On Elementary Abelian TI-Subgroups

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Communicated by Walter Feit

Received January 5, 1976

1. INTRODUCTION

Let $A$ be an elementary abelian $p$-subgroup of the finite group $G$, satisfying the following conditions.

1. $A \cap A^g = A$ or 1 for all $g \in G$,
2. If $N(A) \neq 1$, then $[A, A^g] = 1$.

Then $A$ is called a subgroup of root-type in $G$.

It would be desirable to prove that "essentially" all elementary abelian TI-subgroups are of root-type; that is, to characterize the counterexamples. In this paper we make an attempt in this direction. More precisely we prove

**Theorem 1.** Let $A$ be an elementary abelian 2-subgroup of the finite group $G$. Suppose $A$ satisfies

1. $A$ is a TI-subgroup and $N_G(A)$ contains a 2-Sylow subgroup of $G$.
2. If $A^g \subseteq C_G(A)$, then $A^g$ is of root-type in $C_G(A)$.

Then one of the following holds.

1. $A$ is of root-type in $G$;
2. $A$ is weakly closed in $C_G(A)$ and the normal closure $G^*$ of $A$ in $G$ is (by [11]) of known type;
3. $G^* = \langle A^G \rangle$ contains an elementary abelian normal 2-subgroup $N$, such that $|A \cap N| = 2$ and $G^*/N$ is a covering group of $L_2(2^n)$, $S_4(2^n)$ or $U_3(2^n)$.

In many cases the structure of the weak closure $W_A^C$ of $A$ in $C = C_G(A)$ shows that condition (ii) of Theorem 1 is satisfied. So, for example, in case $W_A^C$ is a 2-group (see Lemma (2.3)). So we get the following.

**Corollary 2.** Let $A$ be an elementary abelian 2-subgroup of the finite group $G$, which is a TI-subgroup. Suppose $(W_A^C)^1 \leq A$ and $N_G(A)$ contains a 2-Sylow subgroup of $G$. Let $G^* = \langle A^G \rangle$. Then one of the following holds.

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(1) $W_A^C = A$ and $G^*$ contains an elementary abelian normal 2-subgroup $N$ such that either $G^*/N = Z(G^*/N)$ or $G^*/N$ is a covering group of $L_u(2^n)$, $U_3(2^n)$, $A_5$, $A_6$, $A_9$, $A_{24}$, $M_{22}$, $M_{23}$ or $M_{24}$.

(2) $W_A$ is the strong closure of $A$ in $N_G(A)$ and either (a) $(W_A^C)' = A$ and $G^*$ is a covering group of $L_u(2^n)$, $G_2(2^n)$ (respectively, $G_2(2^n) \sim U_3(3)$), $3D_4(2^n)$, $A_9$ or $J_2$, or (b) $(W_A^C)' = 1$ and $G^*$ is a central product of covering groups of $L_u(2^n)$, $Sz(2^n)$, $U_3(2^n)$ and 2-nilpotent groups.

(3) $G^*$ contains an elementary abelian normal 2-subgroup $N$ such that $|A : A \cap N| = 2$ and $G^*/N$ is a covering group of $L_u(2^n)$, $Sz(2^n)$ or $U_3(2^n)$.

Corollary 2 generalizes many fusion results known so far. So we get in case $|A : A \cap N| = 2$ Shult's fusion theorem [6] and in case $(W_A^C)' = A$ we get [10, Corollary B], since if we put $A = \Phi(M)$, $M_1 = \langle t^v \mid t^v \in C_G(t) \rangle$ in [10, Corollary B], then $A$ satisfies the hypothesis of our Corollary 2.

The most difficult part of the proof of Theorem 1 (and Corollary 2) is the handling of case (3), which arises in a natural way if there exists an $A^g$ such that $|A : N_G(A^g)| = 2$. So before starting with the proof of Theorem 1 let me give an example where we have (3) but neither (1) nor (2).

Let $X = GL_q(2^n)$, $n > 1$, $N$ elementary abelian of order $2^{3n}$. Suppose $X$ is represented on $N$ in the following way.

\[(a) \quad |C_N(X')| = 2^n.\]
\[(b) \quad X \text{ acts on } N/C_N(X') \text{ in the natural way.}\]
\[(c) \quad \text{There is no } X\text{-invariant complement to } C_N(X') \text{ in } N.\]

(Such a representation of $X$ exists! Set $N = \{(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}) \mid a, b, c \in F_{2n}\}$ and let $X$ act on $N$ by conjugation.) Let $G$ be the split extension of $N$ by $X'$. Let $t$ be an involution of $X$ and put $A = [N, t]$. We show $A$ is a TI-subgroup in $G$.

Let $S \in \text{Syl}_q(X)$ containing $t$, $H = Z(X)$ and $K$ a complement to $S$ in $N_N(S)$. Let $M$ be an $HK$ invariant complement to $C_N(X')$ in $N$. Since $[N, t] = 2^n$ and $H$ normalizes $M \cap [N, t]$ it follows either $M \cap [N, t] = 1$ or $[N, t] \leq M$. But in the second case $[N, S] \leq M$, since $K$ is transitive on $S^e$, whence by Gashutz's theorem there would be an $X$-invariant complement to $C_N(X')$ in $N$. Therefore

$[N, t]^e = \{z, m_i \mid z_i \in C_N(X')^e, m_i \in C_M(S)^e, i = 1, \ldots, 2^n - 1\}$

Let $k \in K^e$ and suppose $z_i m_i = (z_i m_i)^k$. Then $z_i z_j = m_i m_j$, whence $z_i = z_j$ and $m_j = m_k$, whence $t = j$ and $k \in C_G(m_j)$, a contradiction since $K$ acts regularly on $C_M(S)^e$. This shows $[N, t] \cap [N, t]^e = 1$ for each $k \in K^e$, whence $[N, t] \cap [N, \tau] = 1$ if $t, \tau \in S$ but $t \neq \tau$. This implies $A$ is a TI-subgroup in $G$.

Moreover $SN \leq N_G(A)$ and $SN \in \text{Syl}_q(G)$ and $(W_A^C)' = 1$.

The most helpful tool for proving Theorem 1 is the complete determination of the structure of the group $\langle A, A^g \rangle$ in case $A \neq N_G(A^g) \neq 1$, which was done
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in [11], although on a somewhat stronger hypothesis, but exactly the same proof works in our case.

The notation is either standard or self-explanatory (for definition of "root-involutions," see [10]). The natural graph $\mathcal{D}(D)$ of a set $D$ of involutions, is the graph with vertex set $D$ and edges $(e, d)$, where $e, d \in D$ and $ed = de \neq 1$. Further we denote by $Z^*(G)$ the coinage of $Z(G/O(G))$ and by $Z_2(G)$ the coinage of $Z(G/O_2(G))$. Finally all groups considered are finite. Moreover we assume from now on that $A$ is an elementary abelian 2-subgroup of some group $G$ and is there a TI-subgroup

2. Preliminary Results

(2.1) Lemma Let $Y$ be a solvable group generated by a class $D$ of root-involutions. Suppose $\langle C_D(d) \rangle$ is abelian for $d \in D$. Then $Y/O_2(Y) = Z^*(Y/O_2(Y)).$

Proof. Obviously $D$ is degenerate. Let $Y$ be a minimal counterexample to (2.1) Then $O_2(Y) = 1$. Let $d \neq e \in C_D(d)$. Then by hypothesis, $[O(Y), de] = 1$. Let $Q = \langle t \mid t \in D^2 \rangle$. Then $[O(Y), Q] = 1$ as shown and $Q \leq Y$. Let $R = Q \cap O(Y)$. Then $R = O(Q)$ and $R \leq O(Q)$, whence $O_2(Q) \neq 1$. But then $O_2(Y) \neq 1$, a contradiction. This shows $Q = 1$. But then $d = C_D(d)$ and thus $Y = Z^*(Y)$ by the $Z^*$-theorem of Glauberman [5].

(2.2) Corollary. Let $Y$ be as in (2.1) and suppose $Y = \langle d, e, x \rangle; d, x \in D$ and $e \in C_D(d)$. Then $Y = O_2(Y)\langle d, x \rangle$.

Proof. By (2.1), $de \in O_2(Y)$.

(2.3) Lemma. Let $A$ be an elementary abelian TI-subgroup of $G$ and $A^p \leq N_G(A)$. Then $AA^p$ is elementary.

Proof $|AA^p| = |A|^2$ and thus $AA^p = A^p(A^p)^a$ for each $a \in A \in N_G(A^p)

(2.4) Proposition. Let $A$ be an elementary abelian 2-group, which is a TI-subgroup of $G$. Suppose $\bar{A} \sim A$ such that $1 \neq N_A(A) \neq A$. Set $L = N_A(A)$, $M = L\bar{L}$ and $X = \langle A, \bar{A} \rangle$. Then the following properties hold.

(1) $M \leq X$ and if $R = X/M$ then $R \cong L_2(2^n)$, $Sz(2^n)$ or $D_{2m}$; $m \equiv 1(2)$, where $2^n = |A \cdot L|.$

(2) $M = \oplus M_i$, where the $M_i$ are irreducible $F_2R$ modules. Moreover, if $R \cong L_2(2^n)$, $Sz(2^n)$, the $M_i$ are natural $F_2R$-modules

(3) If $a, b$ are two involutions of $X$ not contained in a 2-Sylow-subgroup of $X$, then $C_{M_i}(a) \cap C_{M_i}(b) = 1$. 
(4) $D = \{ t \mid t \sim A^* \text{ in } X \}$ is a set of root-involutions of $X$. Moreover, if $t \in A \neq L$, then $t^X$ is a class of degenerate root-involutions of $X$.

(5) All elements of $A \neq L$ are conjugate in $N_X(A)$.

Proof. $[L, L] \leq A \cap \bar{A} = 1$, thus $M$ is elementary. Now $[A, L] \leq A \cap N_X(\bar{A}) = L$ and similar $[\bar{A}, L] \leq L$. Thus $M < X$. Let now $t \in A \neq L$ and $\tau \in A^\circ - L^\circ$, $g \in X$ such that $[t, \tau] \in M$. Since $t \notin N_A(\bar{A})$ it follows $C_M(t) = L$. By the same reason $C_M(\tau) = L^\circ$, whence $[t, t] \in C_M(\tau) \cap C_M(t) = L \cap L^\circ = L$ or 1. In the first case $L \leq A^\circ = A$. Assume the second case holds. Then $t \in C_M(\tau) < N_A(A^\circ) = N_A(L^\circ)$, since $L^\circ = A^\circ \cap M$. Hence $t$ centralizes an element of $(L^\circ)^\circ$, contradicting $C_M(t) = L$. This shows

(*) $AM/M$ is strongly closed in the centralizer of each of its involutions.

