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J. Differential Equations 228 (2006) 357–376

Journal of Differential Equations

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Nonmonotone travelling waves in a single species reaction–diffusion equation with delay

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Received 4 August 2005; revised 17 February 2006

Available online 13 June 2006

Abstract

We prove the existence of a continuous family of positive and generally nonmonotone travelling fronts for delayed reaction–diffusion equations $u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x))$ $(*)$, when $g \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ has exactly two fixed points: $x_1 = 0$ and $x_2 = K > 0$. Recently, nonmonotonic waves were observed in numerical simulations by various authors. Here, for a wide range of parameters, we explain why such waves appear naturally as the delay h increases. For the case of g with negative Schwarzian, our conditions are rather optimal; we observe that the well known Mackey–Glass-type equations with diffusion fall within this subclass of $(*)$. As an example, we consider the diffusive Nicholson's blowflies equation.

MSC: 35K57; 35R10; 92D25

Keywords: Time-delayed reaction–diffusion equation; Heteroclinic solutions; Nonmonotone positive travelling fronts; Single species population models

1. Introduction

In this paper, we study the existence of positive nonmonotone travelling waves for a family of delayed reaction–diffusion equations which includes, as a particular case, the diffusive Nichol-

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doi:10.1016/j.jde.2006.05.006

son's blowflies equation

$$N_t(t,x) = d\Delta N(t,x) - \delta N(t,x) + pN(t-h,x)e^{-bN(t-h,x)}, \tag{1}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^m$. This problem was suggested in [9,10,12,20,29], where numerical simulations indicated a loss of monotonicity of the wave profile caused by the delay. Equation (1) was introduced in [30] and it generalizes the famous Nicholson's blowflies equation

$$N'(t) = -\delta N(t) + pN(t-h)e^{-bN(t-h)}, \tag{2}$$

intensively studied for the last decade (e.g., see our list of references). After a linear rescaling of both variables N and t, we can assume that $\delta = b = 1$. Equation (1) takes into account spatial distribution of the species, and the mentioned problems reflect the interest in understanding the spatial spread of the growing population [13]. Relevant biological discussion can be found in [1,9,11,12,20,29], where various modifications of (1) were proposed and studied. Here, however, we will concentrate mainly on the mathematical aspects of the dynamics in (1). For the sake of simplicity, we will consider the case of a single discrete delay, but extensions for more general functionals (which additionally can take into account non local space effects) are possible (cf. [6, 21,22]). Since the biological interpretation of N is the size of an adult population, we will consider *only* nonnegative solutions for (1) and for other population models. Actually, our approach allows us to study a more general family of scalar reaction–diffusion equations

$$u_t(t,x) = d\Delta u(t,x) - u(t,x) + g\big(u(t-h,x)\big), \quad u(t,x) \geqslant 0,\ x \in \mathbb{R}^m, \tag{3}$$

related to the Mackey–Glass-type delay differential equations,

$$u'(t) = -u(t) + g\big(u(t-h)\big), \quad u \geqslant 0, \tag{4}$$

with exactly two nonnegative equilibria $u_1(t) \equiv 0$, $u_2(t) \equiv K > 0$ (so that $g(K) = K$, $g(0) = 0$). In particular, with $g(u) = pu/(1+u^n)$ in (4), we obtain the equation proposed in 1977 by Mackey and Glass, to model hematopoiesis (blood cell production). The nonlinearity g is called *the birth function* and thus it is nonnegative, and generally nonmonotone and bounded. Due to these properties of g and the simple form of dependence on the delay in (3), the Cauchy problem

$$u(s,x) = \zeta(s,x), \quad s \in [-h,0],\ x \in \mathbb{R}^m, \tag{5}$$

for Eq. (4) has a unique eventually positive global solution for every $\zeta \neq 0$ taken from an appropriately chosen functional space (e.g., see [27]).

Recently, the existence of travelling fronts connecting the trivial and positive steady states in (1) was studied in [6,31] (see also [9,23] for other methods which eventually can be applied to analyze this problem). In [31], the authors use a monotone iteration procedure coupled with the method of upper and lower solutions. This approach (proposed in [32]) works well if $1 < p/\delta \leqslant e$, since in this case the function g is increasing on $[0, 1/b] \supset [0, K]$, thus $\phi \mapsto -\delta\phi(0) + pg(\phi(-h))$ satisfies the quasimonotonicity condition in [32]. This allows one to establish the existence of *monotone* wave front solutions $N(t,x) = \phi(ct + \nu \cdot x, c)$ for every $p/\delta \in (1,e]$ and $c > 2\sqrt{p-\delta}$ (cf. [25,31]). Moreover, as it was proved in [25], every solution of (1), (5) with $p/\delta \in (1,e]$ converges exponentially to some travelling wave provided that ζ is sufficiently close (in a weighted L^2 norm) to this wave at the very beginning of the propagation.

For the case $p/\delta > e$, clearly g is not monotone on $[0, K]$, and Wu and Zou's method [32] is no longer applicable. In [6], the Lyapunov–Schmidt reduction was used to study systems of delayed reaction–diffusion equations with nonlocal response. We observe that Eqs. (1) and (3) fit into the framework developed in [6]. This approach requires a detailed analysis of an associated Fredholm operator and the existence of heteroclinic solutions of (4) (in [6], the latter was established with the use of the monotone semiflows approach developed by Smith and Thieme [26]). As a result, it was proved in [6] that, even when $p/\delta > e$, (1) possesses a family of travelling waves if $\delta h \in (0, r^*)$ for some $r^* < 1$ (which is given explicitly). The rather restrictive condition $\delta h < r^* < 1$ from [6] was considerably weakened in [8] by invoking a Schauder's fixed-point argument to find heteroclinic solutions of (4). Unfortunately, the main results of [6,8] do not answer the question about the existence (and shape) of *positive* travelling fronts of (1) or (3). We recall here that only nonnegative solutions to (3) are biologically meaningful.

In this paper, inspired by [6,31,32], for a broad family of nonlinearities g (which includes Eq. (1) with $\delta = 1$), we prove that Eq. (3) has a continuous family of positive travelling wave fronts $u(t, x) = \phi(ct + \nu \cdot x, c)$, indexed by the speed parameter $c > 0$, provided that

$$e^{-h} > -\Gamma \ln \frac{\Gamma^2 - \Gamma}{\Gamma^2 + 1}, \quad \Gamma \stackrel{\text{def}}{=} g'(K), \tag{6}$$

and c is sufficiently large: $c > c_*(h, g'(0), g'(K))$. Furthermore, we show that these fronts generally are not monotone: in fact, they can oscillate infinitely about the positive steady state. On the other hand, for large negative values of s, the wave profile $\phi(s, c)$ is asymptotically equivalent to an increasing exponential function. Condition (6) assures the global attractivity of the positive equilibrium of (4), which is required by our approach. It should be noted that this condition is rather satisfactory in the sense that (6) determines a domain of parameters approximating very well the maximal region of local stability for the positive steady state in (4) or (2) (cf. [22]).

Before announcing the main results of the present work, we state our basic hypothesis.

(H) Equation (4) has exactly two steady states $u_1(t) \equiv 0$ and $u_2(t) \equiv K > 0$, the second equilibrium being exponentially asymptotically stable and the first one being hyperbolic. Furthermore, $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and is C^2-smooth in some vicinity of the equilibria, with $p := g'(0) > 1$. The latter implies that the solution $u_1 = 0$ of (4) is unstable for all $h \geqslant 0$.

In the sequel, $\lambda_1(c)$ denotes the minimal positive root of the characteristic equation $(z/c)^2 - z - 1 + p\exp(-zh) = 0$ for sufficiently large c, and λ the unique positive root of the equation $-z - 1 + p\exp(-zh) = 0$, where $p > 1$. As shown later, $\lim_{c\to\infty} \lambda_1(c) = \lambda$. Now we are ready to state our main result.

Theorem 1. *Assume* (H). *If the positive equilibrium K of Eq.* (4) *is globally attracting, then there is $c_* > 0$ such that, for each $\nu \in \mathbb{R}^m$, $\|\nu\| = 1$, Eq.* (3) *has a continuous family of positive travelling waves $u(t, x) = \phi(ct + \nu \cdot x, c)$, $c > c_*$. Furthermore, for some $s_0 = s_0(c) \in \mathbb{R}$, we have $\phi(s - s_0, c) = \exp(\lambda_1(c)s) + O(\exp(2\lambda s))$ as $s \to -\infty$, so that $\phi'(s - s_0, c) = \lambda_1(c)\exp(\lambda_1(c)s) + O(\exp(2\lambda s)) > 0$ on some semi-axis* $(-\infty, z]$. *Finally, if $g'(K)he^{h+1} < -1$ then the travelling profile $\phi(t)$ oscillates about K on every interval* $[z, +\infty)$.

In order to apply Theorem 1, one needs to find sufficient conditions to ensure the global attractivity of the positive equilibrium of (4). Some results in this direction were found in [21,22] for a family of nonlinearities having negative Schwarz derivative (or, more generally, satisfying a generalized Yorke condition [7,21,22]). In particular, [21, Corollary 2.3] implies the following useful version of Theorem 1.

Corollary 2. *Assume* (H) *and* (6). *In addition, suppose that* $g \in C^3(\mathbb{R}_+, \mathbb{R}_+)$ *has only one critical point* x_M *(maximum) and that the Schwarz derivative* $(Sg)(x) = g'''(x)(g'(x))^{-1} - (3/2)(g''(x)(g'(x))^{-1})^2$ *is negative for all* $x > 0$, $x \neq x_M$. *Then all conclusions of Theorem* 1 *hold true.*

Notice that Corollary 2 applies to both the Nicholson's blowflies equation and the Mackey–Glass equation with nonmonotone nonlinearity, see [21].

To prove our main results, we need a detailed analysis of heteroclinic solutions of (4). This study is presented in Section 2, and is crucial for the selection of an appropriate functional space where a Lyapunov–Schmidt reduction is realized. The existence of positive travelling waves is proven in the third section. The main result of Section 3 is given in Theorem 14, which is essentially Theorem 1 without its nonmonotonicity statement. Finally, in the last short section, we show that these waves have nonmonotonic profiles when the delay is over some critical value.

