleologo and Takriti [5], Ahmed and Guan [6] studied the inverse optimal value problem, that is to say, given the optimization problem P, a desired optimal objective value z^*, and a set of feasible cost vectors C, determine a cost vector $c^* \in C$ such that the corresponding optimal objective value of P is closest to z^*. In [6], Ahmed and Guan proved that the inverse optimal value problem is NP-hard and, on the basis of some assumptions, they got the optimal solutions of the inverse optimal value problem by solving a series of linear and bilinear programming problems. However, some assumptions in [6] are rather restrictive, such as cost vectors C being convex.

In order to consider the inverse optimal value problem under wider conditions (cost vectors C needn't be convex), we will transform the inverse optimal value problem into a corresponding nonlinear BLP problem and construct a penalized problem using the duality gap of the lower level problem. On the basis of the concept of partial calmness, we prove that the

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* Corresponding author.
E-mail address: Lvyibing2001@gmail.com (Y.B. Lv).

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penalty function is exact. It is noted that Lv [7] presented a penalty function method for the inverse optimal value problem. However, the main differences between the method of this work and that of the work [7] are in the approach of constructing penalty function and the method of proving the exactness of the penalty function.

Towards these ends, the rest of the work is organized as follows. In Section 2, we will first introduce the inverse optimal value problem, then transform this problem into the corresponding bilevel programming problem. Then, in Section 3 we construct the penalty function and prove that the penalty function is exact on the basis of the concept of partial calmness. The algorithm and the numerical result are presented in Section 4. Finally we conclude this work.

2. Problem statement

Consider the optimal value function of a linear programming in terms of its cost vector,

$$Q(c) = \min_x \{c^{\mathrm{T}}x : Ax \geq b, x \geq 0\} \tag{1}$$

where $x \in R^n, A \in R^{m\times n}, b \in R^m$. Given a set $C \subseteq R^n$ of the objective cost vectors and a real number z^*, then the inverse optimal value problem is to find a cost vector from the set C such that the optimal objective value of the linear programming (1) is "close" to z^*; thus the inverse optimal value problem can be written as [6]

$$\min_c \{f(c) : c \in C\} \tag{2}$$

where $f(c) = |Q(c) - z^*|$.

It is obvious that the inverse optimal value problem (2) is a parameter optimal problem, and going one step further the inverse optimal value problem can be written equivalently as follows [7]:

$$\begin{aligned} \min_c \quad & |Q(c) - z^*| \\ s.t. \quad & c \in C \\ & Q(c) = \min_x \{c^{\mathrm{T}}x : Ax \geq b, x \geq 0\}. \end{aligned} \tag{3}$$

Remark 1. In fact, problem (3) is a special bilevel programming problem called the bilevel programming problem with the optimal value of the lower level problem feeding back to the upper level [8].

From the above description, we can find that the inverse optimal value problems are equivalent to a special class of bilevel programming problems. Then this provides us with an alternative approach to considering the inverse optimal value problem. In the following section, for problem (3), we will construct an exact penalty function.

3. A penalty function approach

There have been many feasible approaches for solving the nonlinear BLP problem, such as the branch-and-bound approach, the descent approach, the penalty function approach [9–11] etc. As the BLP problem (3) has special structure, then in this section, for the special BLP problem (3) we will construct an exact penalty function.

Throughout the rest of this work, we make the following assumptions:

(H_1) The set $\{x \in R^n : Ax \geq b, x \geq 0\}$ is of nonempty bound polyhedra and $\|b\| > 1$.

(H_2) The set of cost vectors C is nonempty and compact.

Remark 2. Unlike in the assumptions A1–A4 in [6], here the cost vectors C needn't be convex. In addition, if the norm $\|b\| < 1$, we can multiply by some constant $m > 1$ on both sides of the inequality $Ax \geq b$. That is, the assumption $\|b\| > 1$ can be satisfied easily.