Further obviously $A \sim \bar{A}$ in $X$.

If now $2^n = |A : L| > 2$ then Shult's fusion theorem [6] and an inspection of the groups listed there, implies $X/M \simeq L_2(2^n)$, $Sz(2^n)$; since $U_3(2^n)$ is not generated by two elementary 2-subgroups. If $|A : L| = 2$, then $X/M$ is generated by two involutions and thus $X/M \simeq D_{2m}$, $m = 1(2)$. This implies (1).

Pick $a \in A \neq L$, $b \in \bar{A} \neq \bar{L}$. Then $C_M(ab) = 1$, since $C_M(a) \cap C_M(b) = L \cap \bar{L} = 1$. Hence in case $R \simeq L_2(2^n)$, $Sz(2^n)$ there is an element of order 3 resp. 5 which acts fixpoint-free on $M$. Hence in this case results of Higman [7] and Martinieu [8] imply (2). If $X/M \simeq D_{2m}$, $m = 1(2)$, then (2) is an easy consequence of Gaschütz's theorem.

Let now $S \in Syl_2(X)$ containing $A$. Then $\Omega_3(S) \leq MA$. Hence if $a$ is an involution of $S$ then $a \in M \cup A$. Especially, if $a \notin M$, then $a \in A$. This implies (3).

To prove (4), let $a \in A \neq L$, $b \in \bar{A} \neq \bar{L}$ and suppose $2 \nmid o(ab)$. Let $1 \neq z \in Z(a, b)$. Then $z \in M$, since $M o(ab)$ is odd. But then $z \in C_M(a) \cap C_M(b) = 1$, a contradiction. Thus $o(ab) \equiv 1(2)$. If now $\varepsilon \in \bar{L}$ then $o(ac) = 4$ and $(ac)^2 \in L \subset D$. This proves (4).

Condition (5) is a consequence of $aL = a[a, M] = a^M$.

(2.5) COROLLARY Let $A$ be an elementary abelian $TI$-subgroup of $G$ and $N$ an abelian normal 2-subgroup of $G$ such that $A \cap N \neq 1$. Set $E = \{ t \mid t \sim A^* \text{ in } G \}$. Then $E$ is a set of root-involution of $\langle E \rangle$ and so $\langle E \rangle$ is (by [10]) of "known type.""\)

Proof. For each pair $A^h$, $A^g$; $h, g \in G$ we can apply (2.4)(4), since $A^g \cap N \neq 1 \neq A^h \cap N$. This proves (2.5)

3. THE CASE $q > 2$

For section 3, let $G$ be a finite group which contains an elementary abelian $TI$-2-subgroup $A$, which satisfies conditions (i) and (ii) of Theorem 1. Suppose
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$G = \langle A^G \rangle$ and statements (1) and (2) of Theorem 1 do not hold. Choose $A^g$ with the following properties.

(a) $1 \neq A^g \cap N_G(A) \neq A^g$.

(b) $|A^g \cap N_G(A)|$ is maximal with (a).

Let $q = |A^g|$: $A^g \cap N_G(A)|$ and $S_0 \in \text{Syl}_2(N_G(A))$ containing $A(A^g \cap N_G(A))$. By hypothesis (i), $S_0 \in \text{Syl}_2(G)$ Since (2) does not hold there exists a $B \sim A$, $A \neq B \leq S_0 \cap C_G(A)$. We choose the following notation. $\bar{A} = A^g$, $A_0 = N_G(A)$, $\bar{A}_0 = N_G(A)$, $M = A_0 \bar{A}_0$ and $X = \langle A, \bar{A} \rangle$. Then all properties of (2.4) hold for $X$.

(3.1). Let $x \in \bar{A}_0^s$ and $b \in B^s$. Suppose $o(xb) \neq 2$ or 4. Then there exists an $s \in \langle x, b \rangle \cap C_G(A)$ such that $\bar{A}_0 \neq \bar{A}_0^s$, $[\bar{A}_0, \bar{A}_0^s] = 1$ and $o(s) = 4$.

Proof. Let $s$ be an element of order 4 in $\langle x, b \rangle$. Since $o(xb) = 2^n$, $n > 3$ it follows $s \in \langle b, b^s \rangle \leq C_G(A)$. Now $\langle x, s \rangle \cong D_5$, whence $[x, s] = 1$. If $x^n \neq \bar{A}_0$ then $xx^s \in \bar{A}_0 \cap Z(x, b)$, whence $b \in N_G(A)$ since $\bar{A}_0 = \bar{A} \cap N_G(A)$, contradicting $o(xb) \neq 2$ or 4. This implies $\bar{A}_0 \neq \bar{A}_0^s$. It remains to show that $[\bar{A}_0, \bar{A}_0^s] = 1$.

Now $\bar{A}_0 = \bar{A} \cap N_G(A)$ is a $TI$-subgroup of $N_G(A)$. Thus $\langle \bar{A}_0, \bar{A}_0^s \rangle$ is of type described in (2.4) if $[\bar{A}_0, \bar{A}_0^s] \neq 1$, since $x \in N_{\bar{A}_0}(\bar{A}_0^s)$. Now for each $y \in \bar{A}_0$ we have by (2.4), $[A_0, y] = 1$ and $[A, y] \leq A_0$. Since $s \in C_G(A)$ the same holds for $ys$. Let now $z, y \in \bar{A}_0$ such that $o(zys) \equiv 1(2)$. Then $zys \in C_G(A)$, since $zys$ centralizes $A_0$ and $A/A_0$. Thus, if $y \in \bar{A}_0 - O_2(\bar{A}_0, \bar{A}_0^s)$ and $z \in (\bar{A}_0 \cap O_2(A_0, A_0^s))$, then $o(zys) \equiv 1(2) \equiv o(zys)$ by (2.4)(3). Hence $yz = (zys)(y^s) \in C_G(A)$ and thus $\bar{A}_0 \cap C_G(A) \neq 1$, a contradiction to (2.4) and the choice of $\bar{A}$. This shows $A_0 \leq O_2(\bar{A}_0, \bar{A}_0^s) = \bar{A}$ and thus $[\bar{A}_0, A_0^s] = 1$.

Choose now $b \in B$ and put $s = b$ if $o(xb) = 2$ or 4 for some $x \in \bar{A}_0$. Otherwise let $s$ be the element of (3.1) let $Y = \langle A, \bar{A}, A^s \rangle$ and $N = A_0 \bar{A}_0 A_0^s$.

(3.2). $N \vartriangleleft Y$. Moreover if $q > 2$, then one of the following holds.

(1) $Y/N \cong S_2(q)$ if $X/M \cong S_2(q)$,

(2) $Y/N \cong L_2(q)$ or $Y/N$ is a perfect central extension of $U_3(q)$ if $X/M \cong L_2(q)$.

Proof. We first show $N \vartriangleleft Y$. If $s = b$ then (2.4) implies $[\bar{A}_0, \bar{A}_0^s] = 1$, since $\langle \bar{A}_0, \bar{A}_0^s \rangle \leq S_0$ and $[x, x^s] = 1$ for some $x \in \bar{A}_0$. Hence (3.1) implies $[\bar{A}_0, \bar{A}_0^s] = 1$ in any case. Thus $N$ is elementary. Now $[A_0, A_0^s] = [A, A_0^s] = A_0^s = A_0$ and $[\bar{A}, \bar{A}_0^s] \leq \bar{A} \cap C_G(A_0, \bar{A}_0) = \bar{A}_0$. Hence $[\bar{A}, \bar{A}_0] \leq A_0^s = A_0$, since $s^s \in Z(\langle x, b \rangle)$ normalizes $\bar{A}_0$. But obviously $[\bar{A}_0^s, A_0] = [\bar{A}, A_0^s] = A_0^s$.

This shows $N \vartriangleleft Y$.

Let now $N_0 = O_2(V)$ and suppose $N_0 \leq N_G(A)$. Then there exists an $u \in N_0$, $u \in A - A_0$, such that $1 \neq uu^s \in N_0$ but $A \neq A^u$. Since $[A_0, A_0^s] = 1$,
(2.4) and the maximality of $|A_0| = |\overline{A}|$ imply $[A, A^a] = 1$. Let now $S \in \text{Syl}_2(G)$ containing $N_0 \overline{A}$. Then by hypothesis (i) there exists an $A^a < S$. Since $A_0 \leq N_0$ there exists an $1 \neq x \in A^a \cap N_0(A)$. Hence $[aa^a, x] \in A^a \cap C_G(A)$, since $aa^a \in C_G(A)$. But then (2.3) implies $[A^a, A] = 1$. On the other hand $[\overline{A}, A^a] = 1$. Hence $A$ is not of "root-type" in $C_G(A^a)$, a contradiction.

This proves $N_0 \leq N_0(A)$. But then immediately $N_0/N \leq Z(Y/N)$.

Let $g, h \in Y$ and consider $\langle A^a, A^a \rangle$. If $[A^a, A^a] \neq 1$, then the maximality of $|A_0| = |A_0|$ implies

$$N_{A^a}(A^a) = A_0^a \quad \text{and} \quad N_{\overline{A}}(A^a) = A_0^h.$$

Hence, if we set $D = \{t \in Y \mid t \sim A \sim A_0 \text{ in } Y\}$, (2.4) and (2.5) imply $D$ is a class of degenerate root-involutions of $Y$. By $q > 2$ there are $a, b \in D$ such that $ab \in D$. Hence $O(Y/N) \leq Z(Y/N)$. But then $S(Y/N) = Z(Y/N)$, where $S(G)$ denotes the maximal solvable normal subgroup.

Now let $d \in A - A_0$ and suppose there exist $e, f \in C_D(d)$ such that $o(ef) = 1(2)$; $e \neq f$. Let $e \in A^a, f \in A^b, g, h \in Y$. Then the maximality of $|A_0|$ implies $\langle A^a, A^b \rangle \leq C_G(A)$, but $A^a$ is not of root-type in $C_G(A)$, a contradiction. This proves $\langle C_D(d) \rangle$ is elementary. But then [3, I] implies $Y/N$ is a perfect central extension of $L_2(2^n), S_2(2^n)$ or $U_3(2^n)$.