2. Heteroclinic solutions of scalar delay differential equations

In this section, we study the existence and properties of heteroclinic solutions to the scalar functional equation

$$x'(t) = -x(t) + f(x_t), \quad x \geqslant 0, \tag{7}$$

where $f : C([-h, 0], \mathbb{R}_+) \to \mathbb{R}_+$ is a continuous functional which takes closed bounded sets into bounded subsets of $\mathbb{R}_+$. Here $C([-h, 0], \mathbb{R}_+)$ is the metric space equipped with the norm $|\phi| = \max_{s \in [-h,0]} |\phi(s)|$. Throughout this paper, we suppose that Eq. (7) has exactly two steady states $x_1(t) \equiv 0$ and $x_2(t) \equiv K$, the second equilibrium being asymptotically stable and globally attractive. Thus, if (7) has a heteroclinic solution $\psi(t)$, it must satisfy $\psi(-\infty) = 0$, $\psi(+\infty) = K$.

We start by proving a general existence result which is valid for the abstract setting of dynamical systems. Let $S^t : X \to X$ be a continuous semidynamical system defined in a complete metric space (X, d). First, we mention the following fact (see, e.g., [16, p. 36]):

Lemma 3. *Suppose that* $\varphi : \mathbb{R} \to X$, $\varphi(0) = x$ *is a complete orbit of* S^t. *If the closure* $\overline{\{\varphi(s),\ s \leqslant p\}}$ *is compact for some* $p \in \mathbb{R}$, *then the* α*-limit set* $\alpha(\varphi) = \bigcap_{q \leqslant 0} \overline{\{\varphi(s),\ s \leqslant q\}}$ *of* φ *is nonempty, compact and invariant* (*this means that for every* $z \in \alpha(\varphi)$ *there exists at least one full trajectory* ψ *with* $\psi(\mathbb{R}) \subseteq \alpha(\varphi)$, $\psi(0) = z$).

For every $A \subset X$ and $h > 0$, let $A(h) \subset A$ denote the set of right endpoints of all orbit segments $S^{[0,h]}z = \{S^u z:\ u \in [0, h]\}$ which are completely contained in A:

$$A(h) = \left\{x \in A:\ x = S^h z \text{ and } S^{[0,h]}z \subset A,\ \text{for some } z \in A\right\}.$$

Next statement shows clearly how to relate the global attractivity property of the positive equilibrium of (4) to the problem concerning the existence of travelling fronts for (3).

Lemma 4. *Assume that $A(h)$ is either empty or precompact, for all bounded sets A and some $h > 0$. Suppose that there exist two disjoint compact invariant subsets K_1, K_2 of X such that $d(S^t x, K_2) \to 0$ as $t \to +\infty$ for every $x \in X \setminus K_1$. If the set $F_\varepsilon = \{x\colon d(x, K_1) = \varepsilon\}$ is not empty for every sufficiently small $\varepsilon > 0$, then there exists at least one complete orbit ψ with $\alpha(\psi) \subset K_1$ and $\omega(\psi) \subset K_2$.*

Proof. Let $\rho = d(K_1, K_2)$ and, for every $n > 2/\rho$, take some $x_n \in F_{1/n}$. Due to the compactness of K_1, we can assume that $x_n \to z$ for some $z \in K_1$. In consequence, if $t_n > 0$ is the minimal real number such that $d(S^{t_n} x_n, K_1) = \rho/2$, then $\lim t_n = +\infty$. Set $w_n = S^{t_n} x_n$. Due to the compactness condition imposed on S^t, we can suppose that $\lim w_n = w$. Let now $\psi_n(u) = S^{u+t_n} x_n$, $u \geqslant -t_n$. We have $S^a \psi_n(t) = \psi_n(a+t)$ for every $a \geqslant 0$, $t \geqslant -t_n$. Since, for every integer $m > 0$ the sequence $\psi_n(-m)$ has a convergent subsequence (say, $\psi_{n_j}(-m) \to b$), we can assume that $\psi_n(t)$ converges uniformly on $[-m, 0]$ to $\psi(t) = S^{t+m} b$. Moreover, we have that $\psi(0) = w$ and $S^a \psi(t) = \psi(a+t)$ for all $a \geqslant 0$, $t \geqslant -m$. In this way, taking $m = 1, 2, 3, \ldots,$ we can use $\psi_n(u)$ to construct a continuous function $\psi : \mathbb{R} \to X$, such that $S^a \psi(t) = \psi(a+t)$ for every $a \geqslant 0$, $t \in \mathbb{R}$. Such ψ defines the complete orbit we are looking for. Since $\psi(\mathbb{R}_-)$ is a subset of the bounded set $B = \{z\colon d(z, K_1) \leqslant \rho/2\}$, we conclude that $\psi(\mathbb{R}_-)$ is precompact. Furthermore, because of $d(\psi(\mathbb{R}_-), K_2) \geqslant \rho/2$, we have $d(\alpha(\psi), K_2) \geqslant \rho/2 > 0$. This means that $\alpha(\psi) \subset K_1$ so that $\lim_{t\to-\infty} d(\psi(t), K_1) = 0$. □

A direct application of Lemma 4 to Eq. (7) gives the following theorem.

Theorem 5. *Let $f : C([-h, 0], \mathbb{R}_+) \to \mathbb{R}_+$ be a continuous functional which takes closed bounded sets into bounded subsets of $\mathbb{R}_+$. Assume further that every nonnegative solution of* (7) *admits a unique extension on the right semi-axis. If $f(0) = 0$, $f(K) = K$ $(K > 0)$ and $x_2(t) \equiv K$ attracts every solution of* (7) *with nonnegative and nontrivial initial function, then there exists a positive complete solution $\psi(t)$ to* (7) *such that $\psi(-\infty) = 0$, $\psi(+\infty) = K$.*

With some additional conditions on f, we can say more about such an orbit ψ.

Lemma 6. *Assume $f(\phi) = g(\phi(-h))$ for some $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ (so that $g(0) = 0$, $g(K) = K$). Assume that $\liminf_{x\to 0+} g(x)/x > 1$, and let p_1, p_2 be such that $1 < p_1 < \liminf_{x\to 0+} g(x)/x \leqslant \limsup_{x\to 0+} g(x)/x < p_2$. Let λ_i be the unique positive real root of the equation $z = -1 + p_i \exp(-zh)$ $(i = 1, 2)$, so we have $0 < \lambda_1 < \lambda_2$. Then for every heteroclinic solution $\psi(t)$ of the equation*

$$x'(t) = -x(t) + g\big(x(t-h)\big) \tag{8}$$

there exist $\tau = \tau(\psi) < 0$, $C_i = C_i(\psi) > 0$ such that

$$C_1 \exp(\lambda_2 t) \leqslant \psi(t) \leqslant C_2 \exp(\lambda_1 t), \quad t \leqslant \tau.$$

Proof. Choose $\delta > 0$ sufficiently small such that $p_1 x \leqslant g(x) \leqslant p_2 x$ for all $x \in [0, \delta)$, and let τ be such that $\psi(t) < \delta$ for all $t \leqslant \tau$. We claim that, for every $s \leqslant \tau$,

$$\psi(t_m) = \min_{u\in[-h,0]} \psi(s+u) \geqslant \frac{\exp(-h)}{p_2} \max_{u\in[-h,0]} \psi(s+u) = \frac{\exp(-h)}{p_2} \psi(t_M). \tag{9}$$

Indeed, if $t_m \geqslant t_M$, then, by the variation of constants formula,

$$\psi(t_m) = \psi(t_M)\exp(t_M - t_m) + \int_{t_M}^{t_m} \exp\bigl(-(t_m - u)\bigr) g\bigl(\psi(u-h)\bigr)\,du \geqslant \psi(t_M)\exp(-h).$$

Finally, suppose that $t_M - h \leqslant t_m < t_M$ so that $\psi'(t_M) \geqslant 0$. Then (9) holds since

$$\begin{aligned}
&\psi(t_M) \leqslant g\bigl(\psi(t_M - h)\bigr) \leqslant p_2 \psi(t_M - h);\\
&\psi(t_m) = \psi(t_M - h)\exp(t_M - t_m - h) + \int_{t_M - h}^{t_m} \exp\bigl(-(t_m - u)\bigr) g\bigl(\psi(u-h)\bigr)\,du\\
&\qquad \geqslant \psi(t_M - h)\exp(-h).
\end{aligned}$$

Next, for every $s \leqslant \tau$ and $u \in [-h, 0]$, we have that

$$\psi(t_m)\exp(\lambda_1 u) \leqslant \psi(s+u) \leqslant \psi(t_M)\exp\bigl(\lambda_2(u+h)\bigr) \leqslant p_2\psi(t_m)\exp\bigl(\lambda_2(u+h)+h\bigr).$$

From the inequalities above and since additionally $p_1 x \leqslant g(x) \leqslant p_2 x$ for all $x \in [0, \delta)$, then for $s + h \leqslant \tau$ and $u \in [-h, 0]$ we have

$$\begin{aligned}
\psi(s+h+u) &= \psi(s)e^{-(h+u)} + e^{-(s+h+u)} \int_s^{s+h+u} e^{\sigma} g\bigl(\psi(\sigma - h)\bigr)\,d\sigma\\
&\leqslant \psi(s)e^{-(h+u)} + e^{-(s+h+u)} p_2\psi(t_M) \int_s^{s+h+u} e^{\sigma + \lambda_2(\sigma - s)}\,d\sigma\\
&= \psi(s)e^{-(h+u)} + e^{\lambda_2 h}\psi(t_M)\bigl[e^{\lambda_2(h+u)} - e^{-(h+u)}\bigr] \leqslant \psi(t_M)e^{\lambda_2(u+2h)} \quad \text{and}\\
\psi(s+h+u) &\geqslant \psi(s)e^{-(h+u)} + e^{-(s+h+u)} p_1\psi(t_m) \int_s^{s+h+u} e^{\sigma + \lambda_1(\sigma - h - s)}\,d\sigma\\
&= \psi(s)e^{-(h+u)} + \psi(t_m)\bigl[e^{\lambda_1(h+u)} - e^{-(h+u)}\bigr] \geqslant \psi(t_m)e^{\lambda_1(u+h)}.
\end{aligned}$$

By repeating the above procedure over intervals of length h, the step by step method implies that, for all $-h \leqslant u \leqslant \tau - s$,

$$\psi(t_m)\exp(\lambda_1 u) \leqslant \psi(s+u) \leqslant p_2\psi(t_m)\exp\bigl(\lambda_2(u+h)+h\bigr).$$

In particular,

$$\frac{\psi(s)}{p_2}\exp\bigl(\lambda_1(\tau - s) - h\bigr) \leqslant \psi(t_m)\exp\bigl(\lambda_1(\tau - s)\bigr) \leqslant \psi(\tau) \leqslant p_2\psi(t_m)\exp\bigl(\lambda_2(\tau - s + h) + h\bigr).$$

Thus, for every $s \leqslant \tau$,

$$p_2^{-1}\psi(\tau)\exp\bigl(\lambda_2(-\tau-h+s)-h\bigr) \leqslant \psi(t_m) \leqslant \psi(s) \leqslant p_2\psi(\tau)\exp\bigl(\lambda_1(-\tau+s)+h\bigr). \qquad \square$$

In what follows, we shall assume that $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $g'(0+) = p > 1$, and use several times the following simple assertion.