Without losing generality, we replace $f(c) = |Q(c) - z^*|$ with $f'(c) = (Q(c) - z^*)^2$; then problem (3) can be written as

$$\begin{aligned} \min_c \quad & (Q(c) - z^*)^2 \\ s.t. \quad & c \in C \\ & Q(c) = \min_x \{c^{\mathrm{T}}x : Ax \geq b, x \geq 0\}. \end{aligned} \tag{4}$$

Using strong duality, $Q(c)$ can be written as

$$Q(c) = \max_u \{u^{\mathrm{T}}b : A^{\mathrm{T}}u \leq c, u \geq 0\} \tag{5}$$

where $u \in R^m$ is a duality variable.

Remark 3. We have that x solves problem (1) and u solves problem (5) if and only if (x, u) is a solution of the following system:

$$\begin{cases} Ax \geq b \\ A^{\mathrm{T}}u \leq c \\ c^{\mathrm{T}}x - u^{\mathrm{T}}b = 0 \\ x, u \geq 0. \end{cases}$$

Then, problem (4) can be equivalently written as the following problem:

$$\begin{array}{ll} \min\limits_{x,c,u} & (u^{\mathrm{T}}b - z^*)^2 \\ s.t. & c \in C \\ & Ax \geq b \\ & A^{\mathrm{T}}u \leq c \\ & c^{\mathrm{T}}x - u^{\mathrm{T}}b = 0 \\ & x, u \geq 0. \end{array} \tag{6}$$

Here, following the constraint condition $c^{\mathrm{T}}x - u^{\mathrm{T}}b = 0$ we can replace $c^{\mathrm{T}}x$ with $u^{\mathrm{T}}b$ in the objective function and this replacement plays a key role in the following theoretical analysis.

Let $Z = \{(x, c, u) \in R^{n+n+m} : c \in C, Ax \geq b, A^{\mathrm{T}}u \leq c, c^{\mathrm{T}}x - u^{\mathrm{T}}b = 0, x \geq 0, u \geq 0\}$ be the feasible region of problem (6), and Z_u be the projection of Z onto R^m, i.e. $Z_u = \{u \in R^m : (x, c, u) \in Z$ for some $x \in R^n, c \in R^n\}$.

Lemma 1. *Let assumptions* (H_1) *and* (H_2) *be satisfied and* $f(u) = (u^{\mathrm{T}}b - z^*)^2$*; then* $f(u)$ *is Lipschitz continuous on* $u \in Z_u$*.*

Proof. Let $u_1, u_2 \in Z_u$; then,

$$\begin{aligned} |f(u_1) - f(u_2)| &= |(u_1^{\mathrm{T}}b - z^*)^2 - (u_2^{\mathrm{T}}b - z^*)^2| \\ &= |(u_1 - u_2)^{\mathrm{T}}b(u_1^{\mathrm{T}}b + u_2^{\mathrm{T}}b - 2z^*)| \\ &= |(u_1 - u_2)^{\mathrm{T}}b| \cdot |(u_1^{\mathrm{T}}b + u_2^{\mathrm{T}}b - 2z^*)| \\ &\leq \|b\| \cdot \|u_1 - u_2\| \cdot |(u_1^{\mathrm{T}}b + u_2^{\mathrm{T}}b - 2z^*)|. \end{aligned}$$

Following assumptions (H_1) and (H_2), $M = \max_{u_1,u_2 \in Z_u} |u_1^{\mathrm{T}}b + u_2^{\mathrm{T}}b - 2z^*|$ is attainable. Thus,

$$|f(u_1) - f(u_2)| \leq \|b\| \cdot M \cdot \|u_1 - u_2\|.$$

Also let $L = \|b\| \cdot M$; we can deduce that $f(u)$ is Lipschitz continuous on $u \in Z_u$ with Lipschitzian modulus L. □

In order to prove the exactness of the penalty function, we need the concept of partial calmness [12]. Firstly, we introduce the following perturbed version of problem (6):

$$\begin{array}{ll} \min\limits_{x,c,u} & (u^{\mathrm{T}}b - z^*)^2 \\ s.t. & c \in C \\ & Ax \geq b \\ & A^{\mathrm{T}}u \leq c \\ & c^{\mathrm{T}}x - u^{\mathrm{T}}b = \varepsilon \\ & x, u \geq 0 \end{array} \tag{7}$$

where $\varepsilon \in R$. From the theory of duality, it is obvious that $c^{\mathrm{T}}x - u^{\mathrm{T}}b > 0$; then we can give the following definition.