Suppose now $2^n > q$. Denote by $*$ the natural homomorphism from $Y$ on $Y/S(Y)$. Then there exists an $a^a, g \in Y$ such that $A^* \neq (A^a)^* \neq 1$, since $|A^*| = 4$. Let $(A^a, (A^a)^* \cong L_2(q)$ resp. $S_2(q)$ if $X/M \cong L_2(q)$ resp. $S_2(q)$. Hence in the first case we find an $a \in A^*,$ $b \in (A^a)^* - A^*$ and $x \in A^*$, such that we get the relations $a \circ - \circ - b$. But then $o(xab) = 4$, a contradiction to $Y^* \cong (A^*, A^a)^*$. Now it is easily seen that all dihedral groups of order 10 are conjugate in $S_2(2^n)$ (see [9]). Hence there exists a $g \in Y^*$ such that $\langle a, x \rangle^a = \langle b, x \rangle$, and $x = x^a$. But by the structure of $S_2(2^n)$ it follows that $C_{Y^*}(x)$ is the 2-Sylow-subgroup of $Y^*$ containing $x$. But then $a^a \neq b$. On the other hand $\langle b, x \rangle$ is contained in an $S_2(2)$ in $Y^*$ and thus there exists an $h \in C_{Y^*}(x)$, such that $a^a h = b$. Thus we may choose $g \in Y^*$ such that $a^g = b, x^g = x$, a contradiction to what we have shown above. This implies $2^n = q$.

It only remains to show $Z(Y/N) = 1$ if $Y/Z^d(Y) \cong L_2(q), S_2(q)$. In the first case this is obvious. In the second case we have $XN/N \cong S_2(q)$ by (2.4). Thus if $Y/N \cong S_2(q), S_2(q)$ would contain a subgroup isomorphic to $S_2(q)$, a contradiction.

(3.3). Suppose $q > 2$. Then $Y \cong X$.

Proof. Suppose first $Y/N$ is a perfect central extension of $U_3(q)$. Since $s^a$ normalizes $\overline{A}$, since $s^a \in Z(x, b)$, $[s^a, A] = 1$ implies $[Y, s^a] \leq N$. Hence $s$ induces an involutory automorphism on $Y/N$. Since $Y/N$ is a perfect central...
extension of $U_q(q) \cong A_0(U_q(q))$. Thus, if $s$ induces an inner automorphism on $Y/N$, there exists an $a \in A - A_0$ such that $[sa, Y] \leq N$. But then $A^s = A^a \mod N$, whence $Y/N = XN/N \cong L_2(q)$

Hence by [11, (2.1)], $s$ induces an involutory field automorphism on $Y/N$. Thus there exists an $A^h$, $h \in Y$ such that $[A, A^h]/A_oA^h \cong L_2(q)$ and $[A, A^h], s] \leq N$. Since $N \leq N_{y}(A, A^h)$ and since we may choose $h \in \langle A, A^h \rangle$ it follows $s \in N_{y}(A^h)$. Hence $[A_oA^h, s] = 1$, since $[A_0, s] = 1$. Thus implies $A_oA^h = C_N(s)$, since if $C_N(s) > A_oA^h$ it follows $A_0 \cap C_N(s) \neq 1$, a contradiction to $A \neq A^s$. Now $[N, s] = 1$. Hence $[A_o, s] \leq C_N(s) \cap C_N(A) = A_oA^h \cap C_N(A) = A_0$. This implies $A_o^s = A_oA^h$ and thus $N = M = M^s = A_oA^h$. Hence $s \in N_{y}(A)$, a contradiction to $Y \neq X$.

Thus we get $Y/N \cong L_2(q)$, $Sz(q)$. But then $Y = NX$. Since $N = A_oA^h \subseteq N_{y}(A)$, it follows $N \leq N_{y}(X)$, and thus $X \leq Y$, since $A \sim A^h$ in $Y$.

(3.4). If $Y = X$ then $s = b$. Moreover $B \leq N_{y}(X)$ if $q > 2$

**Proof.** If $Y = X$ then $s \in N_{y}(X)$ Now $s \notin N_{y}(A)$, if $s \neq b$, since in this case $xx^s \in Z(x, b)$ and thus $b \in N_{y}(A)$ if $xx^s \in A$. Hence if $s \neq b$ there exists an $a \in A - A_0$ such that $[as, X] \leq M$, since $s$ induces by (2.4) an inner automorphism according to $AM/M$ on $X/M$, since $C_N(s) = A_0$. Hence $X \leq N_{y}(M)\langle as \rangle$ and $[M, as] = 1$ But then $\langle s^2 \rangle = \langle (as)^2 \rangle = C_1(M)\langle as \rangle$, whence $X$ centralizes $s^2 = xx^s \in M$, since $x \in A_0$, a contradiction to (2.4).

This shows $s = b \in N_{y}(X)$. If now $q > 2$ then (3.3) implies $Y = \langle A, A^h \rangle = X$ for each $b \in B^s$, since by (2.4), $[B, B^s] = 1$ if $o(xb) = 4$ for one $b \in B^s$. Hence $B \leq N_{y}(X)$ in this case.

(3.5). $B \leq N_{y}(X)$.

**Proof.** Suppose false. If $X/M \cong L_2(q)$ or $Sz(q)$ then $B$ induces inner automorphisms according to $AM/M$ on $X/M$. Suppose now $X/M \cong D_{2n}$. If $b \in B^s$ centralizes an $a^sM/M$ in $X/M$, $a \in A - A_0$, $g \in X$ and $a^s \neq A$, then $b$ centralizes $A_oA^g = M$ and thus $[X, b] = M$. Thus we may assume $b$ normalizes no conjugate $A^g$ of $A$ in $X$ different from $A$. Let $\bar{a} \in A - A_0$ and consider $\langle \bar{a}, b \rangle$. Then there is a $z \in Z(\bar{a}, b)$ such that $xb \sim \bar{a}$ in $\langle \bar{a}, b \rangle$. Hence $xb \in A$ by the above and $[X, (zb)b] = [X, z] = M$. Thus in any case $B$ induces inner automorphisms according to $AM/M$ on $X/M$.

Especially there exists a $B_0 \leq B$, $|B_0| = |A_0|$, such that $[X, B_0] \leq M$. Since $[B_0, A_0] = 1$ it follows $[B_0, M] = 1$, whence $M \leq N_{y}(B)$. On the other hand we have $BA = CA$, where $[C, X] \leq M$ by the above. Hence $[C, M] = 1$, since $[C_0, A_0] = 1$. It follows $[M, B] \leq B \cap [M, CA] \leq B \cap [M, A] = B \cap A_0 = 1$, since $CA = BA$ is abelian. This shows $B \leq N_{y}(A)$ and therefore $[B, A] = 1$ by (2.3). But then $X \leq C_{y}(B)$ and therefore $A$ is not of root-type in $C_{y}(B)$, a contradiction to the hypothesis.
(3.6). If \( q > 2 \) then (1) or (2) of Theorem 1 holds.

In Section 3 we have chosen \( G \) such that neither (1) nor (2) of Theorem 1 holds. If then in addition \( q > 2 \), then (3.4) implies \( B \leq N_\alpha(X) \), but (3.5) implies \( B \leq N_\alpha(X) \), a contradiction.

4. THE CASE \( q = 2 \) AND \( A_0 < N_\alpha(A) \)

Let \( G \) be as in Section 3. Choose the notation as in Section 3, but assume \( q = 2 \). Let \( D = \{ t \mid t \sim A - A_0 \text{ in } Y \} \).

(4.1). \( D \) is a degenerate class of root-involutions of \( Y \). Moreover \( \langle C_\alpha(d) \rangle \) is elementary for \( d \in D \).

Proof. Exactly as in (3 2) one shows \( N < Y \) and \( N \) elementary. Hence the maximality of \( |A_0| \), (2.4), and (2.5) imply \( D \) is a degenerate class of root-involutions of \( Y \). Let \( d \in A - A_0 \); \( e, f \in C_\alpha(d) \) such that \( 1 \neq o(ef) = 1(2) \). Suppose \( e \in A^g, f \in A^h, g, h \in Y \). Then \( A^g \) is not of root-type in \( C_\alpha(A) \), a contradiction.

(4.2) Either \( Y \) is solvable or \( Y/N \cong L_2(2^n), S_\alpha(2^n) \).

Proof. Suppose \( Y \) is not solvable. Then \([3, 1]\) implies \( Y/S(Y) \cong L_2(2^n), S_\alpha(2^n) \) or \( U_3(2^n) \), since \( \langle C_\alpha(d) \rangle \) is elementary. Exactly as under (3.2) one shows \( O_2(Y) \leq N_\alpha(A) \) and thus \( O_2(Y)/N \leq Z(Y/N) \), if \( Y \) is solvable or not.

Denote by \( - \) the natural homomorphism from \( Y \) on \( Y/N \). Then there exist \( \bar{a}, \bar{b} \in D \) such that \( \bar{a}\bar{b} = \bar{b}\bar{a} \neq 1 \), since \( Y \) is not solvable. Suppose \( C_\alpha(\bar{a}) \leq C_\alpha(\bar{b}) \). Let \( \bar{h} \in C_\alpha(\bar{a}) - C_\alpha(\bar{b}) \). Then \( \langle \bar{h}, \bar{b}^\bar{h} \rangle \leq C_\alpha(\bar{a}) \), but \( o(\bar{b}^\bar{h}) = 1(2) \). Let \( a, b, b^h \) be conjugates of \( \bar{a}, \bar{b}, \bar{b}^\bar{h} \). Suppose \( a \in A, b \in A^g, b^h \in A^g, g \in Y \). Then \( \langle A^g, A^g \rangle \leq C_\alpha(A) \), since \( A = A_0 \langle a \rangle \) and \( A^g = A_0 \langle b \rangle \). Hence \( A^g \) is not of root type in \( C_\alpha(A) \), a contradiction.

This shows \( O(\bar{Y}) \leq Z(\bar{Y}) \) and thus \( Y/N \) is a perfect central extension of \( L_2(2^n), S_\alpha(2^n) \) or \( U_3(2^n) \). We show now that the last case does not occur.