Lemma 7. *Suppose that $p > 1$ and $h > 0$. Then the characteristic equation*

$$z = -1 + p\exp(-zh) \tag{10}$$

has only one real root $0 < \lambda < p - 1$. Moreover, all roots $\lambda, \lambda_j, j = 2, 3, \dots$ of (10) are simple and we can enumerate them in such a way that

$$\cdots \leqslant \Re\lambda_3 = \Re\lambda_2 < \lambda.$$

Proof. The last inequality follows from $\Re\lambda_j < -1 + p\exp(-h\Re\lambda_j)$, $j > 1$. $\square$

Lemma 8. *Suppose that $g'(0+) = p > 1$ and that ψ is a heteroclinic solution to (8). Let λ be the positive root of (10). If there exists $g''(0+) \in \mathbb{R}$, then, for each $\delta > 0$ and some $t_0 \in \mathbb{R}$, we have that $\psi(t - t_0) = \exp(\lambda t) + O(\exp((2\lambda - \delta)t))$ at $t \to -\infty$, so that $\psi'(t - t_0) = \lambda\exp(\lambda t) + O(\exp((2\lambda - \delta)t)) > 0$ on some semi-axis $(-\infty, T]$. Moreover, if there exists $g'' : [0, \epsilon) \to \mathbb{R}$ and is bounded for some $\epsilon > 0$, then $\psi(t)$ is unique up to a shift in t.*

Proof. Since $g''(0+)$ is finite, $g(x) = g'(0+)x + O(x^2) = px + O(x^2)$ as $x \to 0$. From Lemma 6, given $\delta > 0$ small, for $p_1 = p - \delta/2$ we have $\psi(t) = O(\exp(\lambda_\delta t))$ at $-\infty$, where λ_δ is the unique positive root of $z = -1 + (p - \delta/2)\exp(-zh)$. It is easy to see that $\lambda_\delta > \lambda - \delta/2$. In fact, let $W(z) := z + 1 - (p - \delta/2)\exp(-zh)$. For $z \in \mathbb{R}$, we have $W(z) < 0$ if and only if $z < \lambda_\delta$. On the other hand, $W(\lambda - \delta/2) = p\exp(-\lambda h)[1 - \exp(\delta h/2)] + \delta/2[\exp(-(\lambda - \delta/2)h) - 1] < 0$. Hence,

$$\psi'(t) = -\psi(t) + p\psi(t-h) + O\bigl(\psi^2(t-h)\bigr), \tag{11}$$

where $O(\psi^2(t-h)) = O(\exp((2\lambda - \delta)t))$ as $t \to -\infty$. Now, consider the linear inhomogeneous delay differential equation

$$x'(t) = -x(t) + px(t-h) + O\bigl(\exp\bigl((2\lambda - \delta)t\bigr)\bigr). \tag{12}$$

The change of variables $x(t) = y(t)\exp((2\lambda - \delta)t)$ transforms it into

$$y'(t) = -(1 + 2\lambda - \delta)y(t) + p\exp\bigl(-(2\lambda - \delta)h\bigr)y(t-h) + O(1). \tag{13}$$

The spectra $\sigma(y)$, $\sigma(x)$ of the linear parts of Eqs. (13) and (12) are related by $\sigma(y) = \sigma(x) - 2\lambda + \delta$, therefore the linear part of (13) has not pure imaginary eigenvalues for all sufficiently small $\delta > 0$ (equivalently, the linearization of Eq. (13) about zero is hyperbolic). In this case, (13) has a bounded solution $y_b(t) = O(1)$ at $t = -\infty$ (e.g., see [4, Lemma 3.2, p. 246] or [17, Section 10.1]. Note that Eqs. (12) and (13) are not autonomous. Nevertheless, the results for autonomous equations near hyperbolic equilibria in [4,17] are valid in this setting, since the linearized equation near zero for Eq. (13) has an exponential dichotomy, cf. [17, p. 312]). Thus Eq. (12) has a solution $x_b(t) = y_b(t)\exp((2\lambda - \delta)t) = O(\exp((2\lambda - \delta)t))$. In consequence, $z(t) = \psi(t) - x_b(t)$ solves the linear homogeneous equation $x'(t) = -x(t) + px(t-h)$ and is bounded at $t \to -\infty$. This is possible if and only if

$$z(t) = C\exp(\lambda t) + \sum_{j=1}^{N} C_j \exp(\lambda_j t),$$

where $\lambda > 0$, $\lambda_j \in \mathbb{C}$, $j = 1, \dots, N$ is a finite set of roots having nonnegative real parts of the characteristic equation (10). Notice that $C \in \mathbb{R}$, $C_j \in \mathbb{C}$ and $\lambda > \Re\lambda_j$ (see Lemma 7). In this way

$$\psi(t) = C\exp(\lambda t) + \sum_{j=1}^{N} C_j \exp(\lambda_j t) + O\big(\exp\big((2\lambda - \delta)t\big)\big).$$

On the other hand, from Lemma 6 we know that $\psi(t) = O(\exp((\lambda - \delta/2)t))$. Since $\lambda > \Re\lambda_j$, this implies immediately that all $C_j = 0$, $C > 0$ and that $\psi(t) = C\exp(\lambda t) + O(\exp((2\lambda - \delta)t))$. By (11),

$$\psi'(t) = C\lambda\exp(\lambda t) + O\big(\exp\big((2\lambda - \delta)t\big)\big) > 0.$$

Observe also that $\mu(t) = \psi(t - \lambda^{-1}\ln C) = \exp(\lambda t) + O(\exp((2\lambda - \delta)t))$ defines another heteroclinic solution of (7).

Finally, suppose that $\mu(t)$, $\nu(t)$ are two heteroclinic solutions to (7) such that

$$\mu(t) = \exp(\lambda t) + O\big(\exp\big((2\lambda - \delta)t\big)\big), \qquad \nu(t) = \exp(\lambda t) + O\big(\exp\big((2\lambda - \delta)t\big)\big).$$

Applying the Lagrange mean value theorem twice, we get $g(x) - g(y) = p(x - y) + (x - y)O(x + y)$ for x, y close to 0. Since $\sigma(t) = \mu(t) - \nu(t) = O(\exp((2\lambda - \delta)t))$ we obtain that

$$g\big(\mu(t-h)\big) - g\big(\nu(t-h)\big) = \sigma(t-h)\big(p + O\big(\exp(\lambda t)\big)\big) = p\sigma(t-h) + O\big(\exp\big((3\lambda - \delta)t\big)\big).$$

Therefore $\sigma(t)$ satisfies

$$x'(t) = -x(t) + px(t-h) + O\big(\exp\big((3\lambda - \delta)t\big)\big), \tag{14}$$

from which, applying the same procedure as above, we deduce that $\sigma(t) = \mu(t) - \nu(t) = O(\exp((3\lambda - \delta)t))$. In this way, we can show that $\sigma(t) = O(\exp((k\lambda - \delta)t))$ for every integer $k \geqslant 2$. This leads us to the conclusion that σ has superexponential decay at $t = -\infty$ (equivalently, σ is a *small solution* at $t = -\infty$, see [4]). We will finalize our proof showing that only the trivial solution of the linear asymptotically autonomous homogeneous equation

$$x'(t) = -x(t) + p(t)x(t-h), \quad p(-\infty) = p > 1, \tag{15}$$

can have superexponential decay at $t = -\infty$ (notice that $\sigma(t)$ satisfies (15) with $p(t) = p + O(\exp(\lambda t))$). Indeed, if $x(t) > 0$ on some semi-axis $(-\infty, z]$, then we can repeat the arguments in the proof of Lemma 6 to find an exponential lower bound for $x(t)$, in contradiction to our assumption of superexponential decay of $x(t)$. Consider now the case of $x(t)$ oscillatory on every semi-axis $(-\infty, z]$, and take z_0 such that $p(t) < p + 1$ for all $t \in (-\infty, z_0]$. Let $t_1 \in (-\infty, z_0)$ be a point of the global maximum of $|x(t)|$: we can assume that $x(t_1) = M > 0$, $x'(t_1) \geqslant 0$. Then $x(t_1 - h) \geqslant M/(p + 1)$, so that $|x(t_2)| = \max_{t \leqslant t_1 - h} |x(t)| \geqslant (p + 1)^{-1} \max_{t \leqslant t_1} |x(t)|$. Analogously, $\max_{t \leqslant t_1 - 2h} |x(t)| \geqslant |x(t_3)| = \max_{t \leqslant t_2 - h} |x(t)| \geqslant (p + 1)^{-1} \max_{t \leqslant t_2} |x(t)| = (p + 1)^{-1} \max_{t \leqslant t_1 - h} |x(t)| \geqslant (p + 1)^{-2} \max_{t \leqslant t_1} |x(t)|$. Thus

