Communication

Limit formulas for $q$-exponential functions

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Communicated by G.E. Andrews
Received 6 August 1993

Abstract
$q$-Analogues are given for the limit formulas $e' = \lim_{n \to \infty} (1 - t/n)^{-n}$ and $e' = \lim_{n \to \infty} (1 + t/n)^n$. The $q$-extension of the first identity is applied in proving that the probability of a $q$-random mapping having no fixed points approaches the reciprocal of a $q$-analogue of the real number $e$.

1. Introduction

The $q$-analogue and $q$-factorial of a non-negative integer $n$ are defined to be the polynomials $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2] \cdots [n]$ where, by convention, $[0] = 0$ and $[0]! = 1$. Two standard $q$-analogues of the exponential function are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]!} \quad \text{and} \quad E_q(t) = \sum_{n=0}^{\infty} \frac{q(t)^n}{[n]!}.$$ 

As $q$ varies from 0 to 1, $e_q(t)$ ranges from $e_0(t) = 1/(1 - t)$ to the exponential $e_1(t) = e^t$. Also note that $E_0(t) = 1 + t$ and $E_1(t) = e^t$.

In analogy with the limit formulas

$$e' = \lim_{n \to \infty} (1 - t/n)^{-n} \quad \text{and} \quad e' = \lim_{n \to \infty} (1 + t/n)^n,$$

the $q$-exponential functions satisfy the following identities.
Theorem 1. For $-1 < q \leq 1$ and $|t| < 1/(1-q)$,

$$e_q(t) = \lim_{n \to \infty} \prod_{i=1}^{n} (1-q^{i-1}t/[n])^{-1},$$

$$E_q(t) = \lim_{n \to \infty} \prod_{i=1}^{n} (1+q^{i-1}t/[n]).$$

For $q = 1$, these formulae respectively reduce to those given in (1). Other properties of the $q$-exponential functions may be found in [2, p. 126–128].

Two proofs of Theorem 1 are offered in Sections 3 and 4. The first reveals an intimate tie with two of Euler's classical results in the theory of partitions. However, before considering either, the probabilistic context that motivated (2) is presented next.

2. An application

Let $\mathcal{F}_n$ denote the set of functions mapping $\{1, 2, \ldots, n\}$ to itself. A function $f \in \mathcal{F}_n$ will be expressed as a list of its range elements: $f=f(1)f(2)\cdots f(n)$.

Suppose the elements of $\mathcal{F}_n$ are generated as follows.

**Bernoulli mapping generator (BMG).** Initialize $f \in \mathcal{F}_n$ as $f=1\cdots 1$. For $1 \leq i \leq n$, the terminal value of $f(i)$ is determined by tossing a coin (with probability $q$ of landing tails up) until heads occurs: Each time tails occurs and $f(i) < n$ the current value of $f(i)$ is to be updated as $f(i)+1$. If the coin lands tails up when $f(i) = n$, then $f(i)$ is reset to 1. When heads occurs, the value of $f(i)$ is locked.

Because of the role played by $q$, a function generated by the BMG is said to be a $q$-random mapping. As an illustration, suppose that $n = 4$. If for $i = 1$ the sequence of tosses is TTTTTTH, then the terminal value of $f(1)$ is 3. If the entire sequence of tosses for $i = 1, 2, 3$ and 4 is TTTTTTH:HTTTH:TTTTH, then the outcome is $f=3141 \in \mathcal{F}_4$.

Although there are many natural directions of inquiry that could be pursued relative to $q$-random mappings, only one question will be considered herein: What
does the probability of a $q$-random mapping $f$ having no fixed points (i.e., $f(i) \neq i$ for $1 \leq i \leq n$) approach as $n \to \infty$?

Let's first determine the probability that $f(i)$ is equal to $j$ for $1 \leq i, j \leq n$. For $0 \leq q < 1$,

$$
P_q \{f(i) = j\} = \sum_{k=0}^{\infty} q^{i-j+k} (1-q) = (1-q)q^{i-1} \sum_{k=0}^{\infty} (q^k)^k = \frac{q^{i-1}}{[n]}$$  \hspace{1cm} (4)

since $[n] = (1-q^n)/(1-q)$. The formula $P_q \{f(i) = j\} = q^{i-1}/[n]$ may be extended continuously to $q = 1$ by defining $P_1 \{f(i) = j\} = 1/n$. From the perspective of the BMG, this extension makes some sense: If $q = 1$, then the coin almost surely never lands heads up. So, $f(i)$ is equally likely (or, more aptly stated, equally unlikely) to be any value from 1 to $n$.

Since the range values are independently generated, the measure induced on $\mathcal{F}_n$ by the BMG is easily determined from the continuous extension of (4).

**Theorem 2.** For $0 \leq q \leq 1$, the probability that $f \in \mathcal{F}_n$ is generated by the BMG is $P_q(f) = q^{s(f)}/[n]$, where $s(f) = f(1) + f(2) + \cdots + f(n) - n$.

Since $P_1(f) = 1/n^n$, $P_q$ is a $q$-analogue of the equiprobable measure on $\mathcal{F}_n$.

The solution to the fixed point problem is easily deduced from (2) and (4). First, observe that one natural way of defining a $q$-analogue of $e$ is by setting $[e] = e_q(1)$. Then (2) with $t = 1$ reduces to the identity

$$
[e] = \lim_{n \to \infty} \prod_{i=1}^{n} (1-q^{i-1}/[n])^{-1}.
$$  \hspace{1cm} (5)

From (4), it follows that the probability of $f \in \mathcal{F}_n$ having no fixed points is equal to

$$
\prod_{i=1}^{n} P_q\{f(i) \neq i\} = \prod_{i=1}^{n} (1 - P_q\{f(i) = i\}) = \prod_{i=1}^{n} (1 - q^{i-1}/[n]).
$$

Combining this fact with (5) establishes the following result.