Obviously \( s^2 \) centralizes \( Y/N \), since it normalizes \( A, \bar{A}, \bar{A}^t \). Hence \( s \) induces an involutory automorphism on \( Y/N \). As under (3.3) we may identify \( s \) with an automorphism of \( U_3(2^n) \). Suppose \( s \) induces an inner automorphism on \( Y/N \). Let \( S \in \text{Syl}_3(Y) \) containing \( A \). Then there exists a \( t \in \Omega_3(S) \) such that \( [st, Y] \leq N, \) since \( s \) centralizes \( A \). Hence \( Y = \langle A, \bar{A}, \bar{A}^t \rangle \mod N \). But by the structure of \( U_3(2^n) \) it follows that \( \langle A, \bar{A}, \bar{A}^t \rangle N/N \cong L_2(2^n) \), since \( \langle \Omega_3(S), \bar{A} \rangle N/N \cong L_2(2^n) \).
Thus we may assume $s$ induces an involutory field automorphism on $Y/N$. Hence there exists (by [11, (2.1)]) an $A^h$, $h \in Y$ such that $[\langle A, A^h \rangle, s] \leq N$ and $[\langle A, A^h \rangle, A_0 A^h] \cong D_{2m}$, $m = 1(2)$. Since $N$ normalizes $A^h$ and without loss $h \in \langle A, A^h \rangle$, it follows $s \in N_G(A^h)$. Therefore $[\langle A, A^h \rangle, s] \leq A_0 A^h$, whence $[A_0 A^h, s] = 1$. This shows $A_0 A^h = C_N(s)$, since $A_0 \cap C_N(s) = 1$ and $N = A_0 A^h A_0$. But obviously $[N, s] = 1$, whence $[A_0, s] \leq C_N(s) = A_0 A^h$. Therefore $[A_0, s] \leq A_0 A^h \cap C_N(A) = A_0$ and thus $A_0 \leq A_0 A_0$. This shows $N = M = A_0 A^h$. But then $s$ centralizes $N$ and thus normalizes $A$, a contradiction to $Y \neq X$. This shows $Y/N$ is a perfect central extension of $L_d(2^n)$ or $Sz(2^n)$.

By [2] it remains to show $Y/N \cong Sz(8)$ But by [2] it is easy to see that $Sz(8)$ is not generated by root-involution. Hence (4.1) implies (4.2).

(4.3). If $Y$ is solvable, then $Y = X$.

Proof. As shown in (4.2), $O_2(Y)/N \leq Z(Y/N)$. Hence (4.1) and (2.2) imply $Y/N = Z^*(Y/N)$. Since $s^2 = xx^s \in A_0 A^s$, $s$ induces an involutory automorphism on $\langle A, A^s \rangle/A_0 A^s$. Thus there exists an $x \in A^s - A_0 A^s$, $g \in \langle A, A^s \rangle$ such that $[\langle A, A^s \rangle, xx^s] \leq A_0 A^s$. Hence $x^s$ is an involution of $x(A_0 A^s)$ and thus by (2.4), $x^s \in A^s$. Thus $s \in N_G(A^s)$. Suppose $A \neq A^s$. Then $[A_0 A^s, s] = 1$, since $[A_0, s] = 1$ and $[\langle A, A^s \rangle, s] \leq A_0 A^s$. Hence, if $Y \neq X$, then $A_0 A^s = C_N(s)$, since $A \neq A^s$ and thus $A_0 \cap C_N(s) = 1$. But then $[A_0, s] \leq C_N(s) \cap C_N(A) = A_0 A^s \cap C_N(A) = A_0$. Hence $A_0 \leq A_0 A^s$ and thus $N = M = A_0 A^s = C_N(s)$, which proves $Y = X$.

Thus we may assume $A^s = A$, whence $x \in A - A_0$. But then $\langle A, A^s \rangle = \langle A, A^s \rangle = X$, which proves $Y = X$.

(4.4). $s = b \in B^a$ and $o(xb) = 2$ or 4 for all $x \in A_0^s$, $b \in B^a$.

Proof. Suppose false. If $Y$ is solvable, then $Y = X$ by (4 3) and thus (4.4) holds by (3.4). Hence by (4 2), $Y/N \cong L_d(2^n), Sz(2^n)$.

Now $s$ induces an involutory automorphism on $Y/N$. If $s$ induces a field-automorphism, then there exists an $A^h$, $h \in Y$ such that $[\langle A, A^h \rangle, s] \leq N$ but $[\langle A, A^h \rangle, A_0 A^h] \cong D_{2m}$, $m = 1(2)$. Arguing as under (4 2), $[A_0 A^h, s] = 1$. Since $xx^s \in Z\langle x, b \rangle$ it follows $s \notin N_G(A)$, if $s^2 \neq 1$. Hence $A_0 A^h = C_N(s)$, since $C_N(s) \cap A_0 = 1$. But then $[A_0, s] \leq C_N(s) \cap C_N(A) = A_0$ and thus $A_0 s \leq A_0 A_0$. Hence $N = M = A_0 A^h = C_N(s)$, a contradiction to $A^s \neq A$.

This proves $s$ induces an inner automorphism according to $\Omega_2(S)$ on $Y/N$, where $S \in \text{Syl}_2(Y)$ containing $A$. Now in each coset of $\Omega_2(S)/N$ lies an element of $D$. Thus there exists a $c \in C_D(A)$ such that $[Y, cs] \leq N$. Let $C \sim A$ in $Y$ containing $c$ and $C_0 = C \cap N$. Since $cs \in C_G(A_0)$ it follows $[N, cs] = 1$. Let $a \in A - A_0$. Suppose $s$ does not centralize $c$ Then $ca \notin N$, since otherwise $cs$ and thus $s$ centralizes $ca$, whence $s$ centralizes $c$. Therefore there exists a $t \in N$ such that $cat \in D$. Now $C_N(a) = A_0 C_0 = C_N(c)$, since $N = A_0 C_0 A_0$. Hence $t \in C_0 A_0$. Let $cat \in A^g$, $g \in Y$. Since $s$ centralizes $C_0 A_0$ it follows $[cat, s] =
But on the other hand $s$ normalizes $A^g$, since $A_0^g \leq A_0 C_0$. Hence $[cat, s] = [c, s] \in C_0 \cap A_0^g$. This shows $C = A^g$ and thus $at = (cat) c \in C$, a contradiction to $|C: C_0| = 2$.

Hence $[s, c] = 1$ and thus $(cs)^2 = c^2 s^2 = s^2$. Now $Y \leq N_G(N\langle cs \rangle)$ and $\langle s^2 \rangle = \langle (cs)^2 \rangle = O_4(N\langle cs \rangle)$. Thus $Y$ centralizes $s^2 = x^x \in \bar{A}_0 A_0^g$, a contradiction to (2.4).

This shows $s^2 = 1$ and thus $s \in B^e$. Hence $o(xb) = 2$ or $4$, where $b = s$, whence (2.4) implies $[B, B^e] = 1$, since $\langle B, B^e \rangle \leq S_0$. This shows $o(xb) = 2$ or $4$ for all $x \in \bar{A}_0^e$, $b \in B^e$.

(4.5). Either part (1) or (2) of Theorem 1 holds for $G$ or there exists a $b \in B^e$ such that $Y = \langle A, \bar{A}, \bar{A}^b \rangle$ is not solvable.

Proof. If $\langle A, \bar{A}, \bar{A}^b \rangle$ is solvable for all $b \in B^e$, then $\langle A, \bar{A}, \bar{A}^b \rangle = X$ for all $b \in B^e$. Hence $B \leq N_G(X)$. But by (3.5), $B \leq N_G(X)$, a contradiction to the choice of $G$.

Therefore we know there exists a $b \in B^e$ such that $\langle A, \bar{A}, \bar{A}^b \rangle = Y$ is not solvable, since by (4.4), $s \in B^e$ and we have chosen $G$ such that (1) and (2) of Theorem 1 do not hold. Hence by (4.2), $Y/N \simeq L_2(2^n)$ or $Sz(2^n)$. Now let $Z = \langle A, \bar{A}^b \rangle$ and $Q = A_0(\prod b \in B^e \bar{A}^b)$.

(4.6). $Q$ is an elementary normal $2$-subgroup of $Z$ and $Z/Q \simeq L_2(2^n)$, $Sz(2^n)$.

Proof. Let $b_1, b_2 \in B$. Then obviously $[\bar{A}_0^b, \bar{A}_0^{b_2}] = [\bar{A}_0, \bar{A}_0^{b_1 b_2}] = 1$, whence $Q$ is elementary. Now $[\bar{A}_0^b, \bar{A}_0^{b_2}] = [\bar{A}, \bar{A}_0^{b_1 b_2}] = 1$, whence each $\bar{A}^b$, $b \in B$, normalizes $Q$. Thus $Q < Z$.

Let $E = \{t \mid t \sim A - A_0 \text{ in } Z\}$. Then (2.5) and $|A : A_0| = 2$ imply $E$ is a degenerate class of root-involutions of $Z$. Now there exists a $b \in B^e$ such that $Y/N \simeq L_2(2^n)$, $Sz(2^n)$, where $Y = \langle A, \bar{A}, \bar{A}^b \rangle$, $N = A_0 \bar{A}_0 \bar{A}_0^b$. Hence there exist $e, f \in D$ such that $(ef) N \in DN/N$. Thus $O(Z/Q) \subseteq Z(Q)$. Now arguing exactly as under (3.2) one shows $O_2(Z/Q) \leq Z(Q)$ and $\langle C_D(d) \rangle$ is elementary for $d \in D$. Hence $[3, 1]$ implies $Z/Q$ is a perfect central extension of $L_2(2^n)$, $S_2(2^n)$ or $U_3(2^n)$.

Suppose now the last case holds. Let $x \in \bar{A} - \bar{A}_0$ and $b \in B^e$ such that $\langle x, x^b \rangle \simeq D_{2m}$, $m \equiv 1(2)$. Hence there exists a $y \sim x$ in $\langle x, x^b \rangle$ such that $y^b = y^b$. Let $A^p \sim A$ in $Z$ such that $y \in A^p$. If $\langle A, A^p \rangle \neq 1$, then $\langle A, A^p \rangle \not\leq C_G(B)$ is not of root-type in $C_G(B)$, a contradiction.

Now let $S_i \in \text{Syl}_2(Z)$, $i = 1, 2$, such that $A \leq S_1, B \leq S_2$. Let $T_i = \Omega_i(S_i)$, $i = 1, 2$. If for $b \in B^e$ follows $[x, x^b] = 1$, then $[\bar{A}, \bar{A}^b] = 1$ and thus $\bar{A}^b \leq T_2$. If $o(xb^b) \equiv 1(2)$, then as shown above, there exists a $y \in A^g \leq T_1$ such that $\bar{A}^b = \bar{A}^b$. Hence $\langle A, \bar{A}^b \rangle \leq \langle A, T_1^b \rangle \leq \langle T_1, T_2 \rangle$, a contradiction to $\langle T_1, T_2 \rangle / Q \simeq L_2(2^n)$ by the structure of $U_3(2^n)$.
Hence $Z/Q$ is a perfect central extension of $L_d(2^n)$, $Sz(2^n)$. But then $Z/Q \cong L_d(2^n)$, $Sz(2^n)$ by [2], since $EQ/Q$ is a class of root-involutions of $Z/Q$.