$$\max_{t \leqslant t_1 - kh} \bigl|x(t)\bigr| \geqslant (p + 1)^{-k} \max_{t \leqslant t_1} \bigl|x(t)\bigr|$$

so that $x(t)$ cannot decay superexponentially as $t \to -\infty$. $\square$

Now, assume (H) and the global attractivity of $x_2 = K$ for Eq. (8), and then take $\lambda > 0$ satisfying (10) and the unique (up to a shift in time) heteroclinic solution ψ described in Lemma 8. Let $\lambda_* \in (0, \lambda)$ be sufficiently close to λ and such that the equation $y'(t) = -(1 + \lambda_*)y(t) + p\exp(-\lambda_* h)y(t - h)$ is hyperbolic. Note that this latter equation is obtained by effecting the change of variables $x(t) = \exp(\lambda_* t)y(t)$ to the linear equation $x'(t) = -x(t) + px(t - h)$. For a fixed $\mu > 0$, we will consider the seminorms $\|x\|^+ = \sup_{\mathbb{R}_+} |x(s)|$, $\|x\|_\mu^- = \sup_{\mathbb{R}_-} e^{-\mu s}|x(s)|$, $\|x\|_\mu = \max\{\|x\|^+, \|x\|_\mu^-\}$ and the following Banach spaces:

$$C_\mu(\mathbb{R}) = \bigl\{x \in C(\mathbb{R}, \mathbb{R}):\ \|x\|_\mu^- < \infty \text{ and } x(+\infty) \text{ exists and is finite}\bigr\},$$

$$C_{\psi,\lambda_*}(\mathbb{R}) = \left\{x \in C_{\lambda_*}(\mathbb{R}):\ \int_{-\infty}^{0} x(s)\psi'(s)\,ds = 0\right\},$$

equipped with the norms $\|x\|_\mu$ and $\|x\|_{\lambda_*}$, respectively (in order to simplify the notation, we shall often write $\|x\|$ instead of $\|x\|_\mu$, etc.). Notice that, due to Lemma 8, we have $\psi, \psi' \in C_{\lambda_*}(\mathbb{R}) \setminus C_{\psi,\lambda_*}(\mathbb{R})$. We shall also need the following integral operator

$$\mathcal{N}:\quad C_{\psi,\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R});\qquad (\mathcal{N}x)(t) = \int_{-\infty}^{t} e^{-(t-s)} q(s)x(s - h)\,ds,$$

where $q(s) = g'(\psi(s - h))$ with $q(-\infty) = g'(0+) = p > 1$, $q(+\infty) = g'(K)$. Observe that $\mathcal{N}$ is well defined, since $(\mathcal{N}x)(+\infty) = g'(K)x(+\infty)$ and, for $t \leqslant h$,

$$\bigl|(\mathcal{N}x)(t)\bigr| = \int_{-\infty}^{t} e^{-(t-s)}\bigl|q(s)\bigr|\|x\|_{\lambda_*}^- e^{\lambda_*(s-h)}\,ds \leqslant \frac{\|x\|_{\lambda_*}^- \sup_{t \leqslant h}|q(t)|}{1 + \lambda_*} e^{\lambda_*(t-h)}.$$

Lemma 9. *If* (H) *is assumed, then* $I - \mathcal{N}: C_{\psi,\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$ *is an isomorphism of Banach spaces.*

Proof. We first prove that $\mathrm{Ker}(I-\mathcal{N})=0$. Indeed, if $y\in\mathrm{Ker}(I-\mathcal{N})$ and $y\neq 0$, then

$$\int_{-\infty}^{t} e^{-(t-s)}q(s)y(s-h)\,ds = y(t).$$

Therefore y is a bounded solution of the linear delay differential equation

$$y'(t) = -y(t) + q(t)y(t-h). \tag{16}$$

Since $g'(x)=p+O(x)$ at $x=0$ and $\psi(t)=O(\exp(\lambda t))$ at $t=-\infty$, we conclude that

$$q(t) = g'\big(\psi(t-h)\big) = p + O\big(\exp(\lambda t)\big), \quad t\to-\infty.$$

Thus $y(t)$ can be viewed as a bounded solution of the inhomogeneous equation

$$x'(t) = -x(t) + px(t-h) + O\big(\exp(2\lambda t)\big), \quad t\to-\infty.$$

Since $y(t)=O(\exp(\lambda_* t))$ at $-\infty$, with $\lambda_*<\lambda$ close to λ, the procedure which has been used before to prove the uniqueness of the heteroclinic $\psi(t)$ allows us to conclude that $y(t) = C\exp(\lambda t) + O(\exp(2\lambda_* t))$ and that $\dim\mathrm{Ker}(I-\mathcal{N})=1$. On the other hand, we know that $\psi'(t)\not\equiv 0$ satisfies (16). Thus we must have $y(t)=c\psi'(t)\notin C_{\psi,\lambda_*}(\mathbb{R})$, $c\neq 0$ constant, a contradiction. Therefore $y(t)\equiv 0$ and $\mathrm{Ker}(I-\mathcal{N})=0$.

We now establish that $I-\mathcal{N}$ is an epimorphism. Take some $d\in C_{\lambda_*}(\mathbb{R})$ and consider the following integral equation

$$x(t) - \int_{-\infty}^{t} e^{-(t-s)}q(s)x(s-h)\,ds = d(t).$$

If we set $z(t)=x(t)-d(t)$, this equation is transformed into

$$z(t) - \int_{-\infty}^{t} e^{-(t-s)}q(s)\big(z(s-h)+d(s-h)\big)\,ds = 0.$$

Hence we have to prove the existence of at least one $C_{\lambda_*}(\mathbb{R})$-solution of the equation

$$z'(t) = -z(t) + q(t)z(t-h) + q(t)d(t-h). \tag{17}$$

First, notice that all solutions of (17) are bounded on the positive semi-axis $\mathbb{R}_+$ due to the boundedness of $q(t)d(t-h)$ and the exponential stability of the homogeneous ω-limit equation $z'(t) = -z(t)+g'(K)z(t-h)$. Here we use the persistence of exponential stability under small bounded perturbations (e.g., see [3, Section 5.2] or [5, Chapter VI (9c)]) and the fact that $q(+\infty) = g'(K)$. Furthermore, since every solution z of (17) satisfies $z'(t) = -z(t) + g'(K)z(t-h) + g'(K)d(+\infty) + \epsilon(t)$ with $\epsilon(+\infty)=0$, we get $z(+\infty) = d(+\infty)g'(K)(1-g'(K))^{-1}$. Next, by effecting the change of variables $z(t) = \exp(\lambda_* t)y(t)$ to Eq. (17), we get a linear inhomogeneous equation of the form

$$y'(t) = -(1+\lambda_*)y(t) + \big[p\exp(-\lambda_* h) + \epsilon_1(t)\big]y(t-h) + \epsilon_2(t), \tag{18}$$

where $\epsilon_1(-\infty) = 0$ and $\epsilon_2(t) = O(1)$ at $t = -\infty$. Since the α-limit equation $y'(t) = -(1+\lambda_*)y(t) + p\exp(-\lambda_* h)y(t-h)$ to the homogeneous part of (18) is hyperbolic, due to the above mentioned persistence of the property of exponential dichotomy, we again conclude that Eq. (18) also has an exponential dichotomy on $\mathbb{R}_-$. Thus (18) has a solution y^* which is bounded on $\mathbb{R}_-$ so that $z^*(t) = \exp(\lambda_* t)y^*(t) = O(\exp(\lambda_* t))$, $t \to -\infty$, is a $C_{\lambda_*}(\mathbb{R})$-solution of Eq. (17). Now, it is evident that $w(t) = z^*(t) - C\psi'(t) = O(\exp(\lambda_* t))$ solves (17) for each $C \in \mathbb{R}$. In consequence, $x(t) = d(t) + z^*(t) - C_d\psi'(t) = ((I-\mathcal{N})^{-1}d)(t)$, if we take $C_d = \int_{-\infty}^0 (d(s) + z^*(s))\psi'(s))\,ds(\int_{-\infty}^0 (\psi'(s))^2\,ds)^{-1}$. □

Remark 10. For $\delta > 0$ small, consider $I - \mathcal{N}_1 : C_{2\lambda-\delta}(\mathbb{R}) \to C_{2\lambda-\delta}(\mathbb{R})$, where $\mathcal{N}_1$ is defined by $(\mathcal{N}_1 x)(t) = p\int_{-\infty}^t e^{-(t-s)}x(s-h)\,ds$ (recall here the discussion after formula (13)). Replacing $\mathcal{N}$ by $\mathcal{N}_1$ in the proof of Lemma 9, we establish similarly that $I - \mathcal{N}_1$ is an isomorphism of the Banach space $C_{2\lambda-\delta}(\mathbb{R})$ onto itself. Since the linear equation $x'(t) = -x(t) + px(t-h)$ is hyperbolic, this situation is actually simpler than the one considered in Lemma 9.