Definition 1 (*Partial Calmness [12]*). Let (x^*, c^*, u^*) solve problem (6). The problem (6) is said to be partially calm at (x^*, c^*, u^*) provided that there exist constants $\mu > 0, \delta > 0$ such that, for all $\varepsilon \in \delta B$ and all $(x, c, u) \in (x^*, c^*, u^*) + \delta B$ that are feasible for problem (7), one has

$$(u^{\mathrm{T}}b - z^*)^2 - ((u^*)^{\mathrm{T}}b - z^*)^2 + \mu(c^{\mathrm{T}}x - u^{\mathrm{T}}b) \geq 0$$

where B denotes the open unit ball in R^{n+n+m}. The constants μ and δ are called the modulus and radius, respectively.

The following lemma shows that problem (6) is that of partial calmness.

Lemma 2. *Let assumptions* (H_1) *and* (H_2) *be satisfied, and* (x^*, c^*, u^*) *be a local solution of problem* (6)*. Then problem* (6) *is partially calm at* (x^*, c^*, u^*) *with modulus* M*.*

Proof. Let $\delta > 0$ be such that (x^*, c^*, u^*) is a local solution of problem (6) in $(x^*, c^*, u^*) + 2\delta B \subset R^n \times R^n \times R^m$. For any $0 < \varepsilon < \delta$, let $(x, c, u) \in (x^*, c^*, u^*) + \delta B$ be feasible for problem (7); then $c^{\mathrm{T}}x - u^{\mathrm{T}}b = \varepsilon$.

For $u' \in Z_u$, we firstly prove that vectors $u' - u$ and b are linearly dependent, i.e., there exists $\lambda \in R$ and $\lambda \neq 0$ satisfying $u' - u = \lambda b$.

In fact, as $u' \in Z_u$, then $c^{\mathrm{T}}x - (u')^{\mathrm{T}}b = 0$, which combines the above equality $c^{\mathrm{T}}x - u^{\mathrm{T}}b = \varepsilon$; we can deduce $(u' - u)^{\mathrm{T}}b = \varepsilon$, and then $\lambda = \frac{\varepsilon}{b^{\mathrm{T}}b}$. This shows that vectors $u' - u$ and b are linearly dependent, which implies that one can choose a $u' \in Z_u$ such that

$$\|u' - u\| = \frac{\varepsilon}{\|b\|}.$$

Since (x, c, u') is feasible for problem (6) and

$$\begin{aligned}\|(x, c, u') - (x^*, c^*, u^*)\| &\le \|(x, c, u') - (x, c, u)\| + \|(x, c, u) - (x^*, c^*, u^*)\| \\ &= \frac{\varepsilon}{\|b\|} + \delta \\ &< \varepsilon + \delta < 2\delta\end{aligned}$$

then $((u')^{\mathrm{T}}b - z^*)^2 \ge ((u^*)^{\mathrm{T}}b - z^*)^2$. Following Lemma 1, $(u^{\mathrm{T}}b - z^*)^2$ is Lipschitz continuous in $u \in Z_u$; we have

$$(u^{\mathrm{T}}b - z^*)^2 - ((u')^{\mathrm{T}}b - z^*)^2 \ge -L\frac{\varepsilon}{\|b\|}.$$

Thus, we can deduce that

$$(u^{\mathrm{T}}b - z^*)^2 - ((u^*)^{\mathrm{T}}b - z^*)^2 + M(c^{\mathrm{T}}x - u^{\mathrm{T}}b) \ge 0.$$

That is, problem (6) is partially calm at (x^*, c^*, u^*) with modulus M. □

On the basis of the concept of partial calmness, we can construct an exact penalized version for problem (6).