(4.7). $A_0$ normalizes each $B \leq S_0$, $B \sim A$ in $G$.

Proof. Suppose false. Choose $B \leq S_0$ such that $A_0$ does not normalize $B$. Then $B$ normalizes $Z = \langle A, A^\beta \rangle$. Since $Z/Q \cong L_d(2^n)$, $Sz(2^n)$ there exist $\alpha \in A, \beta \in C \sim A$ in $Z$; $[C, A] = 1$, and $\gamma \in A^\beta, h \in Z$ such that $\langle \alpha, \beta, \gamma \rangle Q = Z$; since $L_d(2^n)$ and $Sz(2^n)$ may be generated by three involutions. Hence $Q = A_0 C_0 A_0^\beta$, since $A_0 C_0 A_0^\beta \lhd \langle \alpha, \beta, \gamma \rangle Q = Z$. Thus $|Q| = |A_0|^3$.

Suppose now there exists a $b \in B^*$ such that $[Z, b] \lneq Q$. Then $[Q, b] = 1$ and thus $A_0 \lneq Q \lneq N_0(B)$, which is to show. Thus no $b \in B^*$ centralizes $Z/Q$.

Let $S \in \text{Syl}_d(Z)$ containing $A$ and suppose $b \in B^*$ induces an inner automorphism according to $\Omega_d(S/Q)$ on $Z/Q$. Then there is a $c \in C \sim A$ such that $[Z, bc] \lneq Q$. Moreover, $[C, A] = 1$ and $c \not\in Q$. Obviously $[Q, bc] = 1$. Thus if $o(bc) = 4$, then $Z$ centralizes $(bc)^2 \in \Omega_d(\langle bc \rangle)$, a contradiction to $(bc)^2 \in C_0 = C \cap Q$. Thus $b$ centralizes $C_0^\langle c \rangle = C$, whence $[B, c] = 1$ by (2.3). Suppose now there is a $b \in B^*$, which induces an involutory field-automorphism on $Z/Q$. Then there exists an $A^\alpha$, such that $[\langle A, A^\alpha \rangle, b] \lneq Q$, but $[A, A^\alpha \rangle A_0 A_0^\alpha \cong D_{2^r}, r = 1(2)$. Hence $[A_0^\alpha, b] = 1$ and thus $b$ centralizes $A_0 C_0 A_0^\alpha = Q$, since $|Q| = |A_0|^3$. A contradiction to $[Z, b] \lneq Q$. This shows that $B$ induces inner automorphisms according to $\Omega_d(S/Q)$ on $Z/Q$, since at least one element of $B$ induces an inner automorphism.

But then $|B| \lneq 2^{2n} = |\Omega_d(S/Q)|$, since $C_0(Z/Q) = 1$. Now we have $A_0 C_0 \lneq C_0(\langle \alpha \rangle) \cap C_0(\beta)$. Thus $|C_0(\gamma)| \geq |A_0|^3$, whence $|C_0(Z)| \geq |A_0|$, since $|Q| = |A_0|^3$ and $C_0(\alpha) \cap C_0(\beta) \cap C_0(\gamma) \leq C_0(Z)$. Thus $|Q/C_0(Z)| \lneq |A_0|^3 < 2^{2m}$, since $|A_0| < |A| = |B| \lneq 2^m$. But this is a contradiction to [11, (2.7)].

(4.8) Let $B, C \sim A$ in $G$ and suppose $o(bc) = 2$ or $4$ for all $b \in B^*$, $c \in C^e$. Then $[B, C] = 1$.

Proof. Suppose false. Then (2.4) implies there is no $b \in B^*$, $c \in C^e$ such that $o(bc) = 2$. Since $[B, B^e] = 1$ for each $c \in C$ it follows $\prod_{c \in C} B^e$ is elementary. Hence $\langle B, C \rangle$ is a 2-group. Now by hypothesis $\langle B, C \rangle$ normalizes some $A^\alpha$. Hence we may without loss assume that $\langle B, C \rangle \leq S_0$. But then (4.7) implies $\bar{A}_0$ normalizes $B$ and $C$. If $\bar{A} \lneq N_0(B)$ then $\bar{A} \leq C_0(B)$ by (2.3) and thus $A$ is not of root type in $C_0(B)$, a contradiction. Therefore, if $B_0 = N_0(A)$, $C_0 = N_0(\bar{A})$, then $\langle \bar{A}, B \rangle/\bar{A}_0 B_0 \cong D_{2m}$ and $\langle \bar{A}, C \rangle/\bar{A}_0 C_0 \cong D_{2r}$, $r, m = 1(2)$ by (2.4). Now if $c \in C \sim C_0$, then $\langle \bar{A}^e, A \rangle/\bar{A}_0^e A_0$ is dihedral of order $2s$, $s = 1(2)$, since $C \lneq C_0(A)$. Moreover $\bar{A}_0^e \lneq S_0$, whence (4.7) applied for $\bar{A}_0^e$ tell us that $\bar{A}_0^e \lneq N_0(B) \cap N_0(C)$. But then $\bar{A}_0 C_0 = \bar{A}_0^e \bar{A}_0^e \lneq N_0(B)$, whence there exists a $b \in B^e$, $c \in C^e$ such that $o(bc) = 2$, a contradiction.

Now let $H$ be the weak closure of $A_0$ in $N_0(A)$. 


(4.9) Suppose $A_0 \leq Z(H)$. Then the following holds.

(1) $A_0$ is a TI-subgroup. Moreover $H$ is transitive on $A - A_0$ but no element of $A_0$ fuses to $A - A_0$.

(2) If $x \in N_G(A)$ and $x \sim A^a$, then $A_0 = C_A(x)$ or $x \in C_G(A)$

Proof Since $A_0 = A \cap Z(H)$ by $[A, \bar{A}_0] = A_0$, it follows $A_0 \leq N_G(A)$, whence $A_0$ is a TI-subgroup. Since by (2.4), $\bar{A}_0$ is transitive on $A - A_0$ this proves (1).

To prove (2) let $x \in A^a$. Then (2.4) implies $N_{\langle A, A^a \rangle}(A)$ is transitive on $A - C_A(x)$. Thus, if $A_0 \leq C_A(x)$, then $A - C_A(x) \leq A_0$, a contradiction. Hence $A_0 \leq C_A(x)$. But then either $A = C_A(x)$ or $A_0 = C_A(x)$

(4.10) Suppose $A_0 \leq Z(H)$. Let $t \in S_0$ be an involution and $B \sim A$, $B \leq S_0$. Then $t$ normalizes some $A^a \leq BB^t$.

Proof First we show $\langle A^a | A^a \leq S_0 \rangle$ is elementary. Suppose not. Then there exist $A^a$, $A^b \leq S_0$ and $x \in (A^a)^w$, $y \in (A^b)^w$ such that $o(xy) = 4$. By (2.4), $[A^a, A^b]^z = 1 = [A^a, A^b]^w$. Thus $o(xz) = 4$ for all $z \in (A^a)^w$ and $o(yz) = 4$ for all $u \in (A^a)^w$. But then $o(wu) = 4$ for all $v \in (A^a)^w$, $w \in (A^b)^w$, contradicting (4.8).

If $B = B^t$ set $A^a = B$. Thus we may assume $B \cap B^t = 1$. By (4.7), $\bar{A}_0 \leq N_G(B) \cap N_G(B^t)$. Let $B_0 \sim A_0$, $B_0 \leq B$. Then (4.10) implies $N_0 = B_0B_0^t\bar{A}_0$ is elementary.

Consider $R = \langle B, B^t, \bar{A} \rangle$. Then $N_0 \leq R$ by (2.4). Moreover $[B, \bar{A}] \neq 1 \neq [B^t, \bar{A}]$, since otherwise, for example, $\langle A, \bar{A} \rangle \leq C_G(B)$ and thus $A$ is not of root-type in $C_G(B)$. This implies $\langle B, \bar{A} \rangle | B_0 \bar{A}_0$ and $\langle B^t, \bar{A} \rangle | B_0 \bar{A}_0$ are dihedral of order 2 times odd.

Let $E = \{x \mid x \sim B - B_0 \in R\}$. Then (2.4) and (2.5) imply $E$ is a class of degenerate root involutions of $R$. Moreover $\langle C_{\bar{A}}(e) \rangle$ is elementary for $e \in B - B_0$, since otherwise there is some $A^a$ not of root-type in $C_G(B)$. Arguing exactly as under (4.2) this shows either $R$ is solvable or $R/N_0 \simeq L_2(2^a), S_5(2^a)$.

If $R$ is solvable then $bb^t \in O_d(R)$ for $b \in B \cap B_0$. Let $S \in Syl_2(G)$ containing $AO_d(R)$. Let $A^a \leq S$. Let $z \in A^a \cap C_G(bb^t)$, $z \neq 1$. Then $[z, B_0] = 1 = [z, B_0^t]$, since $B_0B_0^t \leq S$, whence $z$ normalizes $B$ and $B^t$. But since $[z, bb^t] = 1$ it follows $[z, b] = [z, b^t] \in B \cap B^t = 1$. Therefore $A^a$ centralizes $B$, since $z$ centralizes $B = B_0 \langle b \rangle$, a contradiction since then $B$ is not of root-type in $C_G(A^a)$.

This shows $R/N_0 \simeq L_2(2^a), S_5(2^a)$. Moreover $bb^t \notin N_0$. Since all involutions of $R/N_0$ are conjugate, there is an $f \in bb^tN_0 \cap E$. Hence $f \in bb^t(B_0 \cap E)$, since $C_{N_0}(f) = B_0B_0^t$ since $o(fa) = 1(2)$ for $a \in A \cap A_0$. Let $A^a \sim A$ in $R$ such that $f \in A^a$. Then $A^a \leq \langle bb^t \rangle B_0B_0^t$, since $A^a \leq \langle bb^t \rangle N_0$. Thus $A^{ab} \leq \langle bb^t \rangle B_0B_0^t$. But since $|B_0| = \frac{1}{2} |A^a|$, it follows $|bb^t| B_0B_0^t | \leq \frac{1}{2} |A^a|$, whence $A^{ab} \cap A^{ab} \neq 1$. But then $A^a = A^{ab}$, which is to show.
As a consequence of the first part of the proof of (4.10) we get

\[(4.11) \quad \langle A^g | A^g \leq S_0 \rangle \text{ is elementary abelian.}\]

(4.12) Suppose \(A_0 \leq Z(H)\). Let \(a \in A \supset A_0\) and \(x \sim A \text{ in } G\) such that \(o(ax) = 4\). Then \(x \in N_G(A)\).