3. Existence of a continuous family of positive travelling waves

In this section, we are looking for travelling waves for (3), that is, solutions $u(x,t) = \phi(\varepsilon\nu \cdot x + t)$, $x, \nu \in \mathbb{R}^m$, $\|\nu\| = 1$, where $c = 1/\varepsilon$ is the wave speed, connecting the two equilibria of (3). We will suppose that ε is sufficiently small. This leads us to the question about the existence of heteroclinic solutions to the singularly perturbed equation

$$\varepsilon^2 x''(t) - x'(t) - x(t) + g\big(x(t-h)\big) = 0, \quad t \in \mathbb{R}, \tag{19}$$

with $x(-\infty) = 0$, $x(+\infty) = K$. Being a bounded function, each travelling wave should satisfy the following integral equation

$$x(t) = \frac{1}{\sigma(\varepsilon)}\left(\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} g\big(x(s-h)\big)\,ds + \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} g\big(x(s-h)\big)\,ds\right), \tag{20}$$

where $\sigma(\varepsilon) = \sqrt{1+4\varepsilon^2}$. For solutions in $C_{\lambda_*}(\mathbb{R})$ with $\lambda_* \in (0,\lambda)$ close to λ, this equation can be written in the shorter form

$$x - (\mathcal{I}_\varepsilon \circ \mathcal{G})x = 0, \tag{21}$$

where $\mathcal{I}_\varepsilon, \mathcal{G} : C_{\lambda_*}(\mathbb{R}) \to C_{\lambda_*}(\mathbb{R})$ are defined by

$$(\mathcal{I}_\varepsilon x)(t) = \frac{1}{\sigma(\varepsilon)}\left(\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} x(s-h)\,ds + \int_{t}^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} x(s-h)\,ds\right),$$

and $(\mathcal{G}x)(t) = g(x(t))$ is the Nemitski operator. (For the sake of simplicity, we write $\mathcal{I}_\varepsilon, \mathcal{G}$ instead of $\mathcal{I}_{\varepsilon,\lambda_*}, \mathcal{G}_{\lambda_*}$.) Throughout all this section, we will suppose that the C^1-smooth function g is defined and bounded on the whole real axis $\mathbb{R}$. Clearly, this assumption does not restrict the generality of our framework, since it suffices to take any smooth and bounded extension on $\mathbb{R}_-$ of the nonlinearity g described in (H). Notice that, since there exists finite $g'(0)$, we have $g(x) = x\gamma(x)$ for a bounded $\gamma \in C(\mathbb{R})$. Set $\gamma_0 = \sup_{t\in\mathbb{R}} |\gamma(x)|$. As it can be easily checked, $\|\mathcal{G}x\| \leqslant \gamma_0\|x\|$ so that actually $\mathcal{G}$ is well defined. Furthermore, we have the following lemma.

Lemma 11. *Assume that $g \in C^1(\mathbb{R})$. Then $\mathcal{G}$ is Fréchet continuously differentiable on $C_{\lambda_*}(\mathbb{R})$ with differential $\mathcal{G}'(x_0): y(\cdot) \to g'(x_0(\cdot))y(\cdot)$.*

Proof. By the Taylor formula, $g(v) - g(v_0) - g'(v_0)(v - v_0) = o(v - v_0)$, $v, v_0 \in \mathbb{R}$. Fix some $x_0 \in C_{\lambda_*}(\mathbb{R})$, then we have

$$\left\|\mathcal{G}x - \mathcal{G}x_0 - g'\big(x_0(\cdot)\big)(x - x_0)\right\| = o\big(\|x - x_0\|\big), \quad x \in C_{\lambda_*}(\mathbb{R}).$$

Clearly, it holds that $\|\mathcal{G}'(x)u\| = \|g'(x(\cdot))u(\cdot)\| \leqslant \sup_{t\in\mathbb{R}} |g'(x(t))|\|u\|$. Since functions in $C_{\lambda_*}(\mathbb{R})$ are bounded and g' is uniformly continuous on bounded sets of $\mathbb{R}$, for any given $\delta > 0$ there is $\sigma > 0$ such that for $\|x - x_0\| < \sigma$ we have $\|\mathcal{G}'(x) - \mathcal{G}'(x_0)\| < \delta$. □

Now, we consider the integral operators $\mathcal{I}_\varepsilon^+, \mathcal{I}_\varepsilon^- : C_\mu(\mathbb{R}) \to C_\mu(\mathbb{R})$ defined as

$$\big(\mathcal{I}_\varepsilon^+ x\big)(t) = \int_t^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} x(s-h)\,ds, \qquad \big(\mathcal{I}_\varepsilon^- x\big)(t) = \int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} x(s-h)\,ds.$$

Lemma 12. *Set $\mathcal{I} = \mathcal{I}_0^-$ and $\mathcal{I}_0^+ = 0$. If $\varepsilon \to 0+$, then $\mathcal{I}_\varepsilon \to \mathcal{I}$ in the operator norm. Moreover, both operator families $\mathcal{I}_\varepsilon^\pm : [0, 1/\sqrt{\mu}) \to \mathcal{L}(C_\mu(\mathbb{R}), C_\mu(\mathbb{R}))$ are continuous in the operator norm.*

Proof. We prove only that $\|\mathcal{I}_\varepsilon - \mathcal{I}\| \to 0$ as $\varepsilon \to 0$, the proof of the continuous dependence of $\mathcal{I}_\varepsilon^\pm$ on ε being completely analogous.

We first establish that $\mathcal{I}_\varepsilon^+ \to 0$ uniformly as $\varepsilon \to 0$. In fact, for all $t \in \mathbb{R}$, we obtain

$$\left|\big(\mathcal{I}_\varepsilon^+ x\big)(t)\right| \leqslant \int_t^{+\infty} e^{\frac{t-s}{\varepsilon^2}} \left|x(s-h)\right| ds \leqslant \varepsilon^2 \|x\|.$$

Furthermore, since $|x(t)| \leqslant \|x\| \exp(\mu t)$ for all $t \in \mathbb{R}$, for $\varepsilon^2 < 1/\mu$ we have

$$\left|\big(\mathcal{I}_\varepsilon^+ x\big)(t)\right| \leqslant \int_t^{+\infty} e^{\frac{t-s}{\varepsilon^2}} \left|x(s-h)\right| ds \leqslant \frac{\|x\|}{\varepsilon^{-2} - \mu} e^{\mu(t-h)}.$$

Hence, for $\varepsilon^2 < (1 - 0.5e^{-\mu h})/\mu$, we obtain that $\|\mathcal{I}_\varepsilon^+ x\| \leqslant 2\varepsilon^2 \|x\|$.

Next, we prove that $\mathcal{I}_\varepsilon^- \to \mathcal{I}$ uniformly as $\varepsilon \to 0$. We have

$$\left|\left(\left(\mathcal{I}_\varepsilon^- - \mathcal{I}\right)x\right)(t)\right| \leqslant \int_{-\infty}^{t} e^{-(t-s)}\left(e^{\frac{\sigma(\varepsilon)-1}{\sigma(\varepsilon)+1}(t-s)} - 1\right)\left|x(s-h)\right| ds.$$

Thus, for $t \leqslant h$, we obtain that

$$\begin{aligned}\left|\left(\left(\mathcal{I}_\varepsilon^- - \mathcal{I}\right)x\right)(t)\right| &\leqslant \int_{-\infty}^{t} e^{-(t-s)}\left(e^{\frac{\sigma(\varepsilon)-1}{\sigma(\varepsilon)+1}(t-s)} - 1\right)\|x\|_\mu^- e^{\mu(s-h)}\, ds \\ &= \frac{\|x\|_\mu^- e^{\mu(t-h)}(\sigma(\varepsilon)-1)}{(2+(\sigma(\varepsilon)+1)\mu)(1+\mu)}\end{aligned}$$

and, for all t,

$$\left|\left(\left(\mathcal{I}_\varepsilon^- - \mathcal{I}\right)x\right)(t)\right| \leqslant \int_{-\infty}^{t} e^{-(t-s)}\left(e^{\frac{\sigma(\varepsilon)-1}{\sigma(\varepsilon)+1}(t-s)} - 1\right)\|x\|\, ds = \|x\|\frac{\sigma(\varepsilon)-1}{2}.$$

Thus $\|\mathcal{I}_\varepsilon^- - \mathcal{I}\| \leqslant 0.5(\sigma(\varepsilon)-1)$, and the proof of the lemma is complete. □

To prove the main result of this section, stated below as Theorem 14, we will make use of the following lemma.

Lemma 13. *Let* $\{z_\alpha, \alpha \in A\}$, *where* $\mathbb{N} \cup \{\infty\} \subset A$, *denote the* (*countable*) *set of roots to the equation*

$$\varepsilon^2 z^2 - z - 1 + p\exp(-zh) = 0. \tag{22}$$

If $p > 1$, $h > 0$, $\varepsilon \in (0, 1/(2\sqrt{p-1}))$ *then* (22) *has exactly two real roots* $\lambda_1(\varepsilon)$, $\lambda_\infty(\varepsilon)$ *such that*

$$0 < \lambda < \lambda_1(\varepsilon) < 2(p-1) < \varepsilon^{-2} - 2(p-1) < \lambda_\infty(\varepsilon) < \varepsilon^{-2} + 1.$$

Moreover:

- (i) *there exists an interval* $\mathcal{O} = (0, a(p,h))$ *such that, for every* $\varepsilon \in \mathcal{O}$, *all roots* $\lambda_\alpha(\varepsilon)$, $\alpha \in A$ *of* (22) *are simple and the functions* $\lambda_\alpha : \mathcal{O} \to \mathbb{C}$ *are continuous*;
- (ii) *we can enumerate* $\lambda_j(\varepsilon)$, $j \in \mathbb{N}$, *in such a way that there exists* $\lim_{\varepsilon\to 0+} \lambda_j(\varepsilon) = \lambda_j$ *for each* $j \in \mathbb{N}$, *where* $\lambda_j \in \mathbb{C}$, *are the roots of* (10), *with* $\lambda_1 = \lambda$;
- (iii) *for all sufficiently small* ε, *every vertical strip* $\xi \leqslant \Re z \leqslant 2(p-1)$ *contains only a finite set of* $m(\xi)$ *roots* (*if* $\xi \notin \{\Re\lambda_j, j \in \mathbb{N}\}$, *then* $m(\xi)$ *does not depend on* ε) $\lambda_1(\varepsilon), \ldots, \lambda_{m(\xi)}(\varepsilon)$ *to* (22), *while the half-plane* $\Re z > 2(p-1)$ *contains only the root* $\lambda_\infty(\varepsilon)$.