Theorem 1. *Suppose assumptions* (H_1) *and* (H_2) *are satisfied and* (x^*, c^*, u^*) *is a local minimum of problem* (6)*. Then there exists* $\mu^* > 0$ *such that* (x^*, c^*, u^*) *is a local minimum of the following penalized problems for all* $\mu \ge \mu^*$*:*

$$\begin{aligned}\min_{x,c,u} \quad & (u^{\mathrm{T}}b - z^*)^2 + \mu(c^{\mathrm{T}}x - u^{\mathrm{T}}b) \\ s.t. \quad & c \in C \\ & Ax - b \ge 0 \\ & A^{\mathrm{T}}u \le c \\ & x, u \ge 0.\end{aligned} \tag{8}$$

Any local minimum of problem (8) *with* $\mu \ge \mu^*$ *with respect to the neighborhood of* (x^*, c^*, u^*) *in which* (x^*, c^*, u^*) *is a local minimum is also a local minimum of problem* (6)*.*

Proof. Suppose that (x^*, c^*, u^*) is a local minimum of problem (6) but not problem (8) for any $\mu > 0$. Then, for each positive integer k, there exists a point $(x_k, c_k, u_k) \in (x^*, c^*, u^*) + (1/k)B$, which satisfies $c_k \in C$, $Ax_k - b \ge 0$, $A^{\mathrm{T}}u_k \le c_k$ and $x_k, u_k \ge 0$, such that

$$(u_k^{\mathrm{T}}b - z^*)^2 + k(c_k^{\mathrm{T}}x_k - u_k^{\mathrm{T}}b) < ((u^*)^{\mathrm{T}}b - z^*)^2 \tag{9}$$

and, then,

$$0 < c_k^{\mathrm{T}}x_k - u_k^{\mathrm{T}}b < \frac{((u^*)^{\mathrm{T}}b - z^*)^2 - (u_k^{\mathrm{T}}b - z^*)^2}{k}.$$

Taking the limit as k goes to infinity, one has $c_k^{\mathrm{T}}x_k - u_k^{\mathrm{T}}b \to 0$ as $k \to \infty$. But the inequality (9) contradicts the fact that problem (6) is partially calm at (x^*, c^*, u^*). Thus for some $\mu^* > 0$, (x^*, c^*, u^*) must be a local minimum of problem (8).

It is obvious that a local minimum of problem (8) with μ^* must be a local minimum for problem (8) with μ whenever $\mu \ge \mu^*$.

Conversely, let $\mu > \mu^*$ and (x_μ, c_μ, u_μ) be a local minimum of problem (8) in the neighborhood of (x^*, c^*, u^*) in which (x^*, c^*, u^*) is a local minimum. Then

$$\begin{aligned}(u_\mu^{\mathrm{T}})^2 + \mu(c^{\mathrm{T}}x - u^{\mathrm{T}}b) &= ((u^*)^{\mathrm{T}}b - z^*)^2 \quad \text{since } (x^*, c^*, u^*) \text{is a local minimum of problem (8)} \\ &\le (u_\mu^{\mathrm{T}})^2 + \frac{1}{2}(\mu + \mu^*)(c_\mu^{\mathrm{T}}x_\mu - u_\mu^{\mathrm{T}}b) \quad \text{since } \frac{1}{2}(\mu + \mu^*) > \mu^*\end{aligned}$$

which implies that

$$(\mu - \mu^*)(c_\mu^{\mathrm{T}}x_\mu - u_\mu^{\mathrm{T}}b) \le 0.$$

Therefore, $c_\mu^{\mathrm{T}}x_\mu - u_\mu^{\mathrm{T}}b = 0$, which implies that (x_μ, c_μ, u_μ) is also a local minimum of problem (6). □

Remark 4. Theorem 1 shows that the penalty function is exact. Moreover, following the properties of the penalty function method, the upper level objective $(u^{\mathrm{T}}b - z^*)^2$ and the penalty term $c_k^{\mathrm{T}}x_k - u_k^{\mathrm{T}}b$ are both monotonically nondecreasing in the penalty value μ. Then, we can get the optimal solutions of problem (3) by solving problem (8).

4. The algorithm

From Theorem 1 and Remark 4, we can propose a penalty function algorithm for problem (8); the algorithm proposed only needs to solve a series of differential nonlinear programs.