Proof. Suppose false. Since \(a^g \in C_G(a)\), (4.9) implies \([A, A^g] = 1\). Let \(x \in A^g\). Since \(o(bx) = 4\) for all \(b \in A^g\) it follows \(\langle A^g, A^g \rangle\) is either elementary or of type described in (2.4). By (4.9) in any case \([A_0^g, A_0^g] = 1\). Since this holds for each \(b \in A^g\) it follows \(\langle A, A_0^g \rangle\) is a 2-group and \(x \in A_0^g\) or \(A^g = A_0^g \langle x \rangle\) and \(\langle A, A^g \rangle\) is a 2-group. In the second case \([A, A^g] = 1\) by (4.11) contradicting \(x \notin N_G(A)\).

Now let \(C < S, C \sim A \text{ in } G\) and \(S \in Syl_2(G)\) containing \(\langle A, A_0^g \rangle\). Then (4.7) applied for \(S\) instead of \(S_0\) and \(A_0^g\) implies \(A_0^g \leq N_G(A)\), a contradiction.

As immediate consequence we get

\[(4.13) \quad \text{Suppose } A_0 \leq Z(H). \text{ Then there exist no } a \in A - A_0, x \sim A \supset A_0 \text{ in } G, \text{ such that } o(ax) = 4.\]

Indeed, if \(x \in A^g\), then (4.12) implies \([a, x] \in A \cap A^g\).

(4.14). Suppose \(A_0 \leq Z(H)\). Then \(A_0 \leq O_2(G)\).

Proof. Suppose false. Choose a subgroup \(U \subseteq G\) with the following properties.

(1) \(U = \langle A, A^g \rangle\) for some \(g \in G\).
(2) \(A_0 \leq O_2(U)\).
(3) \(|U|\) is minimal with (1) and (2).

Since \(A_0 \leq O_2(G)\) a theorem of Baer [1] implies \(U\) exists! Let \(Z = Z(U) \cap O_2(U), Q = O_3(O_2(U)),\) and \(E = \{t \mid t \sim A - A_0 \text{ in } U\}\). We first show

(*) \(E/Z\) is a degenerate class of root-uvolutions of \(U/Z\).

To prove (*) let \(e, f \in E\) such that \(o(ef) = 2n, n = 1(2)\). Let \(e \in A^h, f \subset A^r, h, r \in U\) Suppose \(A^h \cap O_2(A^h, A^r) \neq 1\). Then (4.11) and (4.22) imply \(A^h \cap O_2(A^h, A^r) \leq N_G(A^r)\) and thus by (2.4), \(o(ef) = 1(2)\), a contradiction. Hence the minimality of \(|U|\) implies \(U = \langle A^h, A^r \rangle\). Let \(\langle x \rangle = Z(e, f)\) Then \(ez \sim f \text{ in } e, f\) and \(fx \sim e \text{ in } e, f\). Then (4.9) implies \(ez\) centralizes \(\langle e \rangle A_0^h = A^h\) and \(fx\) centralizes \(A^r\). Thus \(U = \langle A^h, A^r \rangle \leq C_G(z)\), whence \(z \in Z\). This shows \(o(ef) = n \mod 2\). But then (4.13) implies (*).

Now the action of a 4-subgroup of \(A\) on \(O(U)\) implies \(O(U) \leq Z(U)\). We next show

(**) \([Q, U] \leq Z\).
Let $t \in Q$ be an involution and suppose $[A, t] \neq 1$. Then (4.10) and (4.11) imply $AA^t$ is elementary and there is a $B \leq AA^t$, $B \sim A$ in $G$ such that $B = B^t$. If $B \cap [A, t] = 1$ then $A \cap C_d(A)t[A, t] \neq 1$, whence $t \in N_d(A)$ and thus $[A, t] \leq A \cap Q = 1$; since if $A \cap Q \neq 1$ then $U$ is of type described in (2.4) and thus $A_n \leq O_d(U)$ by (4.9).

This shows $B \cap [A, t] \neq 1$, whence $B \cap Q \neq 1$ since $[A, t] \leq Q$. If $B \leq O_d(U)$ then (4.11) and (4.12) imply $B \leq Z$. Therefore $[A, t] = C_{AA}(t) = B \leq Z$.

If $B \leq O_d(U)$, then arguing as above we get $\langle B, A^p \rangle$ is of type described in (2.4) since $B \cap Q \neq 1$. But then $B \cap Q \sim A^p \cap Q$ in $\langle B, A^p \rangle$ by (2.4). Thus $A^p \cap Q \neq 1$, which implies $A \cap Q \neq 1$, since arguing as above $\langle A, A^p \rangle$ is of type described in (2.4). But then $A_n \leq O_d(U)$, a contradiction.

Hence either $[A, t] = 1$ or $[A, t] = B \leq Z$. The same argument shows $[A^p, t] \leq Z$ and then (**) follows, since this holds for every involution $t \in Q$.

Now (4.11) implies $O_d(U)/Q \leq Z(U/Q)$. Hence $1 \leq Z \leq ZQ \leq O_d(U)$ is a part of a central series of $U$.

Now let $a \in A - A_n, e, f \in C_{G}(a)$ such that $o(ef) \neq 2$. Let $e \in A^h$, $f \in A^s$, $h, s \in U$. Then (4.9) implies $\langle A^h, A^s \rangle \leq C_d(A)$. But then by hypothesis $A^h$ and $A^s$ are of root-type in $C_d(A)$ and thus $A^h \cap O_d(A^s, A^h) = 1$. But then the minimality of $|U|$ implies $U = \langle A^h, A^s \rangle$, a contradiction since $A \leq Z(U)$.

This shows $\langle C_{G}(a) \rangle$ is elementary. If $U$ is solvable then (2.1) implies $U/O_d(U) = Z(U/O_d(U))$. But then $A \cap O_d(U) \neq 1$, whence $A_n \leq O_d(U)$, arguing as above, a contradiction.

Hence (3, 1) implies $U/S(U) \simeq L_2(2^n)$, $S\geq 2^n$, since $U_{3, 2^n}$ is not generated by two elementary abelian 2-subgroups.

We next show $S(U) = O_d(U)$.

Let $V$ be the coinage of $O(U/O_d(U))$, $x \in V$, and $a \in A^e$ such that $a \equiv a^x \mod O_d(U)$. Suppose $b \neq b^x \mod O_d(U)$ for some $b \in A$. Since $\langle A, a^x \rangle \leq AO_d(U)$, (4.11) and (4.12) imply $a^x \in N_d(A)$. By the same reason $a \in N_d(A)$. Thus $aa^x \in Q$ by (2.4). Now $[A, A^x] \neq 1$, since $bb^x \neq O_d(U)$. Hence (2.4) and (**) imply $1 \neq [A, aa^x] \leq A \cap Z$, a contradiction to $[A, A^x] \neq 1$.

This shows, if $a \equiv a^x \mod O_d(U)$ for some $a \in A^e$, $x \in V$, then $bb^x \in O_d(U)$ for each $b \in A$. But then the action of a 4-group on a group of odd order implies $[A, V] \leq O_d(U)$, since $|A| \geq 4$ and $A \cap O_d(U) = 1$. Thus $V/O_d(U) \leq Z(U/O_d(U))$. But then one easily shows that $S(U) = Z(U/O_d(U))$. Therefore $S(U) = O_d(U)$, since there are no perfect central extensions of $L_2(2^n)$, $S\geq 2^n$ by groups of odd order.

Now let $K = O_d(U)$. Then $K$ is generated by elements of odd order. Thus $[K, O_d(U)] = 1$, since each element of odd order in $U$ centralizes $O_d(U)$ by what we have shown above. On the other hand $U = K \cdot O_d(U)$, since $U/O_d(U)$ is generated by elements of odd order. Thus $K/K \cap O_d(U) = L_2(2^n)$ or $S\geq 2^n$ and $K$ is a perfect central extension of these groups. Now the minimality of $|U|$ implies $U = KA$.

Suppose $|A| = 4$. Then $|N| = |A_0| = 8$, a contradiction to (4.5) and (4.2). Hence $|A| \geq 8$ and $K \not\simeq SL_2(5)$, since $AO_d(U)/O_d(U) \leq U/O_d(U)$ and
\[ A \cap O_d(U) = 1. \] Suppose now \( K \cong \mathbb{S}^2(8) \). Then \( |A| = 8 \) by the same reason. Let \( S \in \text{Syl}_2(K) \) such that \( A \leq SO_d(U) \). Let
\[ A = \{ \alpha_i \beta_i \mid \alpha_i \in S, \beta_i \in O_d(U), i = 1, \ldots, 7 \}. \]

Then \( \alpha_i \neq 1, i = 1, \ldots, 7 \). Since \( \alpha_i^2 = 1 \) or \( \alpha_i^2 \in Z(K) \), the structure of \( S \) implies \( \alpha_i^2 = 1, i = 1, \ldots, 7 \). Thus \( A_0 = \{ \alpha_i | i = 1, \ldots, 7 \} \cup 1 \) is an elementary subgroup of \( S \), since \( (\alpha_i \beta_i)(\alpha_j \beta_j) = (\alpha_i \alpha_j)(\beta_i \beta_j) \). Now by the structure of \( S \) there exists an \( s \in S \) such that \( \alpha_i^s = \alpha_i z, 1 \neq z \in Z(K) \). On the other hand \( \alpha \mapsto [\alpha, s] \), \( \alpha \in A_0 \) is an homomorphism from \( A_0 \) in \( Z(K) \). Since \( |Z(K)| \leq 4 \), there exists an \( \alpha \in A_0 \) such that \( \alpha^s = \alpha \). Hence \( (\alpha_i \beta_i)^s = (\alpha_i \beta_i) \), and thus \( s \in N_G(A) \). But \( (\alpha_i \beta_i)^s = \alpha_i \beta_i z \), whence \( z \in A \), a contradiction. This proves \( K \cong \mathbb{S}^2(8) \).