Proof. The existence of real roots $\lambda_1(\varepsilon)$, $\lambda_\infty(\varepsilon)$ satisfying $\lambda < \lambda_1(\varepsilon) < \lambda_\infty(\varepsilon)$ is obvious when $\varepsilon \in (0, 0.5/\sqrt{p-1})$. On the other hand, if $z_0 > 0$ is a real root of (22), then $\varepsilon^2 z_0^2 - z_0 - 1 < 0$, $\varepsilon^2 z_0^2 - z_0 - 1 + p > 0$. Hence $z_0 < (1 + \sqrt{1+4\varepsilon^2})/(2\varepsilon^2) < \varepsilon^{-2} + 1$, from which it can be checked easily that

$$\lambda_\infty > \frac{1+\sqrt{1-4(p-1)\varepsilon^2}}{2\varepsilon^2} > \frac{1-2(p-1)\varepsilon^2}{\varepsilon^2}, \qquad \lambda_1 < \frac{1-\sqrt{1-4(p-1)\varepsilon^2}}{2\varepsilon^2} < 2(p-1).$$

We also notice that every multiple root z_0 has to satisfy the system

$$\varepsilon^2 z_0^2 - z_0 - 1 + p\exp(-z_0 h) = 0, \qquad 2\varepsilon^2 z_0 - 1 - ph\exp(-z_0 h) = 0, \tag{23}$$

which implies

$$\big(\varepsilon^2 z_0^2 - z_0 - 1\big)h + 2\varepsilon^2 z_0 - 1 = 0, \qquad p\exp(-z_0 h) = \frac{2+z_0}{2+hz_0}. \tag{24}$$

The first equation of (24) implies that z_0 is real while the second equation of (23) says that $z_0 > 0$. Since z_0 is positive, from the first equation of (24) we obtain $0.5\varepsilon^{-2} < z_0$ (we recall that $\varepsilon^2 z_0^2 - z_0 - 1 < 0$). Let $\zeta_0(p,h)$ be the maximal positive root of the second equation of (24). If $\varepsilon > 0$ is so small that $0.5\varepsilon^{-2} > \zeta_0(p,h)$, system (23) cannot have any positive solution. In consequence, the second assertion of this lemma holds if we set $a(p,h) = 1/\sqrt{2\zeta_0(p,h)}$.

Finally, we prove that the half-plane $\Re z > 2(p-1)$ contains only the root $\lambda_\infty(\varepsilon)$ of (22). For this, let us evaluate $|\varepsilon^2 z^2 - z - 1|$ on the boundary of some rectangle $[2(p-1), b] \times [-c, c] \subset \mathbb{C}$, with b, c being sufficiently large. For $\mu(\varepsilon)$, $\nu(\varepsilon)$ the (real) roots of $\varepsilon^2 z^2 - z - 1 = 0$, we have that

$$\big|\varepsilon^2 z^2 - z - 1\big| = \varepsilon^2\big|z-\mu(\varepsilon)\big|\big|z-\nu(\varepsilon)\big| \geqslant \varepsilon^2\big|\Re z-\mu(\varepsilon)\big|\big|\Re z-\nu(\varepsilon)\big| = \big|\varepsilon^2(\Re z)^2 - \Re z - 1\big|.$$

Thus, for $\Re z = 2(p-1)$, we obtain

$$\big|\varepsilon^2 z^2 - z - 1\big| \geqslant \Re z + 1 - \varepsilon^2(\Re z)^2 > p.$$

If $\Re z > 2(\varepsilon^{-2}+1)$, then

$$\big|\varepsilon^2 z^2 - z - 1\big| \geqslant \varepsilon^2(\Re z)^2 - \Re z - 1 > 8p - 3 > p.$$

Similarly, for $|\Im z| > p/\varepsilon$ fixed, we get

$$\big|\varepsilon^2 z^2 - z - 1\big| = \varepsilon^2\big|z-\mu(\varepsilon)\big|\big|z-\nu(\varepsilon)\big| \geqslant \varepsilon^2(\Im z)^2 > p.$$

Thus, by Rouché's theorem, $\varepsilon^2 z^2 - z - 1 + p\exp(-zh) = 0$ and $\varepsilon^2 z^2 - z - 1 = 0$ have the same number of roots in the half-plane $\Re z > 2(p-1)$, that is exactly one root.

Therefore, for all λ_j with $\Re\lambda_j \in [\xi, 2(p-1)]$ and $\varepsilon \in (0, 0.25/\sqrt{p-1})$, we get

$$pe^{-\xi h} \geqslant \big|\Im\big(\varepsilon^2\lambda_j^2 - \lambda_j - 1\big)\big| = |\Im\lambda_j|\big|1 - 2\varepsilon^2\Re\lambda_j\big| \geqslant |\Im\lambda_j|/2,$$

so that $|\Im\lambda_j| \leqslant 2pe^{-\xi h}$. Hence, applying Rouché's theorem to the functions $\varepsilon^2 z^2 - z - 1 + p\exp(-zh)$ and $-z - 1 + p\exp(-zh)$ along an appropriate rectangle inside $[\xi - 1, 2(p-1)] \times [-3pe^{-\xi h}, 3pe^{-\xi h}] \subset \mathbb{C}$, we prove the last assertion of Lemma 13. $\square$

Theorem 14. *Assume* (H) *and that the positive equilibrium of Eq.* (4) *is globally attractive. Let ψ be some heteroclinic orbit of Eq.* (4): $\psi(-\infty) = 0$, $\psi(+\infty) = K$. *Then, for every $\delta > 0$ there is an interval $\mathcal{E} = (-\varepsilon_0, \varepsilon_0)$ and a continuous family of positive heteroclinic orbits $\psi_\varepsilon : \mathcal{E} \to C_{\lambda-\delta}(\mathbb{R})$ of Eq.* (19) *satisfying the additional conditions $\psi_0 = \psi$ and $\int_{-\infty}^{0} \psi_\varepsilon(s)\psi'(s)\,ds = 0.5\psi^2(0)$. Furthermore, for every $\varepsilon \in \mathcal{E} \setminus \{0\}$ we have that $\psi_\varepsilon(t - t_0) = \exp(\lambda_1(\varepsilon)t) + O(\exp(2\lambda t))$ at $t \to -\infty$ for some $t_0 = t_0(\varepsilon) \in \mathbb{R}$, and that $\psi'_\varepsilon(t - t_0) = \lambda_1(\varepsilon)\exp(\lambda_1(\varepsilon)t) + O(\exp(2\lambda t)) > 0$ on some semi-axis $(-\infty, z]$.*

Proof. We represent the mentioned orbit ψ of (4) as $\psi = \alpha\psi' + \phi_0$, where

$$\phi_0 = (\psi - \alpha\psi') \in C_{\psi,\lambda_*}(\mathbb{R}), \quad \alpha = \psi^2(0)\left(2\int_{-\infty}^{0} \left(\psi'(s)\right)^2 ds\right)^{-1}.$$

For $\delta > 0$ small, consider $\lambda_* = \lambda - \delta$. In virtue of Lemmas 9, 11 and 12, we can apply the implicit function theorem (e.g., see [2, pp. 36, 37] or [28, p. 170]) to Eq. (21) written as $F(\phi, \varepsilon) = 0$, where $F : C_{\psi,\lambda_*}(\mathbb{R}) \times \mathbb{R} \to C_{\lambda_*}(\mathbb{R})$,

$$F(\phi, \varepsilon) = \alpha\psi' + \phi - (\mathcal{I}_\varepsilon \circ \mathcal{G})(\alpha\psi' + \phi), \quad \text{and} \quad \mathcal{I}_0 = \mathcal{I}.$$

Observe that $F(\phi_0, 0) = 0$ and $F_\phi(\phi_0, 0) = I - \mathcal{N}$. In this way, we establish the existence of an interval $\mathcal{E} = (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 \in (0, 1/(2\sqrt{p-1}))$ and a continuous family $\phi_\varepsilon : \mathcal{E} \to C_{\psi,\lambda_*}(\mathbb{R})$ of solutions to $F(\phi, \varepsilon) = 0$. Notice that $\psi_0 = \psi$, $\psi_\varepsilon = \alpha\psi' + \phi_\varepsilon \in C_{\lambda_*}(R)$ satisfy Eq. (21), so that, as it can be checked directly, $\psi_\varepsilon(+\infty) = g(\psi_\varepsilon(+\infty))$. Thus $\psi_\varepsilon(+\infty) = K$ and ψ_ε satisfies all conclusions of the third sentence of the theorem, except its positivity, which is proved below.

Assume now that ε_0 is sufficiently small so that $\lambda_1(\varepsilon) < 0.5\lambda_\infty(\varepsilon)$ for all $\varepsilon \in \mathcal{E} \setminus \{0\}$. Since $g(x) = px + O(x^2)$ as $x \to 0$, and since there exists a constant $C_1 > 0$ such that $|\psi_\varepsilon(t)| \leqslant C_1\exp(\lambda_* t)$, $t \leqslant 0$, $\varepsilon \in \mathcal{E}$, we get

$$\varepsilon^2\psi''_\varepsilon(t) - \psi'_\varepsilon(t) - \psi_\varepsilon(t) + p\psi_\varepsilon(t - h) = \Psi_\varepsilon(t), \tag{25}$$

where $\Psi_\varepsilon(\cdot) = p\psi_\varepsilon(\cdot - h) - g(\psi_\varepsilon(\cdot - h)) \in C_{2\lambda_*}(\mathbb{R})$. Moreover, $\|\Psi_\varepsilon\|_{2\lambda_*} \leqslant C_2$ for some $C_2 > 0$ which does not depend on ε. Now, $C_{2\lambda_*}(\mathbb{R})$-solutions x_ε to

$$\varepsilon^2 x''(t) - x'(t) - x(t) + px(t - h) = \Psi_\varepsilon(t), \tag{26}$$

are solutions to the equation $(I - p\mathcal{I}_\varepsilon)x_\varepsilon = -\mathcal{I}_\varepsilon\Psi_\varepsilon$. Due to Remark 10 and Lemma 12, for $\lambda_* = \lambda - \delta$ close to λ the operator $I - p\mathcal{I}_\varepsilon$ is invertible in $C_{2\lambda_*}(\mathbb{R})$ for all sufficiently small ε. Moreover, Lemma 12 implies that there exists a subinterval $\mathcal{E}_1 \subset \mathcal{E}$ such that $\|(I - p\mathcal{I}_\varepsilon)^{-1}\| \leqslant C_3$ for all $\varepsilon \in \mathcal{E}_1$. Hence, we obtain $\|x_\varepsilon\| \leqslant \|(I - p\mathcal{I}_\varepsilon)^{-1}\mathcal{I}_\varepsilon\|\|\Psi_\varepsilon\| \leqslant C_4$ for all $\varepsilon \in \mathcal{E}_1$. Therefore Eq. (26) has a bounded solution x_ε such that $|x_\varepsilon(t)| \leqslant C_4\exp(2\lambda_* t)$, $t \leqslant 0$, $\varepsilon \in \mathcal{E}_1$. Consequently, $z_\varepsilon(t) = \psi_\varepsilon(t) - x_\varepsilon(t)$ solves the linear homogeneous equation