**Theorem 3.** If $0 \leq q < 1$, then the probability of a $q$-random mapping having no fixed points approaches $1/[e]$ as $n \to \infty$.

In [4], a slight variation of the BMG leads to a $q$-analogue of Euler's product formula for the Reimann zeta function.

3. **A proof of Theorem 1**

The non-trivial case $|q| < 1$ of Theorem 1 may be deduced from a fundamental theorem of Euler's ([1, p. 19]) in the theory of partitions.
Theorem 4 (Euler). For \( |q| < 1 \) and \(|t| < 1\),

\[
1 + \sum_{n=1}^{\infty} \frac{t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{i=1}^{\infty} (1-q^{i-1}t)^{-1},
\]

(6)

\[
1 + \sum_{n=1}^{\infty} \frac{q^n t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{i=1}^{\infty} (1+q^{i-1}t).
\]

(7)

Only the demonstration that (7) implies (3) for \(|q| < 1\) is given.

The first step is to replace \( t \) by \((1-q)t\) in (7). Thus, for \(|q| < 1\) and \(|t| < 1/(1-q)\),

\[
E_q(t) = \prod_{i=1}^{\infty} (1+(1-q)q^{i-1}t).
\]

Next, select \( t_0 \) such that \(|t_0| < 1/(1-q)\). For \( n \geq 1 \), let

\[
u(n) = \prod_{i=1}^{n} (1+(1-q)q^{i-1}t_0) \quad \text{and} \quad v(n) = \prod_{i=1}^{n} (1+q^{i-1}t_0/[n]).
\]

It suffices to show that \( v(n) \) converges to the same limit as does \( u(n) \).

Let \( A = (|t_0|+(1-q)^{-1})/2 \). Since \( A \) is the average of \(|t_0|\) and \((1-q)^{-1}\) as \( n \to \infty \), there exists an integer \( N > 0 \) such that \([n] > A > |t_0| \geq 0\) whenever \( n \geq N \). Setting \( \rho = |t_0|/A \) then leads to the fact that \( |t_0|/[n] < \rho < 1 \) which in turn implies that \( 0 < 1-\rho < 1 + q^{-1}t_0/[n] < 1 + \rho \) for \( n \geq N \). Similarly, \( 1 - \rho < 1 + (1-q)q^{-1}t_0 < 1 + \rho \).

Since the products \( u(n) \) and \( v(n) \) are both strictly positive for \( n \geq N \), the matter reduces to showing that the sum \( \ln v(n) \) converges to the same limit as the sum \( \ln u(n) \). Note that the derivative of \( \ln t \) is bounded by \((1-\rho)^{-1}\) on the closed interval \([1-\rho, 1+\rho]\). This fact together with the Mean Value Theorem permits the following computation for \( n \geq N \):

\[
|\ln v(n) - \ln u(n)| \leq \sum_{i=1}^{n} |\ln(1 + q^{i-1}t_0/[n]) - \ln(1 + (1-q)q^{i-1}t_0)|
\]

\[
\leq \sum_{i=1}^{n} (1-\rho)^{-1} |(1+q^{i-1}t_0/[n]) - (1 + (1-q)q^{i-1}t_0)|
\]

\[
\leq (1-\rho)^{-1} \sum_{i=1}^{\infty} |q|^{i-1}|t_0||1/[n] - (1-q)|
\]

\[
\leq (1-\rho)^{-1}|t_0||1/[n] - (1-q)(1-|q|)^{-1}.
\]

Since \( |1/[n] - (1-q)| \to 0 \) as \( n \to \infty \), the proof of the case \(|q| < 1\) is complete.

In the literature on basic hypergeometric series, the series appearing in Euler’s Theorem are sometimes defined to be \( q \)-analogues of the exponential function. For instance, see [3, p. 9].
4. An alternate proof

For $0 \leq k \leq n$, the $q$-binomial coefficient (also known as the Gaussian polynomial) is defined by

$$\binom{n}{k} = \frac{[n]!}{[k]![n-k]!}.$$ 

G.E. Andrews recently pointed out another proof of Theorem 1 based on the $q$-binomial series and $q$-binomial theorem ([1, p. 36]). Only the proof that the $q$-binomial series implies the case $|q| < 1$ of (2) is sketched.

**Theorem 5 (q-Binomial series).** For $|q| < 1$ and $|t| < 1$,

$$\prod_{i=1}^{n} (1 - q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k.$$ 

First note that $[n+m] \to (1-q)^{-1}$ as $n \to \infty$ for any integer $m$. Thus,

$$\lim_{n \to \infty} \binom{n+k-1}{k} = \lim_{n \to \infty} \frac{[n+k-1][n+k-2] \cdots [n]}{[k]!} = \frac{(1-q)^{-k}}{[k]!}.$$ 

This together with the $q$-binomial series verifies the case $|q| < 1$ of (2):

$$\lim_{n \to \infty} \prod_{i=1}^{n} (1 - q^{i-1}t/[n])^{-1} = \lim_{n \to \infty} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{t}{[n]} \right)^k = \sum_{k=0}^{\infty} \frac{(1-q)^{-k} t^k}{[k]! (1-q)^{-k}} = e_q(t).$$ 

**References**