So we get \( U = K \times Q, |Q| \leq |A| \) and \( Q \) elementary, since \( O_d(U) \cong AK/K \) by \( O_d(U) \cap K = 1 \). Moreover \( K \cong L_2(2^e) \) or \( \mathbb{S}^2(2^e) \) and we may without loss assume that \( SQ \leq S_0 \).

We next show \( K \cap E = \emptyset \). If \( A \cap E \subseteq K \), then \( Q = 1 \) and all elements of \( A \) fuse in \( K \), a contradiction to (4.9). Let by way of contradiction \( a \in A \cap E \cap K \).

Since \( A \cap E \subseteq K \) there exists a \( b \in A \cap E, b \in \Omega_1(S), 1 \neq z \in Q \). Since \( b \sim ab \) in \( K \), it follows \( abz \in E \cap A \). Thus \( \{a, bz, abz\} \subseteq E \cap A \), a contradiction to \( E \cap A = A - A_0 (a \neq b, \) since otherwise \( z = a(ax) \in A \cap Q = 1 \). This shows \( K \cap E = \emptyset \).

Let \( L = \langle A_0^h \mid A^h \leq SQ \rangle \). Then \( N_G(S) \) normalizes \( L \). If \( L \cap K = 1 \) then \( L \leq Q \) and \( N_G(S) \) centralizes \( L \), whence normalizes \( A \). But then the transitivity of \( N_G(S) \) on \( \Omega_1(S)^* \) implies \( \Omega_1(S) \leq A \) But then \( \Omega_1(S) = A \), since \( A \cap Q = 1 \), a contradiction to \( K \cap E = \emptyset \).

Hence \( K \cap L \neq 1 \) and \( \Omega_1(S) \leq L \). Now by (4.7), \( \tilde{A}_0 \) normalizes each \( A^h \leq SQ \) and thus centralizes \( A_0^h \) by (4.9). Hence \( [\tilde{A}_0, L] = 1 \). Let \( ax \in A \cap E, a \in \Omega_1(S)^*, z \in Q^e, \) and \( f \in \tilde{A}_0^e \).

Then \( ax \neq ax' = az' = bz, b \in \Omega_1(S)^*, z \in Q^e \). If \( b = a \) then \( az' = (ax)z = (ax)z \in A \cap Q = 1 \) and thus \( az = bz = (ax')z, \) a contradiction. But we have \( z' = abz = bz \sim ab \) in \( K \). Thus \( z' = abz \in E \). This shows \( x \sim A \sim A_0 \) in \( G \). Let \( z \in A^e \). Now \( \Omega_1(S)Q = L \) since \( U = KA \). Hence \( \tilde{A}_0 \) normalizes \( \Omega_1(S)Q \). But then (4.9) implies \( \langle A_0^e \rangle \) is elementary, since \( z \in \Omega_1(S)Q \). Hence (4.12) implies \( \tilde{A}_0 \leq N_G(A^e) \). This shows
\[ x \in A \cap Q = 1, \] since otherwise \( z \in A^e = A \), contradicting \( A \cap Q = 1 \). But then \( x = x' = (ax)z'y \in A \cap E, \) a contradiction to \( A_0 = C_A(f) \). This proves (4.14).

(4.15). Suppose \( A_0 \leq Z(H) \). Then case(3) of Theorem 1 is satisfied

Proof. (4.11) and (4.12) imply \( A_0 \) normalizes each \( A^g, g \in G \). Hence \( \langle A_0^g \mid g \in G \rangle \) is elementary abelian. Thus (2.4) and (2.5) imply \( E = \)
\{t \mid t \sim A - A_0 \text{ in } G \} \text{ is a degenerate class of root-involutions of } G. \text{ Let } N_0 = \langle A_0^g \mid g \in G \rangle. \text{ Then } Q \leq N_0 \text{ and by (4.6) there exist } a, b \in E \text{ such that } abN_0 \in EN_0. \text{ Hence the action of a 4-group on a group of odd order implies } O(G/N_0) \leq Z(G/N_0), \text{ since } EN_0/N_0 \text{ is a class of root involutions.}

Now by hypothesis \( O_2(G) \leq N_0(A^0) \text{ for some } g \in G. \text{ Thus } [O_2(G), A^q] \leq A_0^q, \text{ whence } O_2(G)/N_0 \leq Z(G/N_0) \text{ since } G = \langle A^0 \rangle.

Let \( A^h, A^g \in C_G(A). \text{ Since } A^h \cap N_0 \neq 1 \neq A^g \cap N_0, \text{ (2.4) implies either } [A^h, A^g] = 1 \text{ or } A^h \text{ is not of root-type in } C_G(A), \text{ a contradiction. This shows } \langle C_G(e) \rangle \text{ is elementary for } e \in E. \text{ Hence } [3, 1] \text{ and the above imply } G/N_0 \text{ is a perfect central extension of } L_2(2^n), Sz(2^n) \text{ or } U_3(2^n), \text{ which is case (3) of Theorem 1.}

5. THE CASE \( q = 2 \text{ BUT } A_0 \text{ NOT NORMAL IN } N_0(A)

We use the notation of Sections 3 and 4. Let \( G \) be as in Section 3 and assume \( q = 2. \text{ Assume moreover } A_0 \leq Z(H). \text{ If } A_0 \leq N_0(A), \text{ then each } A_0^q \leq H \text{ normalizes and thus by (2.4) centralizes } A_0. \text{ Thus } A_0 \leq Z(H). \text{ So we may assume that } A_0 \text{ is not normal in } N_0(A). \)

\( \langle A_0^q \rangle \leq S \text{ containing } A_0. \text{ Then } \langle A_0^q \rangle \text{ is elementary.} \)

\textbf{Proof.} Suppose false. \( \langle A_0^q \rangle \leq S \text{ there exist } a \in A_0^h, b \in A_0, h \in S \text{ such that } o(ab) = 4. \) Since \( \langle A_0, A_0^h \rangle \leq C_G(A_0), \text{ the 3-subgroup lemma implies } (ab)^2 \in C = C_G(A). \text{ Therefore } (ab)^2 \notin A \text{ and } (ab)^2 \notin A^h. \text{ Let } X = \langle A, A^a \rangle. \text{ Since } o(bb^a) = 2, \text{ } X \text{ is of type described in (2.4). Moreover } \langle A_0, A_0^a \rangle \leq S \text{ implies } \langle A_0, A_0^a \rangle \leq O_2(X) \text{ and is therefore elementary abelian.} \text{ Hence } o(ab) = 4 \text{ for all } e \in A_0^q. \text{ Similarly } o(bf) = 4 \text{ for all } f \in (A_0^h)^q \text{ and thus } o(ef) = 4 \text{ for all } e \in A_0^q, f \in (A_0^h)^q, \text{ since } [e, f] \in C \text{ by the 3-subgroup lemma.} \text{ Let } M = A_0A_0^a. \text{ Then by (2.4), } X/M \cong D_{2m}, m = 1(2), \text{ since } X \text{ is abelian.}

\text{ Then (4.7) applied to a 2-Sylow-subgroup } B \text{ of } G \text{ containing } \langle A_0, A_0^a \rangle \text{ implies } \overline{A} = A_0 \text{ since a centralizes } N_0(A). \text{ For } C \prec S, C \sim A \text{ in } G. \text{ Hence (2.4) implies there is an } \overline{A}, g \in G \text{ such that } a \in N_G(A_0). \text{ Since } C_2(a) = [A_0, a] \text{ and } \overline{A}_0 \cap C_2(a) \neq 1 \text{ it follows } 1 \neq [A_0, a] \cap \overline{A}_0 \leq C. \text{ Hence } \overline{A}_0 \text{ centralizes } A. \text{ Moreover } [A_0, a] = C_2(a) = \overline{A}_0^a, \text{ since no element of } \overline{A}_0^a \text{ centralizes } A. \text{ This shows } \overline{A}_0 \text{ is a conjugate of } A \text{ in } X \text{ normalized by } a.

\text{ Let } x \in \overline{A} - A_0. \text{ Then } o(xa) = 2n, n = 1(2) \text{ since by (2.4), } o(xa^a) = n = 1(2) \text{ and } a \in X. \text{ Thus there is } c \sim x \text{ in } \langle x, a \rangle \text{ such that } o(ca) = 2. \text{ Hence by the above } c \in \overline{A}. \text{ But then } a \text{ centralizes } \overline{A}_0^a/c = \overline{A} \text{ and thus } \langle A, \overline{A}_0 \rangle \leq C_2(A), \text{ a contradiction since then } A \text{ is not of root-type in } C_G(A^0).

(5.2). \text{ } M \text{ is the direct sum of irreducible } F_4(X/M)\text{-modules.}

\textbf{Proof.} By } q = 2, X/M \cong D_{2m}, m = 1(2) \text{ Let } M_0 \text{ be a minimal } X \text{ invariant
submodule of $M$ and $C_0$ be a complement of $C_{M_0}(a)$ in $C_M(a)$ for $a \in A - A_0$. Then $[C_0, b] C_0$ is a $b$-invariant complement to $M_0$ in $M$, where $b \in \overline{A} - \overline{A}_0$. Hence Gaschütz’s theorem implies there is an $X$-invariant complement to $M_0$ in $M$.

(5.3) $X/M \simeq \Sigma_3$.

Proof. Suppose false. Since $A_0$ is not normal in $N_0(A)$ there exists a $g \in N_0(A)$ such that $[\overline{A}_0^g, A_0] \neq 1$. Now $[\overline{A}_0^g, A] = A_0^g$ and $|A_0 : A_0 \cap A_0^g| = 2$. Let $t \in A - A_0^g$ and $\chi_t : \overline{A}_0^g \to A_0^g$ defined by $x \to [x, t]$ for $x \in A_0^g$. By [11, (2.3)], $\chi_t$ is an isomorphism, whence $F = \chi_t^{-1}(A_0 \cap A_0^g)$ is a hyperplane in $A_0^g$ satisfying $[F, A] \leq A_0 \cap A_0^g$. Hence $F \leq N_F(A_0)$. Moreover $F = 1$ implies $|A| = 4$ and thus $X \simeq \Sigma_4$ by (2.4). Thus $F \neq 1$.

Let $S \in \text{Syl}_2(N_F(A_0))$ containing $F$. Since $S \cap C_F(A_0)$ is a 2-Sylow-subgroup of $C_F(A_0)$, $S$ contains some conjugate of $A_0$. Thus, by changing notation if necessary, we may assume $A_0 \leq S$.