$$\varepsilon^2 z''(t) - z'(t) - z(t) + pz(t - h) = 0, \quad t \in \mathbb{R},$$

and is bounded as $t \to -\infty$. This is possible if and only if

$$z_\varepsilon(t) = A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)t\bigr) + B_\varepsilon \exp\bigl(\lambda_\infty(\varepsilon)t\bigr) + \sum_{j=2}^{N} C_{j,\varepsilon} \exp\bigl(\lambda_j(\varepsilon)t\bigr),$$

where $\lambda_j(\varepsilon) \in \mathbb{C}$, $j = 1, \ldots, N$, and $\lambda_\infty(\varepsilon)$ are the roots with nonnegative real parts of the characteristic equation (22), $A_\varepsilon, B_\varepsilon \in \mathbb{R}$, $C_{j,\varepsilon} \in \mathbb{C}$, $\varepsilon \in \mathcal{E}_1 \setminus \{0\}$. In consequence,

$$\psi_\varepsilon(t) = A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)t\bigr) + B_\varepsilon \exp\bigl(\lambda_\infty(\varepsilon)t\bigr) + \sum_{j=2}^{N} C_{j,\varepsilon} \exp\bigl(\lambda_j(\varepsilon)t\bigr) + x_\varepsilon(t).$$

It follows from Lemma 13 that $\Re\lambda_j(\varepsilon) < \lambda_* < \lambda < \lambda_1(\varepsilon) < 0.5\lambda_\infty(\varepsilon)$, provided ε is small (say, $\varepsilon \in \mathcal{E}_2 \subset \mathcal{E}_1$) and λ_* is sufficiently close to λ. Since $\psi_\varepsilon(t) = O(\exp(\lambda_* t))$, this implies immediately that $C_{j,\varepsilon} = 0$ and

$$\psi_\varepsilon(t) = A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)t\bigr) + B_\varepsilon \exp\bigl(\lambda_\infty(\varepsilon)t\bigr) + x_\varepsilon(t), \quad t \in \mathbb{R},\ \varepsilon \in \mathcal{E}_2 \setminus \{0\}.$$

To prove the positivity of ψ_ε for ε small, we first establish that $\limsup_{\varepsilon\to 0} |B_\varepsilon|$ is finite, from which we deduce that the constants A_ε are positive; in fact, we will find that $A_\varepsilon > 1 - 4\delta$. Let us suppose already that $1 - 5\delta > 0$ and $\lambda_1(\varepsilon) < 2\lambda_* - \delta$ for all $\varepsilon \in \mathcal{E}_2$. Since $\psi_\varepsilon \in C_{\lambda_*}(R)$, for all $t \leqslant 0$, $\varepsilon \in \mathcal{E}_2 \setminus \{0\}$, we get

$$\bigl|A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)t\bigr) + B_\varepsilon \exp\bigl(\lambda_\infty(\varepsilon)t\bigr)\bigr| \leqslant \bigl|\psi_\varepsilon(t)\bigr| + \bigl|x_\varepsilon(t)\bigr| \leqslant C_5 \exp(\lambda_* t),$$

where $C_5 = C_1 + C_4$. In particular, taking $t = 0$ and $t = -1$, we obtain

$$|A_\varepsilon + B_\varepsilon| \leqslant C_5, \qquad \bigl|A_\varepsilon + B_\varepsilon \exp\bigl(\lambda_1(\varepsilon) - \lambda_\infty(\varepsilon)\bigr)\bigr| \leqslant C_5 \exp\bigl(\lambda_1(\varepsilon) - \lambda_*\bigr) \leqslant C_5 \exp(\lambda),$$

hence $|B_\varepsilon|(1 - \exp(\lambda_1(\varepsilon) - \lambda_\infty(\varepsilon))) \leqslant C_5(1 + \exp(\lambda)) := C_6$, for $\varepsilon \in \mathcal{E}_2 \setminus \{0\}$.

Noting that $\exp(\lambda_1(\varepsilon) - \lambda_\infty(\varepsilon)) \to 0$ as $\varepsilon \to 0$, we deduce that there is $\mathcal{E}_3 = (-\varepsilon_3, \varepsilon_3) \subset \mathcal{E}_2$ such that $|B_\varepsilon| \leqslant 2C_6$ for $\varepsilon \in \mathcal{E}_3 \setminus \{0\}$, so that

$$\bigl|B_\varepsilon \exp\bigl(\lambda_\infty(\varepsilon)t\bigr)\bigr| \leqslant 2C_6 \exp\bigl(\lambda_\infty(\varepsilon)t\bigr) \leqslant 2C_6 \exp(\delta t) \exp\bigl((2\lambda_* - \delta)t\bigr),$$

for $t \leqslant 0$, $\varepsilon \in \mathcal{E}_3 \setminus \{0\}$. Set $y_\varepsilon(t) = B_\varepsilon \exp(\lambda_\infty(\varepsilon)t) + x_\varepsilon(t)$. By Lemma 8, we have $\psi(t) = \exp(\lambda t) + z(t)$ with $z(t) = O(\exp(2\lambda_* t))$ at $t = -\infty$. Since $\lim_{t\to-\infty} C_6 \exp(\delta t) = 0$, we now conclude that there is $s_0 = s_0(\delta) < 0$ such that for $t \leqslant s_0$ and $0 < |\varepsilon| < \varepsilon_3$ we have

$$\bigl|y_\varepsilon(t)\bigr| \leqslant \delta \exp\bigl((2\lambda_* - \delta)t\bigr), \qquad \bigl|y_\varepsilon(t) - z(t)\bigr| \leqslant \delta \exp\bigl((2\lambda_* - \delta)t\bigr).$$

On the other hand, for $\delta_0 = \delta \exp((\lambda - \lambda_*)s_0) = \delta \exp(\delta s_0)$, there exists $\varepsilon_4 = \varepsilon_4(\delta) \in (0, \varepsilon_3]$ such that, for $|\varepsilon| < \varepsilon_4$, we have $|\psi_\varepsilon(t) - \psi(t)| \leqslant \delta_0 \exp(\lambda_* t)$. Taking $t = s_0$ we obtain for $0 < |\varepsilon| < \varepsilon_4$

$$\bigl|A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)s_0\bigr) - \exp(\lambda s_0)\bigr| \leqslant \bigl|\psi_\varepsilon(s_0) - \psi(s_0)\bigr| + \bigl|y_\varepsilon(s_0) - z(s_0)\bigr| \leqslant 2\delta \exp(\lambda s_0),$$

hence

$$\begin{aligned}\psi_\varepsilon(s_0) &= A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)s_0\bigr) + y_\varepsilon(s_0)\\ &\geqslant \exp(\lambda s_0) - \bigl(\bigl|A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)s_0\bigr) - \exp(\lambda s_0)\bigr| + \bigl|y_\varepsilon(s_0)\bigr|\bigr) > (1-3\delta)\exp(\lambda s_0).\end{aligned}$$

Therefore, for all $0 < |\varepsilon| < \varepsilon_4$,

$$A_\varepsilon > -y_\varepsilon(s_0)\exp\bigl(-\lambda_1(\varepsilon)s_0\bigr) + (1-3\delta)\exp\bigl(\bigl(\lambda - \lambda_1(\varepsilon)\bigr)s_0\bigr) \geqslant (1-4\delta) > 0.$$

Thus, for $0 < |\varepsilon| < \varepsilon_4$ and $t \leqslant s_0$ we get $\psi_\varepsilon(t) \geqslant \exp(\lambda_1(\varepsilon)t)[(1-4\delta)-\delta)] > 0$. Since $\lim_{\varepsilon\to 0}\psi_\varepsilon = \psi$ uniformly on $\mathbb{R}$ and ψ is bounded from below by a positive constant on $[s_0,\infty)$, we conclude that ψ_ε is positive on $\mathbb{R}$, for all ε small.

Finally, for every fixed $\varepsilon \in \mathcal{E}_2 \setminus \{0\}$, we have that

$$g(x) = px + q(x)x^2, \qquad \psi_\varepsilon(t) = A_\varepsilon \exp\bigl(\lambda_1(\varepsilon)t\bigr) + b_\varepsilon(t)\exp(2\lambda_* t),$$

where $q \in C[0,+\infty)$ and b_ε is bounded on $(-\infty,0]$. Hence, $\psi_\varepsilon(t) = A_\varepsilon \exp(\lambda_1(\varepsilon)t) + O(\exp(2\lambda_* t))$ at $-\infty$ and

$$g\bigl(\psi_\varepsilon(t-h)\bigr) = A_\varepsilon p \exp\bigl(-\lambda_1(\varepsilon)h\bigr)\exp\bigl(\lambda_1(\varepsilon)t\bigr) + c_\varepsilon(t)\exp(2\lambda_* t), \quad \varepsilon \in \mathcal{E}_2 \setminus \{0\},$$

where $c_\varepsilon(t)$ is bounded: $|c_\varepsilon(t)| \leqslant c_0(\varepsilon)$, $t \leqslant 0$. Differentiating (20), we obtain

$$\begin{aligned}\psi_\varepsilon'(t) &= \frac{1}{\sigma(\varepsilon)}\left(\frac{-2}{1+\sigma(\varepsilon)}\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}} g\bigl(\psi_\varepsilon(s-h)\bigr)\,ds\right.\\ &\quad\left. + \frac{1+\sigma(\varepsilon)}{2\varepsilon^2}\int_t^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}} g\bigl(\psi_\varepsilon(s-h)\bigr)\,ds\right)\\ &= \frac{A_\varepsilon p\exp(-\lambda_1(\varepsilon)h)}{\sigma(\varepsilon)}\left(\frac{-2}{1+\sigma(\varepsilon)}\int_{-\infty}^{t} e^{\frac{-2(t-s)}{1+\sigma(\varepsilon)}}\exp\bigl(\lambda_1(\varepsilon)s\bigr)\,ds\right.\\ &\quad\left. + \frac{1+\sigma(\varepsilon)}{2\varepsilon^2}\int_t^{+\infty} e^{\frac{(1+\sigma(\varepsilon))(t-s)}{2\varepsilon^2}}\exp\bigl(\lambda_1(\varepsilon)s\bigr)\,ds\right) + O\bigl(\exp(2\lambda_* t)\bigr)\\ &= A_\varepsilon\lambda_1(\varepsilon)\exp\bigl(\lambda_1(\varepsilon)t\bigr) + O\bigl(\exp(2\lambda_* t)\bigr), \quad t \to -\infty. \qquad (27)\end{aligned}$$