Step 0 Set $i = 0$; choose $\mu > 0$ (μ large), $\lambda > 0$.
Step 1 Solve problem (8); obtain the optimal solution $(x^i_\mu, c^i_\mu, u^i_\mu)$
Step 2 Compute $\alpha^i_k = (c^i_\mu)^{\mathrm{T}} x^i_\mu - (u^i_\mu)^{\mathrm{T}} b$.
Step 3 (i) If $\alpha^i_k > 0$, set $k = k + \lambda$, $i = i + 1$; go to Step 1.
(ii) If $\alpha^i_k = 0$, the optimal solution of problem (8) is (x^i_k, c^i_k, u^i_k).

Theorem 2. *Let* (H_1) *and* (H_2) *be satisfied; then the sequence* $\{(x^k, c^k)\}$, *which comes from the above algorithm, converges to the optimal solution of problem* (3).

Proof. In fact, following Theorem 1, we know that the penalty function constructed is exact. Then, the sequence $\{(x^k, c^k)\}$, which comes from the above algorithm is finite. From the termination of the algorithm, it is obvious that the last point in the sequence $\{(x^k, c^k)\}$ solves problem (3). □

To illustrate the algorithm, we solve the following inverse optimal value problem.

Example. Consider the following inverse optimal value problem, where $c \in R^2$, $x \in R^2$:

$$\min_c \{|Q(c) - z^*| : 10 \le c_1^2 + c_2^2 \le 13, c_1 \ge 0, c_2 \ge 0\} \tag{10}$$

where $z^* = 14$ and $Q(c) = \max_x\{c_1x_1 + c_2x_2 : x_1 + 2x_2 \le 8, x_1 \le 4, x_2 \le 3, x \ge 0\}$.

Following problem (4), we can write the above inverse optimal value problem as the following nonlinear bilevel programming:

$$\begin{aligned} \min_{c\ge 0} \quad & (c_1x_1 + c_2x_2 - 14)^2 \\ s.t. \quad & 10 \le c_1^2 + c_2^2 \le 13 \\ & Q(c) = \max_{x\ge 0}\{c_1x_1 + c_2x_2 : x_1 + 2x_2 \le 8, x_1 \le 4, x_2 \le 3\}. \end{aligned} \tag{11}$$

Step 0 Set $i = 0$, $\mu = 100$, $\lambda = 10$.
Step 1 Solve the following nonlinear program:

$$\begin{aligned} \min \quad & [(8u_1 + 4u_2 + 3u_3 - 14)^2 + 100(8u_1 + 4u_2 + 3u_3 - x_1c_1 - x_2c_2)] \\ s.t. \quad & c_1^2 + c_2^2 \ge 10 \\ & c_1^2 + c_2^2 \le 13 \\ & x_1 + 2x_2 \le 8 \\ & x_1 \le 4 \\ & x_2 \le 3 \\ & u_1 + u_2 \ge c_1 \\ & 2u_1 + u_2 \ge c_2 \\ & x, c, u \ge 0. \end{aligned} \tag{12}$$

The optimal solution of problem (12) is $x^0_{100} = (4, 2)$, $c^0_{100} = (2, 3)$, $u^0_{100} = (1.5, 0.5, 0)$.
Step 2 Compute $\alpha^0_{100} = (c^0_{100})^{\mathrm{T}} x^0_{100} - (u^0_{100})^{\mathrm{T}} b = 0$.
Then, the algorithm is terminated and the inverse optimal value problem (10) is $c^* = (2, 3)$, $x^* = (4, 2)$.
Through some simple validating calculations, it is shown that the algorithm proposed in this work is feasible for the inverse optimal value problem.

5. Conclusion

In this work, we transform the inverse optimal value problem into a corresponding nonlinear BLP problem, and propose an algorithm for the inverse optimal value problem, which can resolve the inverse optimal value problem under wider conditions. The main feature of this work is the approach of proving the exactness of the penalty function. On the other hand, the algorithm proposed in this work can also be applied to a class of BLP programming with the structure that the lower level problem is the parameter linear programming problem. As the algorithm proposed in this work needs only to solve a series of differential nonlinear programming problems, it has attractive computation perspectives.

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