Let $a \in F^s$. Then $\langle A_0, A_0^s \rangle$ is elementary by (5.1) Suppose $\langle A, A_0^s \rangle$ is elementary. Then as under (5.1), (4.7) implies $a \in N_F(\hat{A})$. Let $t \in A_0 - A_0^g$ and $\chi_t : \overline{A}_0^g \to A_0^g$ defined by $x \to [x, t]$ for $x \in A_0^g$. By [11, (2.3)], $\chi_t$ is an isomorphism, whence $F = \chi_t^{-1}(A_0 \cap A_0^g)$ is a hyperplane in $A_0^g$ satisfying $[F, A] \leq A_0 \cap A_0^g$. Hence $F \leq N_F(A_0)$. Moreover $F = 1$ implies $|A| = 4$ and thus $X/M \simeq \Sigma_3$ by (2.4)(2), which is to prove.

Let $X_1 = \langle \overline{A}, \overline{A}_0^g \rangle$, $M_1 = \overline{A}_0 \overline{A}_0^g$. Then by (2.4) and the above $X_1/M_1 \simeq D_{2m}$, $m \equiv 1(2)$. If $a \in X_1$ then $\overline{a} \in \overline{A}_0^g$, $\overline{a}_0 \in X_1$ and thus $\overline{A}_0^g = \langle a \rangle$ $\overline{A}_0^a \leq X_1$. Thus $[A, \overline{A}_0^g] = 1$ by (2.3), a contradiction to $[A, a] \neq 1$. Thus $a \notin X_1$ and if $x \in A_0 - A_0^g$ then $o(xa) = 2n$, $n \equiv 1(2)$, since $o(xa) = n$. Hence there exists a $c \sim x$ in $\langle x, a \rangle$ such that $1 \neq ca \in Z(x, a)$. Let $c \in \overline{A}^r$, $r \in X_1$. We will show $\overline{A}^r \leq C_G(\hat{A})$.

Let $F_1 = \{u \in \overline{A}_0^g | [u, A] \leq A_0 \cap A_0^g \}$. Then, arguing as above, $F_1$ is a hyperplane in $\overline{A}_0^g$ and $F_1 \neq 0$. Now $|C_{M_1}(a)| = |M_1 : \overline{A}_0^g| = |\overline{A}_0^g|$, since $\overline{A}_0^g \cap C_{M_1}(a) = 1$. Hence $[F_1, a]$ is a hyperplane in $C_{M_1}(a) = [\overline{A}_0^g, a]$, since $|[F_1, a]| = |F_1|$. Suppose first $[A_0] > 4$. Then $|\overline{A}_0^g| > 4$ and thus $|C_{M_1}(a)| > 4$, since $a \in N_F(\hat{A})$. Hence $C_{M_1}(a) \cap [F_1, a] = 1$, since both groups are contained in $C_{M_1}(a)$. But the 3-subgroup lemma implies $[F_1, F_1] \leq C_G(\hat{A})$. Hence $\overline{A}^r$ contains a nonidentity element centralizing $A$ and thus $[A, \overline{A}_0^g] = 1$.

Suppose now $|A_0| = 4$. Then $|M| = 16$ and thus $X/M \simeq D_{10}$. Hence $A_0^X$ is a partition of $M$ and thus all elements of $M^s$ fuse to $A_0^s$. Now $M \leq C_G(\overline{A}_0^s)$, since $[\overline{A}_0^s, M] = 1$ and thus $[M, M_1] = 1$. Hence $M : C_M(\overline{A}^s) = 4$, since $|\overline{A}^s| = 2$. By (2.3) this implies $C_M(\overline{A}^s) = A_0^s$ for some $s \in X$. Suppose $A_0^s \neq A_0$. Then $M = A_0 A_0^s$. Since $A_0^s A_0^{s^a} \leq MMM = MM$ and is therefore elementary abelian, it follows $[A^s, A_0^{s^a}] = 1$, since $\langle A^s, A_0^{s^a} \rangle \leq C_G(\overline{A}^r)$ and so $A^s$ is of root-type in $C_G(\overline{A}^r)$. Hence $\langle A^s, \overline{A}^r \rangle \leq C_3$.
contained in $N_G(\bar{A}^b)$. Thus by (4.7) applied on a 2-Sylow-subgroup $S$ of $N_G(\bar{A}^b)$ containing $\langle A^b, \bar{A}^b, a \rangle$ it follows $a \in N_G(A^b)$. Hence $a \in N_G(X)$ and thus $A^a \leq X$. But then $X = X_1$, whence $A = A^r$ since $A^r \leq C_X(A)$. Moreover $A$ is the only conjugate of $A$ in $X$ normalized by $a$. Thus $[M, a] = C_{M}(a) = A_0$, a contradiction since $C_{A}(a) = A^a \neq A_0$. 

This shows $A^a = A_0$ and $A$ centralizes $\bar{A}^r$ in any case. Suppose a centralizes $\bar{A}^r$. Then $A^r = \bar{A}^r \langle c \rangle \leq C_G(a)$ and thus $\langle A^r, \bar{A}^r \rangle \leq C_G(\bar{A}^r)$ and $A$ is not of root type in $C_G(\bar{A}^r)$, a contradiction. Since $[a, F_1] \leq C_{M}(A) = A_0$, this shows $|\bar{A}^r : C_{A_0}(ac)| = 2$. Since $[X_1, ac] \leq M_1$ this implies $|M_1 : C_{M_1}(ac)| = 2$, whence $|\bar{A}^r : C_{A_0}(ac)| = 2$. Moreover $[A, ac] = [A, a] \leq A_0 \cap A^a$ and $[A, ac] \leq A \cap M_1 - A_0$. This proves $[X, ac] \leq M$ and thus $|M : C_M(ac)| = 2$, since $|\bar{A}^r : C_{A_0}(ac)| = 2$. Hence (5.2) implies $M = C_M(ac) \oplus N$, where $N$ is $X$-invariant and $|N| = 4$. Since $N$ is (by (2.4)) not centralized by an element of odd order in $X$, this implies $X/M \simeq \Sigma_3$.

(5.4) $G$ satisfies (1) or (2) of Theorem 1.

Proof. (5.3) shows, that if ever $|A^p : N_{A_0}(A^b)| = 2$, then $\langle A^p, A^b \rangle \leq O_2 \langle A^p, A^b \rangle \simeq \Sigma_3$.

Consider $Y = \langle A, \bar{A}, \bar{A}^b \rangle$, $b \in B^a$. By (4.5) we may choose $b$ such that $Y$ is not solvable, since otherwise $G$ satisfies (1) or (2) of Theorem 1. Let $\alpha \in A - A_0$, $\beta \in \bar{A} - \bar{A}_0$, and $\gamma \in \bar{A}^b - \bar{A}_0^b$. Since $N = A_0 \bar{A}_0 \bar{A}_0^b \triangleleft Y$ we get (by (2.4) and (5.3)) the relations

$$\alpha \circlearrowleft \beta \gamma.$$

Hence [4] implies $\langle \alpha, \beta, \gamma \rangle$ is solvable. Hence $Y$ is solvable, since $Y = N\langle \alpha, \beta, \gamma \rangle$, a contradiction to the choice of $b$.

6. Proof of Theorem 1 and Corollary 2

Proof of Theorem 1. Let $G$ be a group of minimal order containing an elementary abelian $T$-subgroup $A$, which satisfies conditions (i) and (ii) of Theorem 1, but does not satisfy the conclusion of Theorem 1. Then $G = \langle A^c \rangle$ by the minimality of $G$, since all $A^g, g \in G$ satisfy (i) and (ii) in $\langle A^c \rangle$. Hence $G$ satisfies the hypothesis of Section 3 and with the notation introduced in Section 3 we have either $q = 2$ or $q > 2$. By (3.6), $q = 2$. Hence if $A_0 \leq Z(H)$, then we get a contradiction to (4.15). Thus the hypothesis of Section 5 is satisfied. But then (5.4) implies $G$ satisfies Theorem 1.
Proof of Corollary 2. Let G be a group which satisfies the hypothesis of Corollary 2. Since $(WAC)' \not\leq A$ it follows $WAC = \langle A^g | A^g \not\leq C_0(A) \rangle$ is a 2-group. Hence (2.4) implies each $A^g \leq WAC$ is of root type in $WAC$. Thus $G$ satisfies the hypothesis of Theorem 1. If case(2) of Theorem 1 holds, then $WAC = A$ and we get (by [11]) case(1) of Corollary 2. If case(3) of Theorem 1 holds we get case(3) of Corollary (2). Thus we may assume $A$ is of root-type in $G$. But then $WAC$ is the strong closure of $A$ in $N_G(A)$. If $(WAC)' \neq 1$ then there are $A^g, A^h \leq WAC$ such that $1 \neq [A^g, A^h] \leq A$. But no element $t \in (A^h)^\circ$ centralizes an element of $(A^g)^\circ$, whence $|[A^g, A^h]| \geq |A|$. Thus implies $(WAC)' = A$. Let $D = \{ t | t \sim A^\# \text{ in } G \}$. If $t \in A^\#$ and $t \in (tG \cap C_0(t)) \cap S_i \neq 1$ it follows $A$ normalizes each $S_i$, $t = 1, \ldots, r$. Hence $[A, S_i] = 1$ since $C_0(A) \leq N_G(A)$. This implies $[A, Ag] < H$ for each $g \in G$. But then $G^* = \langle A^g \rangle = H \cdot Q$, which is a special case of (2)(β). But then again the structure of the automorphism group of $X_1$ implies $A$ induces inner automorphisms on $X_1$. Hence $A \leq HC_0(H)$ in this case.

Now obviously $A \cap C_0(H) = 1$, whence $[A, C_0(H)] = 1$ since $C_0(H) \leq N_G(A)$. This implies $[A, A^g] \leq H$ for each $g \in G$. But then $G^* = \langle A^g \rangle = H \cdot Q$, which is a special case of (2)(β). But then again the structure of the automorphism group of $X_1$ implies $A$ induces inner automorphisms on $X_1$. Hence $A \leq HC_0(H)$.
\( Q \leq C_G(H) \) and \( Q/Q \cap H \) is elementary abelian if we put \( Q = G^* \cap C_G(H) \). Since \( Q \cap H \leq Z(H) \) this is exactly the assertion of Corollary 2, (2)(β).

REFERENCES

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