Hence, $\psi_\varepsilon, \psi_\varepsilon' \in C_{\lambda_1(\varepsilon)}(\mathbb{R})$ so that $\Psi_\varepsilon \in C_{2\lambda_1(\varepsilon)}(\mathbb{R})$ in (25). Therefore, in view of [24, Proposition 7.1] and the inequality $\lambda < \lambda_1(\varepsilon)$, $\varepsilon \in \mathcal{E}_2 \setminus \{0\}$, we get from (25) that $\psi_\varepsilon(t) = A_\varepsilon \exp(\lambda_1(\varepsilon)t) + O(\exp(2\lambda t))$ and $\psi_\varepsilon'(t) = A_\varepsilon\lambda_1(\varepsilon)\exp(\lambda_1(\varepsilon)t) + O(\exp(2\lambda t))$, $t \to -\infty$. □

4. Nonmonotonicity of travelling waves

As it was noticed in [1,9,10,12,20,29], various investigators have studied numerically the case of a large delay in the Nicholson's blowflies equation, and noted a loss of monotonicity of the wave front as the delay increases, "*with the front developing a prominent hump*" whose height "*is bounded above by a bound that does not depend on the delay*," see [1, p. 308] from which the above citation was taken. It is not difficult to explain the second phenomenon, since at every point of local maximum σ of $\psi(t,c)$ we have $\psi'(\sigma,c)=0$, $\psi''(\sigma,c)\leqslant 0$ so that $\psi(\sigma,c)\leqslant g(\psi(\sigma-h,c))\leqslant \max_{x\geqslant 0} g(x)$. Here we explain also the first phenomenon, easily getting the oscillation of the travelling waves about K as $t\to+\infty$ stated in Theorem 1 from the next two lemmas.

Lemma 15. *Let $g'(K)<0$, $h>0$ and $|g'(K)|he^{h+1}>1$. Then the equation*

$$\varepsilon^2z^2-z-1+g'(K)\exp(-zh)=0 \tag{28}$$

has no negative real roots, for all sufficiently small ε. Moreover, if the equilibrium K of (4) *is hyperbolic, then, for all small ε, there are no roots of* (28) *on the imaginary axis.*

Proof. Set $\Delta_\varepsilon(z)=\varepsilon^2z^2-z-1+g'(K)\exp(-zh)$. We first prove that the lemma is valid for $\varepsilon=0$ (see also [14]). Let z_0 be the maximum point of $\Delta_0(z)$ on $\mathbb{R}$, i.e., $z_0\in\mathbb{R}$ is such that $\Delta_0'(z_0)=-1-hg'(K)\exp(-z_0h)=0$. Note that $\Delta_0(-\infty)=-\infty$ and $\Delta_0(z)<0$ for $z\geqslant 0$. If there is a negative zero of $\Delta_0(z)$, then $z_0<0$ and $\Delta_0(z_0)=-z_0-1-1/h\geqslant 0$, implying that $0=\Delta_0'(z_0)\geqslant -1+|g'(K)|he^{h+1}$, which contradicts the hypothesis $|g'(K)|he^{h+1}>1$.

If $h|g'(K)|\geqslant 1$, then $\Delta_\varepsilon'(z)\geqslant 2z\varepsilon^2-1+e^{-zh}>z(2\varepsilon^2-h)>0$ for all $z<0$ and $\varepsilon^2<h/2$, hence $\Delta_\varepsilon(z)<0$ for $z\leqslant 0$. Now, let $h|g'(K)|<1$, so that $z_0<0$. For $|\varepsilon|>0$ small, by the implicit function theorem we conclude that there is a negative root $z(\varepsilon)$ of the equation $\Delta_\varepsilon'(z)=0$ with $z(0)=z_0$; moreover, $z(\varepsilon)$ is the absolute maximum point of $z\mapsto\Delta_\varepsilon(z)$ on $(-\infty,0]$. Since $\delta(\varepsilon):=\Delta_\varepsilon(z(\varepsilon))$ depends continuously on ε and $\delta(0)<0$, for $\varepsilon>0$ small we have $\Delta_\varepsilon(z)<0$ for all $z\leqslant 0$.

We now prove that (28) has no roots on the imaginary axis. First, notice that $|\Delta_\varepsilon(ib)|\geqslant |g'(K)|>0$ if $b>2|g'(K)|$. For $\varepsilon=0$, Eq. (28) does not have roots on the imaginary axis, therefore $|\Delta_0(ib)|>0$, $|b|\leqslant 2|g'(K)|$. Hence, $|\Delta_\varepsilon(ib)|>0$ for all ε small and $|b|\leqslant 2|g'(K)|$, which implies the hyperbolicity of Eq. (28). □

The next lemma can be considered as an extension of the linearized oscillation theorem from [15] to the second order delay differential equation

$$\varepsilon^2x''(t)-x'(t)-x(t)+g\big(x(t-h)\big)=0,\quad t\in\mathbb{R}. \tag{29}$$

Lemma 16. *Assume* (H) *and that $g'(K)he^{h+1}<-1$. For small $\varepsilon>0$, set $(Dx)(t)=\varepsilon^2x''(t)-x'(t)-x(t)+g'(K)x(t-h)$. Then every nonconstant and bounded solution $x:\mathbb{R}\to\mathbb{R}$ of* (29) *such that $x(+\infty)=K$ oscillates about K.*

Proof. Consider some nonconstant solution $x:\mathbb{R}\to\mathbb{R}$ of (29) such that $x(+\infty)=K$. If we suppose for a moment that, for some $\eta\in\mathbb{R}$, it holds $x(s)=K$ identically for all $s\geqslant\eta$, then we obtain easily that $x(s) = K$ for all $s \in [\eta - h, \eta]$. Hence x should be a constant solution, in contradiction with our initial assumption. Therefore either $\sigma(t) = x(t) - K$ oscillates about zero or is eventually nonconstant and nonnegative, or nonpositive. In order to get a contradiction, assume that σ is not oscillatory. Notice that σ satisfies the following linear asymptotically autonomous delay differential equation

$$\varepsilon^2\sigma''(t) - \sigma'(t) - \sigma(t) + \gamma(t)\sigma(t-h) = 0, \quad t \in R,\ \gamma(+\infty) = g'(K) < 0, \tag{30}$$

where $\gamma(t) = g'(K) + c_0(t)$ and $c_0(t) = g'(K + \theta(t)\sigma(t-h)) - g'(K)$ for some $\theta(t) \in (0,1)$ given by the mean value theorem. Since $x(t)$ is bounded on $\mathbb{R}$ and $x(t) \to K$ as $t \to +\infty$, we can use the integral representations (20) and (27) to prove that $\lim_{t\to+\infty} x'(t) = 0$. From Lemma 15, it follows that the equation $(Dx)(t) = 0$ is hyperbolic, hence the equilibrium $(K, 0)$ of the system $x'(t) = v(t)$, $\varepsilon^2 v'(t) - v(t) - x(t) + g(x(t-h)) = 0$ is hyperbolic for all sufficiently small ε. Thus the trajectory of $x(t)$ belongs to the stable manifold of the hyperbolic equilibrium K of (29), so that we can find $a > 0$ such that $\sigma(t) = O(e^{-at})$, $\sigma'(t) = O(e^{-at})$ at $t = +\infty$. Therefore we have $c_0(t) = O(\sigma(t-h)) = O(e^{-at})$ at $+\infty$. From [24, Proposition 7.2] (see also [18, Proposition 2.2]), we conclude that: (i) either there are $b \geqslant a$, $\delta > 0$ and $u(t)$ a nontrivial eigensolution of the limiting equation

$$\varepsilon^2 u''(t) - u'(t) - u(t) + g'(K)u(t-h) = 0 \tag{31}$$

on the generalized eigenspace associated with the (nonempty) set Λ of eigenvalues with $\Re e\lambda = -b$, such that $\sigma(t) = u(t) + O(\exp(-(b+\delta)t)$, $t \to +\infty$; (ii) or $\sigma(t)$ decays superexponentially at $+\infty$. However, this latter condition is not possible: as it was established in [19, Lemma 3.1.1], if $\gamma(+\infty) \neq 0$ then every eventually nontrivial and nonnegative solution of (30) does not decay superexponentially (see also [18, Lemma A.1] for the case $\gamma(+\infty) > 0$). On the other hand, from Lemma 15 we know that there are no real negative eigenvalues of (31): hence $\Im m\,\lambda \neq 0$ for all $\lambda \in \Lambda$. From [18, Lemma 2.3], we conclude that $\sigma(t)$ is oscillatory. □

Finally, we observe that due to the exponential stability of the positive steady state, which implies fast convergence, numerical heteroclinic solutions $\psi(t,c)$ exhibit only one or two well pronounced humps, see [9, Fig. 2].

Acknowledgments

This research was supported by FONDECYT (Chile), projects 7040044 (Teresa Faria) and 1030992 (Sergei Trofimchuk), by FCT (Portugal), program POCTI/FEDER, under CMAF (Teresa Faria), and by CONICYT (Chile) through PBCT program ACT-05 (Sergei Trofimchuk). This work was partially written while T. Faria was visiting the University of Talca, and she thanks the University for its kind hospitality.

The authors express their gratitude to the anonymous referee, whose valuable comments helped to improve the original version of this paper.

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